§ 4 Namba - forcing

IN' is a variation of the Namba conditions IN which is discussed at length in [PIF] and by us in [LF] § 6 and [SPSC]. We define IN as the set of all subtrees T of 
\((\omega_2 \times \omega_2 = \text{the set of monotone} f: m \rightarrow \omega_2)

such that \(m \leq \omega_1\) if 

whenever \(T \in T\), then \(\{ t | t \leq T \} \) 

has size \(\omega_2\). 

IN' is the set of \(T \in \text{IN} \) s.t. for some \(S \in T\), 

\(T = \{ t | t \leq s \lor s \leq t \} \) 

and if \( t \geq s \) in \( T \), then \( t \) has \( \omega_2 \) many immediate successors.

In both cases the partial ordering of the conditions is inclusion.

An [PIF] it was shown that, assuming CH, we have:

- IN' adde no reals
- \( \text{IN'} \) is different from \( \text{IN} \) in the
sense that no IN’-generic contain an IN-generic sequence and conversely.

(Note We say that \( c = \langle s_i : i < \omega \rangle \) is IN-generic iff \( c = c_G = UNG \), where \( G \) is IN-generic. \( G \) is then recoverable from \( c \) as the set \( G_c \) of \( T \in IN \) in which \( c \) is a branch. Similarly for IN’.)

An [SPSC] we showed that, assuming \( CH + 2^{\omega_1} = \omega_2 \), IN’ is subproper, (hence can be iterated without collapsing \( \omega_2 \)). Here we assume only CH and show that IN’ is \( \omega_1 \)-subproper and Dee-subproper. Hence it can be iterated without adding reals.

From now on assume CH.
$\mathcal{N}'$ satisfies a weak amalgamation lemma:

Definition: $\text{rtm}(T) = \text{the stem of } T = \text{the maximal}\ n \in T \text{ s.t. } \forall t \in T \ (s \leq t \Rightarrow s \leq t)$

Lemma 1: Let $T \in \mathcal{N}'$, $u = \text{rtm}(T)$. Let $\langle Tu \mid u \in (\omega_2, \omega) \rangle$ be $\mathcal{T}$.

- $Tu \in \mathcal{N}'$ and $\emptyset = T$
- $|Tu| = |\omega| + |u|$ where $u = \text{rtm}(Tu)$
- $u^<v \neq u^<\gamma$ for $v < \gamma < \omega_2$

Then $T' \in \mathcal{N}'$, where

$$T' = \bigcap_{m < \omega} Tu = \bigcup_{m < \omega} T_{\text{fin}}^m$$

(Note: $\text{rtm}(T') = \text{rtm}(T \restriction \omega_1)$)

The refinement lemma reads:

Lemma 2: Let $T \in \mathcal{N}'$, $f : T \to \omega_1$.

There exist $T' \leq T$, $g : \omega \to \omega_1$ s.t.
- $\text{rtm}(T') = \text{rtm}(T)$
- $f(t) = g(\text{rtm}(t))$ for $t \in T'$.

This is proven with a game-theoretic argument due to Shelah, a proof is given in [LF]§6. Using these two lemmas one can show:
Lemma 3 \( \text{IN}' \) adds no reals.

A proof of this is also given in [LR] 86.

We now prove:

Thm 4 \( \text{IN}' \) is Dee.-subproper.

The proof stretches over many sublemmas. We first define the completeness system which will verify Dee.-subproperness. Let \( \overline{N} = \cup \overline{L} \) be a countable \( \text{ZFC} \)-model and let \( \overline{\text{IN}} \)'s \( \overline{N} \) satisfy the definition of \( \text{IN}' \) in \( \overline{N} \). Let \( \overline{C} = \langle \overline{c}_i \mid i < \omega \rangle \) be monotone and cofinal in \( \overline{\omega}_2 = \omega_2 \overline{N} \).

Set:

- \( A_{\overline{C}} \) is the set of \( G \) s.t.
  - \( G \) is \( \overline{\text{IN}} \)'-generic over \( \overline{N} \)
  - \( \text{cx} = \forall \overline{G} \text{ is } \overline{\text{IN}} \text{-generic over } \overline{N} \),

Let \( \overline{G} = \langle \overline{g}_i \mid i < \omega \rangle = \overline{c}_0 = \cup \overline{G} \). Then there is \( m < \omega \) s.t. for all \( i > m \), there is \( \overline{u}_i < \omega \) s.t.

At \( u \in \text{H}_{\overline{G}} \), \( \text{a} \) such a \( \overline{c} \), we simply set:

- \( \overline{A}_{\overline{u}} \) is the set of \( G \) s.t. \( G \) is \( \overline{\text{IN}} \)'-generic over \( \overline{N} \).
Set:\n\[ \mathcal{D}(\bar{N}, \bar{N}') = \{ A_u \mid u \in \omega_1 \} \].

Lemma 4.1 \( \mathcal{D} \) is a complete new system.

Proof:\nWe verify (a) - (d) in the definition of "complete new system". (a), (d) are trivial. We verify (b), (c).

(b) Let \( T \in \bar{N}' \), \( \pi = \text{Atm}(T) \), \( n = |T| \).

Claim: There is \( G \in T \) s.t. \( G \in \mathcal{D}(\bar{N}, \bar{N}') \).

Proof:\nLet \( c = \langle \bar{s}_i \mid i < \omega \rangle \) be monotone and cofinal in \( \bar{\omega}_2 = \bar{\omega}_2^{\bar{N}} \). We show:

Claim 1: There is \( G \in T \) s.t.

- \( G \in \bar{N}' \)-generic over \( \bar{N} \)
- \( \forall i \geq m \forall j_i : (\bar{s}_i \leq \bar{s}_j_i \leq \bar{s}_{j_i+1} \leq \bar{s}_{j_i+1}) \)

where \( c = \langle \bar{s}_i \mid i < \omega \rangle = \bar{c}_G \).

The following easily by:
Claim 2 Let $T \in \overline{N}$, $x = \text{itm}(T)$. Let $\Delta \in \overline{N}$ be strongly dense in $\overline{N}$. There is $T' \leq T$ in $\overline{N}$ s.t. $T' \in \Delta$ and, let $\text{itm}(x') = \text{itm}(T')$.

$(*)$, for $1 \leq i < i+1 \leq |x'|$, then $\forall i \; x_i \leq x_i' < x_{i+1}' \leq x_{i+1}$.

Proof

For $t \in T$ set:

$$m(t) = \begin{cases} 1 & \text{if } \exists T' \leq T \text{ s.t. } \text{itm}(T') = t \wedge T' \in \Delta \\ 0 & \text{if not} \end{cases}$$

By the refinement lemma there is $T^* \leq T$ s.t. $\text{itm}(T^*) = \text{itm}(T)$ and $m(t) = g(1+1)$ for $t \in T^*$.

But then $g(m) = 1$. For some $m$, since there is $T' \leq T^*$ s.t. $T' \in \Delta$.

Let $g(m) = 1$. Clearly $|x| \leq m$.

where $x = \text{itm}(T)$. But then there is $x' \in T^*$ s.t. $|x'| = m$ and $(*)$ holds for $x'$.

QED (Claim 2)
This proves (b). We prove (c). Let \( C_i^* = \langle \bar{s}_i^* \mid i < \omega \rangle \) be monotone and cofinal in \( \bar{\omega}_2 \) for \( i < \omega \). Set \( s_i^* = \sup \{ \bar{s}_j^* \mid \bar{h}, j \leq i \} \) for \( i < \omega \). Then \( C^* = \langle s_i^* \mid i < \omega \rangle \) is monotone and cofinal in \( \bar{\omega}_2 \). Clearly, however, \( A_{C^*} \subset \bigcap_i \bar{A}_{C_i}^* \).

QED (Lemma 4.4)

Now let \( \Theta > 2^{2^{\omega_2}} \). We show:

Main Claim \( \langle \Theta, \bar{D} \rangle \) witnesses the \( \text{Dec}_{\text{subproperness}} \) of \( \bar{N} \).

Let \( N = L^\alpha \) be a \( \text{ZFC}^- \) model s.t. \( H_\Theta \subset N \) and \( \Theta < \bar{\omega} \). Let \( \pi : \bar{N} \prec N \) s.t. \( \bar{N} \in \text{countable and full} \), where \( \bar{\omega} \in N \), \( \bar{N} \) is countable and full, and \( \pi(\bar{\omega}, \bar{N}^\prime) = \Theta, N^\prime \). We claim that there is a \( \bar{C} = \langle \bar{s}_i \mid i < \omega \rangle \) monotone and cofinal in \( \bar{\omega}_1 = \omega_1 \bar{N} \) s.t. whenever \( \bar{C} \in \bar{A}_C \) and \( \bar{x} \in \bar{N} \), \( \bar{x} = \pi(\bar{C}) \),
Then there is $T \in \text{IN}',$ forcing that whenever $G \in T$ is $\text{IN}',$ generic, then there is $\sigma \in V[G]$ s.t.

(a) $\bar{T} \in \text{IN}$

(b) $\sigma((\bar{B},\bar{B'},\bar{\pi}) = \bar{B},\bar{B'},\bar{\pi}$

(c) $\mathcal{C}_{\text{in}}^2 \left( \text{rng} \sigma \right) = \mathcal{C}_{\text{in}}^2 \left( \text{rng} \bar{\pi} \right)$

(d) $\sigma" T \subseteq \mathcal{G}.$

An order to find such a $\sigma,$ we work in $V[G^*],$ where $G^* \in \text{IN}',$ generic.

Note that $H_{\omega_2}^V = H_{\omega_1}^V.$

**Def.** Let $c = \langle \delta_i : i < \omega \rangle$ be monotone and cofinal in $\omega_2.$ $c$ is good iff whenever $F \in V, F : \omega_2 \to \omega_2,$ then $\forall m \exists i \geq n \delta_i > \sup_{h < i} F(\delta_h^m).$

**Lemma 4.1.** There is a good $c \in V[G^*].$

**Proof.**

Let $c = G^* = \wedge UG^*.$ Then $c$ is good. To see this, note that if $F \in V, F : \omega_2 \to \omega_2,$ and
$T \in \mathbb{N}'$, then $T' \in \mathbb{N}'$ where $T'$ is the set of $t \in T$, $t'$. Letting $m = \text{tm}(T)$,

$$\forall i \in [m, m+1) \quad t_i' > \sup_{h < i} F(t_h).$$

Thus $T' \leq T$ in $\mathbb{N}'$ and $T' \models \forall i \exists \delta_i \exists s_i > \sup_{h < i} F(s_h').$

QED (Lemma 4.2)

(Note This shows that every $\mathbb{N}'$-generic sequence is good. However, no $\mathbb{N}'$-generic sequence is good.)

Until further notice let $C$ be a fixed good sequence. We note that if $C' = <\delta_i' | i < \omega>$ is monotonic and cofinal in $\omega_1$ and $\forall m \forall i \exists \nu_{<m} \forall j \delta_i' \leq \delta_i < \delta_{i+1}'$, then $C'$ is trivially good.

Define $N' = \langle N, \emptyset, \omega, \mathbb{N}', \delta, \tau, \pi \rangle$

$\sigma^* : \mathbb{N}' \to \mathbb{Z}$, where $\mathbb{Z}$ is the smallest

$Z < \mathbb{N}'$ such that $C \subset Z$.

Let $\mathbb{N}' = \langle \tilde{\mathbb{N}}, \tilde{\emptyset}, \tilde{\mathbb{N}}', \tilde{\delta}, \tilde{\tau}, \tilde{\pi} \rangle$. 


Then \( \sigma^* : \tilde{N}' \rightarrow N' \).

**Definition**

Let \( D = \{ \sigma^* \text{ of } \tilde{N}' \text{ definable } \} \subset \omega_2 \),

\[ \tilde{D} = (\sigma^*)^{-1}[D] \]

\[ F = \sigma^* \cap \tilde{D} \]

Then \( F \) bijects \( \tilde{D} \) onto \( D \).

**Lemma 4.3**

\( F \in \mathcal{V} \)

**Proof**

Clearly \( D \in \mathcal{V} \) is countable. Let \( \langle \tilde{x}_i \mid i < \omega \rangle \)
enumerate \( D \). Set \( \tilde{x}_i = (\sigma^*)^{-1}(x_i) \).

Then \( \langle \tilde{x}_i \mid i < \omega \rangle \in \mathcal{H}_{\omega_1} \subset \mathcal{V} \). Hence

\[ F = \{ \langle \tilde{x}_i, \tilde{x}_i \rangle \mid i < \omega \} \subset \mathcal{V}, \text{ QED (4.3)} \]

**Definition**

Let \( f : \tilde{\omega}_2 \rightarrow \omega_2 \) cofinally, where \( \tilde{\omega}_2 = \text{pr} \omega_2 \), \( f \) is smooth iff for each \( X \in \tilde{D} \) and all \( \tilde{i}, \tilde{j}, \tilde{m} < \tilde{\omega}_2 \),

\( \tilde{x}_1, \ldots, \tilde{x}_n \in X \rightarrow \langle f(\tilde{x}_1), \ldots, f(\tilde{x}_n) \rangle \in F(X) \),

(2.1 being Gödel tuple function on ordinals).

**Lemma 4.4**

Let \( f \) be smooth. There is a unique extension \( f^* : \tilde{N}' \rightarrow N' \).
Proof of Lemma 4.4

(1) \( N' = \Phi[\tilde{\nu}] \) \( \iff \) \( N' = \Phi[f(\tilde{\nu})] \)

for \( \nu_1, \ldots, \nu_m < \tilde{\omega}_2 \).

Proof.

There is \( \gamma \in \mathcal{D} \) s.t.

\( \forall \tilde{\nu} \in \gamma \iff N' = \Phi[\tilde{\nu}] \) for \( \nu_1, \ldots, \nu_m < \tilde{\omega}_2 \).

Set \( \tilde{\gamma} = (\gamma^*)^{-1}(\gamma) \). Then

\( \forall \tilde{\nu} \in \tilde{\gamma} \iff \tilde{N}' = \Phi[\tilde{\nu}] \) for \( \nu_1, \ldots, \nu_m < \tilde{\omega}_2 \).

The conclusion is immediate. \( \varnothing \) \( \mathcal{E} \) \( \mathcal{D} \) \( \gamma \)

But each \( x \in \tilde{N}' \) is \( \tilde{N}' \)-definable from

ordinals \( \nu_1, \ldots, \nu_m < \tilde{\omega}_2 \). Let \( x = \tilde{g}(\tilde{\nu}) \), \( \tilde{g} \) be a definable function, set:

\( f^*(x) = g(f(\tilde{\nu})) \),

where \( g \) is \( \tilde{N}' \)-definable by the same formula. It is apparent from (1) that

this definition is independent of the choice of \( \tilde{\nu} \) and the defining formula

of \( \tilde{g} \). Now let \( \nu_1, \ldots, \nu_m \in \tilde{N}' \), with

\( \nu_1 = \tilde{g}_1(\tilde{\nu}) \) for \( i = 1, \ldots, m \). By (4) we

have:

\( \tilde{N}' = \Phi[\tilde{x}] \iff \tilde{N}' = \Phi[\tilde{g}(\tilde{\nu}), \ldots, \tilde{g}_m(\tilde{\nu})] \iff \)

\( \iff \tilde{N}' = \Phi[g(f(\tilde{\nu})), \ldots, g_m(f(\tilde{\nu}))] \iff \)

\( \iff \tilde{N}' = \Phi[f^*(\nu_1), \ldots, f^*(\nu_m)] \).
This proves existence. To see uniqueness, let \( f' \) have the same property. Then
\[
 f'(\tilde{\nu}(\nu)) = g(f(\nu)) = f^*(\tilde{\nu}(\nu)),
\]
QED (Lemma 4.4)

**Lemma 4.5** Let \( f^* \) be as above. Set:
\[
 C = C^{N_\omega}_{\omega_2}(\text{rng } \tilde{\pi}), \quad \tilde{C} = C^{\tilde{N}_\omega}_{\omega_2}(\text{rng } \tilde{\pi}).
\]
Then \( \tilde{C} \subseteq \tilde{N} \) and \( f^*(\tilde{C}) = C \).

**Proof.**
Let \( \Pi : \tilde{N} \leq N^* \) cofinally. Then
\[ N^* = N \setminus \lambda \in N \text{ and } N^* \not\subseteq N. \]
where
\[ \lambda = \text{min} N^*. \]
Hence \( C = C^{N^*_\omega}_{\omega_2}(\text{rng } \tilde{\pi}) \).
Similarly \( \tilde{C} = C^{\tilde{N}^*_\omega}_{\omega_2}(\text{rng } \tilde{\pi}) \), where
\( \tilde{\pi} : \tilde{N} \leq \tilde{N}^* \) cofinally. But \( f^*(\tilde{N}^*) = N^* \)
and \( f^*(\tilde{\pi}) = \tilde{\pi} \).
QED (4.5)

**Lemma 4.6** There is \( \tilde{\sigma} : \tilde{N} \leq \tilde{N} \) s.t.

- \( \tilde{\sigma} \) takes \( \tilde{\omega}_2 = \omega_2^{\tilde{N}} \) cofinally to \( \omega_2 \).
- \( C^{\tilde{N}}_{\omega_2}(\text{rng } \tilde{\sigma}) = C^{\tilde{N}}_{\omega_2}(\text{rng } \tilde{\pi}) \).
- \( \sigma(\tilde{\sigma}, \tilde{N}, \tilde{\pi}) = \tilde{\sigma}, \tilde{N}, \tilde{\pi} \)
Let \( \mathcal{N} \) be a model of \( L_{\infty\omega} \).

Then \( \vec{\omega} = \vec{\omega}_{\vec{\omega}} \) has the derived

proposition. OED (Lemma 4.6)

(\( \vec{\omega} = \vec{\omega}_{\vec{\omega}} \)) is modelled by:

Then \( \vec{\omega} \) is consistent since the

corresponding language \( L_{\infty\omega} \) on \( \mathcal{N}_{\infty\omega} \). For instance:

\[
\vdash \neg \exists x (x \in \mathcal{N}_{\infty\omega} \land x \cdot x = 1)
\]

\[
\vdash \neg \exists x (x \in \mathcal{N}_{\infty\omega})
\]

The map \( \vec{\omega} \to \vec{\omega}_0 \) is the lift of \( \vec{\omega} \to \vec{\omega}_0 \).

Let \( \vec{\omega}_0 \in \mathcal{N} \)

\( \vec{\omega}_0 \). Let \( \vec{\omega}_0 \in \mathcal{N} \).

The lift \( \vec{\omega}_0 \to \vec{\omega}_0 \) commutes with \( \vec{\omega}_0 \).

Language on \( L_{\infty\omega} \):

\[
\vdash \neg \exists x (x \in \mathcal{N}_{\infty\omega})
\]

\[
\vdash \neg \exists x (x \in \mathcal{N}_{\infty\omega} \land x \cdot x = 1)
\]
(Note: N is called full if N is a second order ZFC- model in \( L_{\omega_1 (N)} \), where \( L_{\omega_1 (N)} \) satisfies ZFC-.)

The concepts "lift-up" and "almost full" are elucidated in [SPLC] §3.

From now on let \( \bar{\sigma} \) be a fixed map \( \bar{\sigma} : \bar{\omega} \to \bar{\omega} \) satisfying Lemma 4.6.

Lemma 4.7 Let f be a smooth and set \( \bar{\sigma} = f^{\ast} \bar{\sigma} \). Then \( C_{\omega_2} (\omega \cup \bar{\sigma}) = C_{\omega_2} (\omega \cup \bar{\sigma}) \).

Proof:

(1) \( \text{rng} \bar{\sigma} = f^{\ast} \text{rng} \bar{\sigma} \subset f^{\ast} (C_{\omega_2} (\omega \cup \bar{\sigma})) = f^{\ast} (C_{\omega_2} (\omega \cup \bar{\sigma})) = C_{\omega_2} (\omega \cup \bar{\sigma}) \) by Lemma 4.6.

Hence \( C_{\omega_2} (\omega \cup \bar{\sigma}) \subset C_{\omega_2} (\omega \cup \bar{\sigma}) \).

(2) Let \( x \in C_{\omega_2} (\omega \cup \bar{\sigma}) \), Then \( x = \pi (f) (\tilde{x}) \) where \( \tilde{x} \in \omega \) and \( f \in \bar{\omega} \). Hence \( \pi (f) (\tilde{x}) \in C_{\omega_2} (\omega \cup \bar{\sigma}) \). Then \( x = \pi (f) (\tilde{x}) = \sigma (h) (f (\tilde{x})) (\tilde{h}) \in C_{\omega_2} (\omega \cup \bar{\sigma}) \), where \( \tilde{h} \in \omega \).

\( \square \)
Remark $\mathcal{F} = \sigma^* \mathcal{F}$ is smooth with $\mathcal{F} \mathcal{F} = \sigma^* \mathcal{F}$. Hence $\sigma = \sigma^* \sigma$.  

Lemma 4.8. Let $\sigma$ be as above. There is $c' \in V[\mathcal{G}^*]$ s.t. $c'$ is good and $c' \subseteq \text{rng} \sigma$. 

Proof. Let $c = \langle \delta_i | i < \omega \rangle \in V[\mathcal{G}^*]$ be good. Define $c' = \langle \delta_i' | i < \omega \rangle$ by $\delta_0' = 0$, $\delta_i' = \text{the least} \delta \in \text{rng} \sigma$ s.t. $V_i \delta' \leq \delta < \delta_{i+1}$, $\delta < \delta_i$. QED (4.8)

Now let $\sigma = \sigma^* \sigma$. Assume that $c \in V[\mathcal{G}^*]$ is not only good but that $c \subseteq \text{rng} \sigma$. Set $\hat{c} = (\sigma')^{-1} c$. 

We show: Claim $\hat{c} \in A_\Omega$. There is $\mathcal{T} \in \text{IN}$ satisfying the conclusion of the Main Claim.

Let $\hat{c}' = \langle \delta_i' | i < \omega \rangle = \sigma \hat{c} = \mathcal{T} \cup \hat{c}$. 

Set $\hat{c}' = \sigma^* \hat{c}'$. Let $L$ be the following language over the admissible structure $\langle H_\Theta, < \rangle$, where $<$ well orders $H_\Theta$. 


Predicate: $\varepsilon$, Constants $f$, $x$ ($x \in H_\omega$)

Axiom: ZFC - $\text{H}_\omega = \text{H}_{\text{Suc}}$, $\forall x (x \in \text{H}_{\omega_1} \rightarrow \forall u \in \text{H}_{\omega_1} u \text{ is smooth } x = u)$,
$f: \omega_2 \rightarrow \omega_2$ cofinally.

$\exists \omega \in \omega_2$ such that $\omega$ is smooth

$\text{Ax}_\text{hm} \land \omega_2 \in \text{H}_{\omega_2} \
\forall x \in \text{H}_{\omega_2} (\forall u \in \text{H}_{\omega_2} u \rightarrow \langle f(u), x \rangle \in E(x))$

for $m < \omega$.

Clearly this is consistent, since it is modeled by $\langle H_\omega, <, E \rangle$.

We now define a subtree $T' \subset T$ by:

Def: $T' \subset T$ iff $x = \langle \tilde{c}, i \rangle$ in consistent

where $\tilde{c} = \langle \tilde{c}_i \mid i \in \omega \rangle = \tilde{c} \upharpoonright \tilde{c}'$.

Thus $\tilde{c}' = \langle \tilde{c}_i \mid i \in \omega \rangle$ is a branch through $T'$.

Lemma 4.1: There is $T' \subset T$ s.t.

- $T' \in \text{H}'$
- $x = \langle \tilde{c}_i \mid i < 121 \rangle$ where $x = \text{stm} (T')$

Proof:

For $m < \omega$ consider the following game $G_m$: In the $i$-th move player I picks an $\gamma_i$ s.t. $\gamma_i > \gamma_j$ for $j < i$.

Player II then picks $\gamma_i$ if possible - a $\gamma_i$ s.t. $\langle \gamma_m, i \rangle \in T$.

If $i < m$ he must pick $\gamma_i = \tilde{c}^i$.

If $i \geq m$ then $\gamma_i > \tilde{c}^i$. 
At some \( i \) II has no move, then I wins. Otherwise II wins.

**Claim** For arbitrarily large \( m \), II has a winning strategy.

**Proof.**

Suppose not. Then I has a winning strategy for all \( m \geq m_0 \). Let \( D \subseteq w_2 \) be the set of \( S \) s.t. for all \( m \geq m_0 \) and all plays \( <\nu_0, \ldots, \nu_m> \) by II s.t. \( \nu_m < S \), we have \( S(\nu_0, \ldots, \nu_i) < S \) for \( i \leq m \), where \( S_m \) is I's winning strategy. Then D \( \in \) club in \( w_2 \). Set:

\[ F(3) = \text{the least } S \in D \text{ s.t. } 3 < S. \]

Since \( D \) is good, there is \( m \geq m_0 \) s.t.

\[ \forall i \geq m \, \delta_i > \sup_{h<i} F(\delta_h). \]

But then

\[ <\delta_i | i < \omega> \] defeats the strategy \( S_m \). Contradiction! QED (Claim).

Now let \( S \) be II's winning strategy for \( G_m \). Let \( T \) be the set of all \( S(\gamma_0, \ldots, \gamma_i) \) where \( <\gamma_0, \ldots, \gamma_i> \) is any possible play by I. Then...
Let $T'$ be as in Lemma 4.9. We show that $T'$ satisfies the conclusion of the Main Claim. Let $G \in T'$ be $\mathbb{N}'$-generic. We claim that there is $\sigma \in V[G]$ satisfying (a1)-(d1) of the Main Claim. Let $C'' = \langle \delta_i'' | i < \omega \rangle = C_G = \bigwedge U G$. For $i < \omega$, let:

- $f_i'' = \text{the } N\text{-least } f : \omega_1 \rightarrow (\delta_i'' + 1)$
- $\tilde{f}_i' = \Pi_{N'} f : \tilde{\omega}_1 \rightarrow (\tilde{\delta}_i' + 1)$

Set $f_i = f_i'' \circ (\tilde{f}_i', -1)$. Then $f_i : \tilde{\delta}_i' + 1 \rightarrow \delta_i'' + 1$. But if $M$ is a $N'$-valid model of $\mathcal{L} + \exists \tilde{\delta}_i' = \tilde{\delta}_i' / \tilde{\delta}_i$ and $f = \tilde{f}_i'$, then $f$, being smooth, extends to $f^* : \tilde{N}' \subseteq N'$. Hence $f^*(\tilde{f}_i') = f_i''$ and $f_i = f^*(\tilde{\delta}_i' + 1)$. Hence $f_i < f_i$ for $i < \omega$. Set $f = \bigcup f_i$. 


Then:
(1) \( f: \tilde{\omega} \to \omega \) so finally and \( f(\tilde{\delta}_i^*) = \delta_i^* \).
Moreover:
(2) \( f \) is smooth.

Proof:
Let \( \tilde{\omega}_1, \ldots, \tilde{\omega}_n \) be \( \tilde{\omega} \). Then \( \tilde{\omega}_i \) for some \( i \).
Let \( \tilde{\delta}_i \) be a solid model of \( \tilde{\delta} \), \( \tilde{f}(\tilde{\delta}_i^*) = \delta_i^* \).
Then \( \tilde{f} \) is smooth and \( \tilde{f}(\tilde{\delta}_i^*) = \tilde{f}^*(\delta_i^*) \).

By the above argument. Hence, for \( X \subseteq \tilde{A} \), we have:
\[
\{ \tilde{\omega}_i \} \subseteq X \iff \{ \tilde{f}(\tilde{\omega}_i) \} \subseteq \tilde{f}(X) \subseteq \tilde{f}(\tilde{A}) \subseteq \tilde{F}(\tilde{X}) \]
\[ \text{P.E.D. (2)} \]

But then \( f \) extends to:
(3) \( f^*; \tilde{N} \rightarrow \tilde{N} \),
and we get \( \tilde{\omega} = f^* \tilde{\omega} \). We show that
(1)-(d) \( \tilde{\omega} \) the main Claim hold for the
\( \tilde{\Omega} \). (a), (b) are immediate by (3) and
\( \tilde{\omega}(\tilde{\delta}, \tilde{N}^*, \tilde{\omega}) = \tilde{\omega}, \tilde{N}^*, \tilde{\omega} \).
\( (c) \) follow by
Lemma 4.7. Finally, (1d) follows, since
\( \tilde{C} = C \), \( \tilde{C}' = \tilde{C} \), and \( f^*(\tilde{C}') = \tilde{C} \).

P.E.D (Theorem 4)
It remains only to show that \( N \) is \( \omega_1 \)-subproper. Before doing so, we attempt to distill some more information from the proofs just given. From now on let \( \Theta \) be \( (2^{\omega_2})^+ \). At \( i \) enough, however, to take \( \Theta \) as having this meaning in the sense of an \( N \) which is \textit{viable} in the following sense:

\[
\text{Def: } \ N \text{ is viable iff } N = L^A_i \cup \text{ a model of } ZFC^- + CH
\]

- There are \( \Theta, H \in N \) s.t.
  \[
  L_\varphi(N) = (\Theta = (2^{\omega_2})^+ \land H = H_\Theta)
  \]

When dealing with a fixed viable \( N \) we shall write \( \omega_1, \omega_2, \Theta, H_\Theta, N', \text{etc.} \) to denote the relativization of the concept to \( N \). All that we did in the proofs just given goes through for an arbitrary viable \( N \).
Assuming \( N = L^\alpha \) to model \( 2 \models C^+ + CH \),
\( N \) will be viable in either of the cases:
- \( 2^{\omega_2} = \Theta \) in \( V \) and \( V^N \subset N \)
- \( N \) is almost full.

For transitive \( M, C \subset M \), let:
\[
M[c] = \bigcup_{\delta \in \text{On}_M} L_{\text{On}_M} \cap (Tc \cup \{c\})
\]

We define:

**Def.** Let \( N \) be viable and \( c = \langle \gamma_i \mid i < \omega \rangle \)
monotone and cotfinal in \( \omega_2 \).

\( c \) is tame if the following hold:
- \( \delta_N = \delta_N[c] \)
- \( H^N_{\omega_1} = H_{\omega_1} \cap L_{\delta} (N[c]) \) \( (\delta = \delta_N) \)
- \( c^+ = (2^{\omega_2})^+ \) in \( L_{\delta} (N[c]) \)
- \( H^N_{\Theta} [c] = H^N_{\Theta} \cap L_{\delta} (N[c]) \)

\( c \) is good if for all \( F : \omega_2 \to \omega_2 \) in \( N \)
we have:
\[
\forall m \forall i \exists n \forall j > \sup F(\gamma^*_n) \text{ s.t. } \gamma^*_i > \gamma^*_n,
\]

Clearly, any \( L^N \) - generic sequence over \( N \) in both good and tame.
Lemma 5.1  Let $c$ be tame. Let $c' \in N[c]$ s.t. $c' = \langle g_i' : i < \omega \rangle$ is monotone and cofinal in $\omega_2$. Then $H_\theta[c] = H_\theta[c']$. (Hence $c'$ is tame.)

Proof. $c' \in H_\theta[c]$, since $c$ is tame.

Claim $c \in H_\theta[c']$

Let $f$ map $\omega_1$ onto $\omega_2$, $f \in H_\theta[c']$, $s + c = f^{-1} \circ c$. Then $\bar{c} \in H_{\omega_1}$ and $c = f \circ \bar{c} \in H_\theta[c']$. QED (5.1)

Now let $\pi : \tilde{N} \leq \tilde{N}$ where $\pi \in N$ and $\tilde{N} = \tilde{L}_{\tilde{c}}$ is countable and full in $N$.

Let $\pi(\tilde{c}, \tilde{N}, \tilde{\tau}) = \theta, \tilde{N}', \tilde{\tau}, \tilde{c} \in \tilde{N}$ being arbitrary. We can literally repeat our previous proof: Setting $N' = \langle N, \theta, N', \tilde{\tau}, \tilde{\pi} \rangle$, let $Y = \{ \text{the smallest } Y \leq N', \text{ s.t. } c \leq Y \},$ where $c$ is tame for $N'$. Then $Y \in L_\delta(N[c])$ where $\delta = \delta_N$. Hence $\tilde{N}', \tilde{\tau} \in L_\delta(N[c])$ where
\( \sigma \ast \tilde{\mathcal{N}} \to \tilde{Y} \) and \( \tilde{\mathcal{N}} = \langle \tilde{N}, \tilde{\sigma}, \tilde{\mathcal{N}}, \tilde{x}, \tilde{\kappa} \rangle \) is transitive. Hence \( \tilde{\mathcal{N}} \in H_{\omega_1} \). Define \( P, Q, F = \sigma \ast \tilde{\mathcal{N}} \) exactly as before. As before we have \( P, Q, F \in H = H_{\omega_2} \). Define the notion of a smooth \( f : \tilde{\omega}_2 \to \omega_2 \) exactly as before. A smooth \( f \) then again extends uniquely to \( f^* : \tilde{\mathcal{N}} \to \mathcal{N} \) s.t. \( f \ast \tilde{\omega}_2 = f \). As before we get the existence of \( \tilde{\sigma} \in H_{\omega_1} \) s.t. \( \tilde{\sigma} : \tilde{\mathcal{N}} \to \mathcal{N} \), \( \tilde{\sigma}(\tilde{\omega}, \tilde{\mathcal{N}}, \tilde{x}) = \tilde{\sigma}, \tilde{\mathcal{N}}, \tilde{x} \), and

\[
C_{\tilde{\omega}_2}(\text{sing} \tilde{\sigma}) = C_{\omega_2}(\text{sing} \tilde{\sigma}).
\]

Thus setting \( \sigma = f \ast \tilde{\sigma} \), where \( f \in \mathcal{N}[c] \) is smooth, we get:

\[
C_{\omega_2}(\text{sing} \sigma) = C_{\omega_2}(\text{sing} \tilde{\sigma}),
\]

But then:

Lemma 5.2 \( \sigma \in \mathcal{N}[c] \).

**Proof:**

Clearly \( \sigma \in L_{\omega}[N[c]] \). Let \( k : N^* \to C_{\omega_2}(\text{sing} \tilde{\sigma}) \).

Since \( \text{sing} \sigma \subset C_{\omega_2} \), we can let \( U = k^{-1}(\text{sing} \sigma) \). Then \( U \subset N^* \) is countable, where \( N^* \in H_{\omega_1} \). Hence
\[ u \in H_0 \mathbb{C} = H_0^{N[C]}, \text{ Hence,} \]
\[ \text{ring } \sigma = k''u \in N[C], \text{ Q.E.D. (5.2)} \]

The main thing we now distill from this is:

**Lemma 5.3** Let \( c \) be tame. There is \( \sigma \in N[C] \) s.t.
- \( \sigma \mathcal{N} \mathcal{N} < \mathcal{N} \)
- \( \sigma (\bar{\mathcal{N}}, \mathcal{N}, \mathcal{N}) = \mathcal{N}, \mathcal{N}, \mathcal{N} \)
- \( C_{\omega_2}^N (\text{ring } \sigma) = C_{\omega_2}^N (\text{ring } \mathcal{N}) \).

But by the proof of Lemma 4.8:

**Lemma 5.4** Let \( \sigma, c \) be as above, where \( c \) is both tame and good. There is \( c' \in N[C] \) s.t. \( c' \mathcal{C} \text{ ring } \sigma \) and \( c' \) is tame and good (hence \( N[C] = N[C'] \)).

Note: It might appear that our formulation of Lemma 5.3 throws out valuable information since, in order to get \( \sigma \), we first defined \( \bar{\sigma}, \bar{\mathcal{N}}, \bar{\sigma} \).

Using Lemma 5.4, however, we can quickly restore the missing information:
Def \( \langle c, \sigma \rangle \) is excellent for \( N, \pi \) iff

1. \( c \) is tame and good
2. \( \sigma \in \mathbb{N}[c] \)
3. \( \sigma; N \leq N \)
4. \( \sigma(N, \sigma(N)) = \theta, N' \)
5. \( C_{\omega_2}^{N}(\text{rng } \sigma) = C_{\omega_2}^{N}(\text{rng } \sigma') \)
6. \( c \in \text{rng } \sigma \)

Lemma 5.3, 5.4 Then say:

Cor 5.5 Let \( c' \) be tame and good, let \( \pi(\bar{a}) = a \).

There is an excellent \( \langle c, \sigma \rangle \) s.t. \( \sigma(\bar{a}) = a \)
and \( N[c] = N[c'] \).

From now on suppose that \( \langle c, \sigma \rangle \) is excellent, \( \bar{a} \in \bar{N} \), and \( \sigma(\bar{a}) = 2 \).

(\( \bar{a} \) is not necessarily the same as our previous \( \bar{a} \) with \( \pi(\bar{a}) = 2 \).) Set:

\( N' = \langle N, \theta, N', \pi, a \rangle \). Since \( \text{rng } \sigma \subset C_{\omega_2}^{N}(\text{rng } \sigma') \), each \( x \in \text{rng } \sigma \) is \( N' \)-definable from elements of \( \omega_1 \cup C \).

Since \( \sigma \) is countable in \( L_\omega(N[c]) \), there is \( a \leq \omega_1 \) s.t. each \( x \in \text{rng } \sigma \) is \( N \)-definable in parameters from \( \omega_1 \cup C \).
Let \( Y = \) the smallest \( Y \lessdot N \) s.t. \( \forall U \subset C \subset Y \).
Then \( Y \in L_\sigma(N) \). Hence \( \sigma^* \tilde{Y} \in L_\sigma(N) \),
where \( \sigma^* \tilde{Y} \lessdot Y \) and \( \tilde{Y} = \langle \tilde{N}, \tilde{O}, \tilde{N}', \tilde{F}, \tilde{S} \rangle \)
is transitive. Hence \( \tilde{N}' \in H_{\omega_1} \). Set \( \tilde{\sigma} = (\sigma^*)^{-1} \sigma \).
Letting \( D \) be the set of \( N' \)-definable \( X \subset \omega_1 \) and \( \tilde{D} = (\tilde{\sigma})^{-1}(D) \),
\( F = \sigma^* \tilde{D} \), we again have that \( D \in H_\emptyset \)
is countable in \( N \) (arguing in \( L_\sigma(N) \)).
Now follow as before that \( B, F \) are countable in \( N \),
we define the notion of smooth function \( f : \tilde{\alpha}_1 \to \omega_1 \)
before and again conclude that each smooth \( f \) extend uniquely to an
\( f^* : \tilde{N}' \to N' \). As before, we have
\( f^*(\tilde{C}) = \tilde{C} \), where \( \tilde{C} \in C^{\tilde{N}}(\text{rng}\ \tilde{\omega}_1) \),
\( \tilde{C} = C^{\tilde{N}_{\tilde{\omega}_1}}(\text{rng}\ \tilde{\omega}_1) \). Now set \( \tilde{\sigma} = (\sigma^*)^{-1} \sigma \).
Then \( \tilde{\sigma} : \tilde{N} \lessdot \tilde{N} \). Moreover, \( \tilde{C} = C^{\tilde{N}}(\text{rng}\ \tilde{\sigma}) \).
(To see this note that \( C^{\tilde{N}}(\text{rng}\ \tilde{\omega}_1) = C \);
\( \sigma^* \text{ is injective on } \omega_1 \).}
\[ C^{\omega_2}(\text{rng}\ \tilde{\sigma}) = (\sigma^*)^{-1}(C) = C \)
if \( f \) is smooth and \( \sigma^* = f^* \tilde{\sigma} \), we can
conclude as before that \( \mathcal{N}_{\omega_2} \cap \text{rng } \sigma' = \emptyset \).

Now let \( \bar{\xi} = \sigma^{-1} \xi' \). Define \( A_{\bar{\xi}} \) as before (relative to \( N \)).

**Lemma 5.6** Let \( \mathcal{T} \in \mathcal{N}' \), \( \tau = \text{stm}(\mathcal{T}) \), \( m = \mu \).

Then \( \exists \bar{\xi} \in A_{\bar{\xi}} \mid \mathcal{T} \in \mathcal{G} \) is dispersed at \( \tau \).

We now prove a technical lemma.

**Lemma 5.7** Let \( A \subseteq A_{\bar{\xi}} \) be dispersed at \( \tau \).

Let \( \sigma(\bar{\xi}) = \tau \). There are \( \bar{\xi} \in A \), \( \mathcal{T} \in \mathcal{N}' \), such that

(a) \( \text{stm}(\mathcal{T}) = \sigma(\bar{\xi} \cap m) \)

(b) \( \mathcal{T} \cap \mathcal{G} \subseteq \mathcal{N}' \) is generic over \( N \), then

\( \sigma' \in N[N[G]] \) such that

- \( \sigma' \in \mathcal{N} \)
- \( \sigma'(\bar{\xi}, \mathcal{N}, \bar{\xi}) = \theta, \mathcal{N}', \varphi \)
- \( C_{\omega_2} (\text{rng } \sigma') = C_{\omega_1} (\text{rng } \sigma) \)
- \( \sigma' \cap G \subseteq G \)
proof of 5.7,
Define $L$ on $\langle H_\theta, < \rangle$ as before:

Predicates: $C_i$; Constants: $f^i$, $x$ ($x \in H_\theta$)

Axioms: $\varepsilon \in \varepsilon \subset H_\theta$, $\omega_1 = H_{\omega_1}$, $F_n \in H_\theta$,

$\forall \nu, \varepsilon \in \omega_1 \Rightarrow \forall \nu, \varepsilon \in \omega_1, f^i_\nu \omega_2 \rightarrow \omega_2$ coherently;

"$f$ is smooth" - i.e. $x$ for $n < \omega$!

$\forall \nu, m < \omega_2 \forall \varepsilon \in \omega \exists \nu \in \nu \exists \delta : (\langle \varepsilon, \nu \rangle \in x \Rightarrow \langle f^i_\nu \rangle \in F(x))$.

Set $A^\ast = \{ d \mid \forall \varepsilon \in A \ d = \varepsilon - 3 \}$, $F_n \in A^\ast$.

let $\hat{\nu} = \varepsilon$, and define $T_d \subset (\omega_2 < \omega$ as before:

Def $t \in T_d \Rightarrow \exists + \nu \cdot \nu \in d$ and define $T_d \subset (\omega_2 < \omega$

Claim: There is $d \in A^\ast$, $T \in T_d$, s.t.

- $T \in \mathbb{N}'$
- $t = \sigma(d \uparrow \nu)$ where $t = \sigma \nu_m (T)$

We define $\forall d \in A^\ast$ a game $q(d)$ as follows. At stage $\gamma$;

- I plays an $\nu < \omega_2$ s.t. $\gamma > \sup_{h < \gamma} \gamma_h$.
- II plays, if possible, a $\nu_i$ s.t. $\langle \nu_0, \ldots, \nu_i \rangle \in T_d$.

If $i < m$, II must choose $\nu_i = \sigma(d_\nu_i)$.

If $i > m$, II must choose $\nu_i > \gamma_i$.

At some $i$, II has no move, then I wins.

Otherwise II wins.
Subclaim II: There is a winning strategy for some \(d \in A^*\).

**Proof.**

Suppose not. For every \(d \in A^*\), \(I\) then has a winning strategy \(S_d\). Let \(C\) be the set of \(\lambda < \omega_2\) s.t. for all \(d \in A^*\),

\[
(\forall m < \lambda) \ni (\forall v \in A_m) \ni \exists \omega_2 \ni \lambda \not< \lambda.
\]

Then \(C\) is a club in \(\omega_2\). But then, letting

\[
F(\bar{s}) = \text{the least } \lambda \in C \ni \bar{s} \not< \lambda,
\]

\(<\xi_i : i < \omega = C \text{ being an } \omega \text{ good sequence with } \bar{c} = \sigma^{-1}(C) \). Since \(A\) is dispersed at \(m\), there \(\exists d \in A^*\) s.t.

\[
\delta_m \geq \bar{s}_m \land (\forall i \geq m) \forall v (\delta_i \leq \bar{s}_i < \bar{s}_{i+m} \leq \delta_{i+m}),
\]

But then \(<\sigma(\delta_i) : i < \omega >\) is a sequence of moves which defeats \(S_d\). Contradiction!

**QED (Subclaim)**

Now let \(S\) be \(I\)'s winning strategy for \(d \in A^*\). Let \(T\) be the set of finite sequences obtained by applying \(S\) to a finite sequence of moves by \(I\),

\(T\) has the desired properties.

**QED (Claim)**
Now let $G \in T$ be $1^N$-generic over $N$. As before, let $\mathcal{A} = \langle \delta_i : i < \omega \rangle$, $\mathcal{A}' = \langle \delta_i' : i < \omega \rangle$.

There is a unique smooth $f$ such that $f(\delta_i) = \delta_i'$ for $i < \omega$. Moreover, $f \in N[\mathcal{A}]$ (hence $f^* \in N[\mathcal{A}]$). But then $\tilde{\sigma} = f^* \circ \tilde{\sigma}$ has the desired properties.

\textit{G.E.D. (Lemma 5.7)}
We are now ready to prove:

**Thm 6.** \( N' \in \omega_1 \)-subproper.

The proof will again stretch over several sublemmas. In the following let \( N \) be viable, \( \alpha < \omega_1 \), and let \( \pi = \langle \pi^i \mid i < \alpha \rangle \) be a tower for \( N \) with \( N^i = \overline{N^i} \) and \( \pi^i(\Theta, i^N_i) = \Theta_i N_i^i \) for \( i < \alpha \). We must show that, for any finite \( u \subseteq N^\alpha \) and any \( \pi \in N' \), \( \pi \) has a \( \langle \Theta, N' \rangle \), \( IN', G \)-revision \( \sigma \) which coincides with \( \pi \) on \( u \) and \( \pi \) at \( \pi \in G \).

We first show the existence of a somewhat simpler sort of revision:

**Def.** By an excellent revision of \( \pi \) we mean a pair \( \langle c, \sigma \rangle \) s.t.
- \( c = \langle c^i \mid i \leq \alpha \rangle \)
- \( \sigma = \langle c^i \mid i \leq \alpha \rangle \) is a revision of \( \pi \)
- \( \langle c^\alpha, \sigma^\alpha \rangle \) is excellent for \( N \)
- \( \langle c^i, \sigma^i, i^+ \rangle \) is excellent for \( N^i+1 \) (\( i < \alpha \))
- \( \sigma \in N[C^\alpha] \)
- \( \langle \sigma^h, i^+ \mid h \leq i \rangle \in N^{i+1}[C^i] \) (\( i < \alpha \)).

(\textit{Hence } \langle c^\alpha \rangle_{i < \alpha} \text{ is good and tame for } N \text{ and } c^i \subseteq \omega_2 \text{ for } N^{i+1} \text{ is good and tame for } N^{i+1}.)

It follows that \( \langle c^i \mid i < \alpha \rangle \subseteq N \) and \( \langle c^i \mid i < \alpha \rangle \subseteq N^{i+1} \text{ for } i < \alpha \).\)
Lemma 6.1 Let $c' \in \omega_2$ be good and tame for $N$. Let $a \in N^d$ be finite. There is $<c, \sigma> \in N[c']$ which is an excellent revision of $\bar{\pi}$ coinciding with $\bar{\pi}$ on $a$.

(Hence $N[c'] = N[c]$ by Lemma 6.1.)

Proof: By induction on $a$.

Case 1: $a = 0$. Trivial by Cor. 5.5.

Case 2: $a = \beta + 1$

Let $<c', \sigma'>$ be excellent for $N$, $N^d$ with $\sigma' : \mu = \bar{\pi}^d \mu$, (Thue exist. in $N[c']$ by Cor 5.5.)

Set $\bar{\sigma} = (\pi^d)^{-1} \sigma' \mu$, $\bar{\pi} < \pi^d: i \leq \beta$.

There exists $c' \in H_{\omega_1}$ which is good and tame for $N^\alpha$ (e.g., an $IN^\alpha$-generic sequence). Hence, by the induction hypothesis, there is $<\bar{c}, \bar{\sigma}> \in H_{\omega_1}$ which is an excellent revision of $\bar{\pi}$ coinciding with $\bar{\pi}$ on $\bar{\sigma}$.

$c' = \begin{cases} c' & \text{if } i = a \\ \bar{c}' & \text{if } i < a \end{cases}$

$\bar{\sigma} = \begin{cases} \sigma' & \text{if } i = a \\ \bar{\sigma}' & \text{if } i < a \end{cases}$

$<c, \sigma>$ has the desired properties.

QED (Case 2)
Case 3 $\alpha$ is a limit ordinal.

Let $\langle \alpha_i : i < \omega \rangle$ be monotone and cofinal in $\alpha$ s.t. $\alpha_0 = 0$ and $\alpha_i = \beta_i + 1$ for $i > 0$.

We successively pick $\langle c(i), \sigma(i) \rangle$ ($i < \omega$) s.t. $\langle c(i), \sigma(i) \rangle$ is an excellent revision of the tower $\pi(i) = \langle \pi^\nu, c_i + \nu \mid a_i \leq \nu < c_i + 1 \rangle$ for $N^{c_i + 1}$. Setting

$\tilde{\pi}(i) = \langle \pi^\nu, c_i + \nu \mid \nu \leq \beta_i + 1 \rangle$ and defining

$\langle \tilde{c}(i), \tilde{\sigma}(i) \rangle$ by:

$\langle \tilde{c}(0), \tilde{\sigma}(0) \rangle = \langle c(0), \sigma(0) \rangle$

$\tilde{c}(i+1) = \{ \tilde{c}(i+1) \mid \nu < \alpha_i \}$

$\tilde{\sigma}(i+1) = \{ \tilde{\sigma}(i+1) \mid \nu \leq \beta_i \}$

We see that $\langle \tilde{c}(i), \tilde{\sigma}(i) \rangle$ is an excellent revision of $\tilde{\pi}(i)$ for $i < \omega$.

While doing this construction we ensure that $\tilde{\sigma}(i)$ coincides with $\pi(i)$ on $U_i$, where $U_i \subseteq N^{\beta_i + 1}$ is a finite set chosen so as to satisfy the conditions.
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- $U_i = U_i$ for $h \leq i$

- $(\prod_{h+1}^{i+1} \beta_h \lambda d)^{-1} \circ U_i \subseteq U_i$

- Letting $\langle x_i \mid i < \omega \rangle$ be a fixed enumeration of $\mathbb{N}^\omega$, we have:

  $(\prod_{h+1}^{i+1} \beta_h \lambda d)^{-1} (x_h) \subseteq U_i$

for $h < i$, whenever defined.

- Letting $\langle z_i, \nu \mid i < \omega \rangle$ be a fixed enumeration of $\langle x_i \rangle_\nu \mid \nu < \omega_1 \land \nu \in \mathbb{N}^\omega \rangle$, we have:

  if $h < i$, $\nu < \delta_i$, then

  $\prod_{h+1}^{i+1} \beta_h \lambda d (\sum_{i+1}^{(i+1) \beta_h \lambda (z_h)}) \subseteq U_i$.

At this point, we can set:

- $\mathcal{E}^i = \mathcal{E}(i) \nu$ for $\nu < \delta_{i+1}$

- $\mathcal{E}^i \nu \mathcal{E} = \mathcal{E}(i) \nu \mathcal{E}^i$ for $\mathcal{E} \leq \mathcal{E} < \delta_{i+1}$.

- $\langle \mathcal{E} \nu \mathcal{E}^i \mid \nu \leq \mathcal{E} < \delta \rangle$ is then a commutative, continuous system. By our construction, we have ensured that:

- At $\mathcal{E} \in \mathbb{N}^\omega$, there are $\nu < \delta, \mathcal{E} \in \mathbb{N}^\nu$

  such that $\mathcal{E} = \prod_{\mathcal{E}}^{\nu \mathcal{E}} (\mathcal{E})$ and

  $\prod_{\mathcal{E}}^{\nu \mathcal{E}} (\mathcal{E}) = \mathcal{E} \nu \mathcal{E}^i (\mathcal{E})$ for $\mathcal{E} \leq \mathcal{E} < \delta$,
At \( x \in N^r, r < a \), there is \( \tau \geq r \) s.t. \( \tau < a \) and for all \( \bar{z} \), if \( \tau \leq \bar{z} < a \), then
\[
\left( \bar{c} \right)^{\bar{z}}(x) = \left( \bar{c} \right)^{\tau}(x) = \left( c \right)^{\tau}(x).
\]

Arguing as before (cf. the proof of §2 Theorem 5), it follows that \( \langle N^r | r < a \rangle, \langle \left( \bar{c} \right)^{\bar{z}}, \bar{z} \leq \bar{z} < a \rangle \) has a unique directed limit of the form:

\( N^*, \langle \left( \bar{c} \right)^{\bar{z}}, \bar{z} < a \rangle. \)

Now let \( \langle c^*, \sigma^* \rangle \in N[c'] \) be excellent. Set:

\[
\begin{align*}
c^r & = \begin{cases} c^* \text{ if } r = a \\ \left( \bar{c} \right)^{\tau} \text{ if } r < a \end{cases} \\
\sigma^r & = \begin{cases} \sigma^* \text{ if } r = a \\ \sigma^*, \left( \bar{c} \right)^{\tau} \text{ if } r < a \end{cases}.
\end{align*}
\]

Then \( \langle c, \sigma \rangle \) has the desired properties.

\[\Box\text{ED (Lemma 6.1)}\]

We now define ?.
Def: \( \langle G, \sigma \rangle \) is a superb revision of \( \overline{\Pi} \) iff the following hold:

- \( \sigma \) is a revision of \( \Pi \)
- \( G \in N' \) - generic over \( N \)
- \( G' = \sigma^{-1} \) if \( G \in N' \) - generic over \( N' \) (i \( \leq \) d)
- Set: \( C^d = C \) and \( C' = C_{i+1} \) for \( i < \alpha \). Then \( \langle c, \sigma \rangle \) is an excellent revision of \( \overline{\Pi} \).

Remark: \( \langle G, \sigma \rangle \) is a superb revision of \( \overline{\Pi} \), then \( \sigma \) is a \( \langle \theta, IN' \rangle, IN', G \rangle - \) revision of \( \overline{\Pi} \). Hence the assertion of Thm 6 will follow from the conjunction of Lemma 6.1 and:

Lemma 6.2: Let \( \langle c, \sigma \rangle \) be an excellent revision of \( \overline{\Pi} \). Set \( \overline{C}^d = (\sigma^d)^{-1} \) and \( \overline{C}' = \sigma^{-1} \). Define \( A\overline{C} \) relative to \( N \) and \( \overline{A} \overline{C}' \) relative to \( N' \) for \( i < d \). Let \( u \in N^k \) be finite. Let \( \overline{T} \in IN' \). Then there exist \( T \in IN', \overline{c} \in A\overline{C} \) s.t.

- \( \sigma^0 (T) = \sigma^0 (\overline{T}) \)
- If \( G \in T \) is \( N' \) - generic over \( N \), then there is \( \sigma' \in N[G] \) s.t. \( \langle G, \sigma' \rangle \) is a superb revision of \( \overline{\Pi} \) s.t. \( G^0 = \overline{G} \), and \( \sigma' \) coincides with \( \sigma \) on \( u \).
We prove Lemma 6.1.2 by induction on \( \alpha \).

**Case 1** \( \alpha = 0 \).

Then \( c = (c^0,0) \), \( \sigma = (\sigma^0,0) \) where \( (c^0,\sigma^0) \) is excellent for \( N \). We know that

\[ A = \{ G \in A_{c^0} | \exists \tau \in G \} \] is dispersed at \( m = |\text{atm}(\tau)| \). The conclusion follows by Lemma 6.1.7.

**Case 2** \( \alpha = \beta + 1 \).

By the induction hypothesis there are \( \widetilde{G} \in A_{c^0} \), \( \widetilde{\tau} \in N^d \), \( \tau \in \widetilde{G} \) and

- \( \text{atm}(\tau) = \sigma^0 \cdot \alpha(\text{atm}(\tau)) \)
- \( \text{At} \widetilde{\sigma} \exists \widetilde{\tau} \in \text{IN}^{d,
} \widetilde{\tau} \in \text{IN}^{d_{\alpha}} - \text{generic over } N_{\beta^d} \) for \( \beta \geq 1 \).

There is \( \sigma' \in N_{\beta^d}[\widetilde{G}] \) s.t. \( \langle \widetilde{G}, \sigma' \rangle \) is a super generic version of \( \widetilde{G} = \{ \tau^i_d | i \leq \beta \} \) on \( \widetilde{\sigma} = (\sigma^0 \cdot \alpha)^{-1} \sigma \).

But \( \tilde{A} = \{ G \in A_{c^0} | \tau \in G \} \) is dispersed at \( m = |\text{atm}(\tau)| \). Since \( (c^d,\sigma^d) \) is excellent for \( N \), there are \( \widetilde{G} \in \tilde{A} \), \( \tau \in \text{IN}^{d_{\alpha}} \), \( \text{atm}(\tau) = \sigma^0 \cdot \alpha(\text{atm}(\tau)) \)

- \( \text{At} \sigma \exists \tau \in \text{IN}^{d_{\alpha}} - \text{generic over } N \), then there is \( \sigma' \in N[\sigma] \) s.t. \( \sigma' \in N^d \cdot N \); \( \sigma'(\theta, \text{IN}_{d^\alpha}) = \sigma_{\text{IN}_{d^\alpha}} \)
- \( \tau' \in N = \sigma^d \cdot \tau, C_{\omega^d} (\text{rng } \sigma') = C_{\omega^d} (\text{rng } \sigma'_{\text{IN}_{d^\alpha}}) \)
- \( C_{\omega^d} (\text{rng } \sigma') = C_{\omega^d} (\text{rng } \sigma_{\text{IN}_{d^\alpha}}) \), and \( \sigma' \) is \( \widetilde{G} \subset G \).
Now let $G \in T$ be $\mathbb{N}^-$-generic over $N$. Let $\tilde{x} \in N[G]$, $\tilde{G} \in A\tilde{G}$ be as above. Since $\tilde{G} \in T$, there is $\tilde{y} \in N^x[G]$, $\tilde{G} \in A\tilde{G}$ as above.

Set: \[ \sigma''^{\nu} = \begin{cases} \sigma', & \nu = 1 \\ \sigma^{\nu} & \text{if } \nu < 2 \end{cases} \]

Then $\langle G, \sigma''^{\nu} \rangle$ is a superb revision of $\pi$ and $\sigma''^{\nu}$ coincides with $\sigma$ on $U$. Q.E.D. [Case 2]

**Case 3** $\alpha$ is a limit ordinal.

We know that there is $d \in \omega_1$, which is good and tame for $N$ and $A\tilde{G}$, $d \in N[c^d]$, $d \in \text{rank}(\sigma^\circ)$. Hence $N[c^d] = N[d]$ and we may suppose w.l.o.g. that $c^d = d$. Set:

\[ d = \langle \delta_i \mid i < \omega \rangle \]

\[ d^{\nu} = (\sigma^\circ)^{-1}(d) = \langle \delta_i^{\nu} \mid i < \omega \rangle \quad (\nu < 2). \]

The notion of a superb $A$-revision $\langle \tilde{G}, \tilde{\sigma} \rangle$ is defined like that of a superb revision with $N^d$ in place of $N$ and $\tilde{\pi} = \langle \pi^{\cdot d} \mid i \leq d \rangle$ in place of $\tilde{\pi}$.

Equivalently:
\[ \text{Def: } \langle \tilde{G}, \tilde{\sigma} \rangle \text{ is a super $\delta$-revision of } \tilde{\pi} = \langle \pi, \delta \mid i \leq d \rangle \text{ if}
\]

- $\tilde{G}$ is $\mathcal{N}$-generic over $\mathcal{N}$
- $\tilde{\sigma} = \langle \tilde{\sigma}^i \mid i < d \rangle$ where $\tilde{\sigma}^i : \mathcal{N}^i \to \mathcal{N}$
- Set $\tilde{G}^i = (\tilde{\sigma}^i)^{-1} \tilde{G} (i < d)$ and $\tilde{G}^i = (\tilde{\sigma}^i)^{-1} \circ \tilde{G}^i (i \leq i < d)$.

Then $\langle \tilde{G}^i \mid i < d \rangle$ is a super $\delta$-revision of $\langle \pi \mid i < d \rangle$ for $i < d$.

**Lemma 6.2.1** Let $\tilde{\pi} \in \mathcal{N}^{i_0}$. Let $u \in \mathcal{N}$ be finite. There are $\tilde{G} \in \mathcal{A}_{\tilde{\pi}}$ and $\langle \tilde{G}, \tilde{\sigma} \rangle$ s.t.

(i) $\tilde{\pi} \in \tilde{G}$

(ii) $\langle \tilde{G}, \tilde{\sigma} \rangle$ is a super $\delta$-revision of $\tilde{\pi}$

(iii) Let $m = \mathcal{L} + \mathcal{L} (\tilde{\pi})$. Let $\tilde{G} = \langle \tilde{G}^i \mid i < \omega \rangle$.

Then $\tilde{G} \uplus m = \sigma : \delta_d (\tilde{G} \uplus m)$ and

\[ \bigwedge \tilde{G}^i \bigvee_{j} (\sup \tilde{\sigma}_h \leq \tilde{\delta}_j < \delta_{\tilde{\delta}_j + 1} \leq \tilde{\delta}_i) . \]

(iv) $\tilde{\sigma}$ coincides with $\langle \tilde{\sigma}^i \mid i < d \rangle$ on $u$ (i.e. $(\tilde{\sigma}^i)^{-1} \uplus u \subseteq (\tilde{G}^i)^{-1} \uplus u$ for $i < d$).

(\text{Note By (c) we have: } \tilde{G} \in \mathcal{A}_{\tilde{\pi}} . )
proof of Lemma 6.2.1

We successively construct \( \langle T_i, G_i, \sigma_i \rangle \) (\( i < \omega \)) s.t.

(a) \( T_0 = \bar{T} \in G_0 \), \( \sigma_0 = \emptyset \)

(b) \( T_{i+1} \in G_{i+1} \) and \( \langle G_{i+1}, \sigma_{i+1} \rangle \) is a superp recursion of \( T_i = \langle T_{i-1}, \sigma_{i-1} \mid \sigma_i \rightarrow T_i, \sigma_{i+1} \rangle \), i.e., \( G_{i+1}^{d_i} = G_i \) and \( \sigma_{i+1} \) coincides with \( \sigma_i \), \( \langle \sigma_i, d_{i+1} \rangle \mid \sigma_i \rightarrow T_i, \sigma_{i+1} \rangle \) on \( U_i \subseteq N^{d_{i+1}} \). (\( U_i \) must still be determined.)

(c) \(|\text{atm}(T_i)| = m + i\)

(d) \( G_{i+1}^{m+i} = \sigma_i \cdot d_i \cdot d_{i+1} \cdot (\text{atm}(T_i)) \)

\( T_0 = \bar{T} \) is given. Now let \( T_i \) be given and let \( \langle T_{i-1}, G_{i-1}, \sigma_{i-1} \rangle \) be given for \( n < i \) satisfying (a1 - ld). We pick \( G_i, T_i \) s.t.

\( G_i \in A^{d_i} \) and \( T_i \in N^{d_{i+1}} \) s.t.

(e) \( \text{atm}(T_i') = \sigma_i \cdot d_i \cdot d_{i+1} \cdot (\text{atm}(T_i)) \)

(f) Let \( G \in T_i \) be \( N^{d_{i+1}} \)-generic over \( N^{d_{i+1}} \). There is \( \sigma' \in N^{d_{i+1}}[G] \) s.t.

- \( \langle G, \sigma' \rangle \) is a superp recursion of \( T_i \)

- \( G^{d_i} = G_i \)

- \( \sigma' \) coincides with \( \sigma_i \) on \( U_i \).
At $i = 0$, we set $\sigma_0 = \emptyset$. At $i = h+1$, let $\sigma_i \in N^{d_i} [G_i]$ s.t. $\langle G_i, \sigma_i \rangle$ is a superreversion of $T_{ih}$, $G^{d_i}_i = G_{ih}$, and $\hat{\sigma}$ coincides with $\hat{\sigma}_h$ on $U_h$. (Such $\sigma_i$ exists, since (b) holds at $h$.)

We then define $T_{i+1}$ as a sub-tree of $T_i$ as follows: Let $i' = \text{rtm} (T_i)$, let $s' < \nu \in T_i$ s.t.

$$V_i \sup_{h < i} \delta_i^{d_i < h} \leq \delta_i^{d_i+i+1} \leq \nu.$$

Set $T_{i+1} = T_i (s' < \nu)$. The set of $t \in T_i$ s.t. $t \leq s' < \nu$ or $s' < \nu \leq t$ in $T_i$.

This defines $\langle T_i, G_i, \sigma_i, T_i \rangle$ ($i \leq \omega$) modulo the choice of the finite set $U_i \subset N^{d_i+i}$, which we must still determine.
At we then set 
\[ \tilde{\sigma}^h_{i+1} = \begin{cases} \tilde{\sigma}^h_{i+1} & \text{if } d_i \leq h < d_{i+1} \\ \tilde{\sigma}^{-d_i} \circ \tilde{\sigma}^h_{i+1} & \text{for } h < d_i \end{cases} \]
we see that \( \langle \tilde{\sigma}^h_{i+1}, \tilde{\sigma}^{i+1}_i \rangle \) is a subpart revision of \( \tilde{\sigma}^h_i = \langle \tilde{\sigma}^{-d_i} \circ \tilde{\sigma}^h_{i+1} \circ \tilde{\sigma}^{i+1}_i \mid h \leq \beta_i + 1 \rangle \).
At it is easily seen that we can define \( \tilde{\sigma}^h = \tilde{\sigma}^h_i \) for \( h < d_i \),
\( \tilde{\sigma}^{i+1}_i \) for \( h \leq i < d_i \).
The choice of \( i \) being immaterial.
Then \( u_i \subset N^{\beta_i + 1} \) are chosen so as to satisfy:
\( h < i \),
\( \tilde{u}_i \subset U_i \), where \( \tilde{u}_i = (\sigma_{\beta_i + 1})^{-1} U_i \).
Let \( \langle x_i \mid i < \omega \rangle \) enumerate \( N^d \).
Then \( (\sigma_{\beta_i + 1})^{-1} (x^h_i) \in U_i \) if defined and \( h < i \).
\( \langle (x_i, y_i) \mid i < \omega \rangle \) enumerate \( \langle x_i, y_i \mid i < \omega \rangle \) enumerate \( \langle x_i, y_i \mid i < \omega \rangle \) for \( \langle x_i, y_i \mid i < \omega \rangle \) with \( \xi \in N \). At \( h < i \) and \( x^h_i \), then \( \tilde{\sigma}^{i+1}_{\beta_i + 1} (x^h_i) \in U_i \).
Just as before, the directed system
\( \langle N^h | h < \alpha \rangle, \langle \tilde{\sigma}^h | h \leq \xi < \alpha \rangle \) has a
direct limit of the form:
\[ N^\alpha, \langle \tilde{\sigma}^h | h < \alpha \rangle. \]

Set \( \tilde{G} = \bigcup_{h < \alpha} \tilde{G}^h, \tilde{G} = \tilde{G}^0. \)

**Claim** \( \tilde{G}, \tilde{G}, \tilde{\sigma} = \langle \tilde{\sigma}^h | h < \alpha \rangle \) satisfy
(i) - (iv) of Lemma 6.2.1.

**Proof**
\( \tilde{G} \) is easily seen to be \( \text{IN}^\alpha \) - generic
over \( N^\alpha \), given that \( \tilde{G}^h \) is \( \text{IN}^{\alpha^h} \)
genetic over \( N^{\alpha^h} \). But then (ii) follows trivially, (i), (iii), (iv) are also
straightforward. We prove (iii):

Let \( G^h = \langle \tilde{G}^h | i < \omega \rangle, G^{\alpha^h} = \langle \tilde{G}^{\alpha^h} | i < \omega \rangle \)
for \( h < \alpha \). Then \( \tilde{\sigma}^h (\tilde{G}^h) = \tilde{G}^h \). Now let:

(1) \( \nu = \tilde{\sigma}^{d_{i+1}} \) is the maximal point in \( \mathfrak{N}^{m+1}(T_{i+1}) \).

**By our construction:**

(2) \( \tilde{\sigma}^{d_{i+1}+1} h (\nu) = \tilde{\sigma}^{d_{i+1}+1} h (\nu) \)

for \( d_{i+1} \leq h < \alpha \).

Let \( \rho = \tilde{\sigma}^{d_{i+1}+1} h (\nu) \) for \( d_{i+1} \leq h < \alpha \).
Then, for $h < \alpha$:

(3) $\tilde{\sigma}^h(n_h) = \sigma^h(n_h) = \nu \overset{\circ} = \nu^m + i$

Hence:

(4) $\tilde{\sigma}^h(n_{h+1}) = \sigma^h(n_{h+1})$ for $h < \alpha$

(To see this, let $f_h$ be the $N^h$-loop map of $\nu^h$ onto $n_{h+1}$ for $h \leq \alpha$.
Then $\tilde{\sigma}^h(f_h) = \sigma^h(f_h) = f_\alpha$. Hence $\tilde{\sigma}^h(n_{h+1}) = \sigma^h(n_{h+1}) = f_\alpha \circ (f_h^{-1})$).

We arranged, however, that

$$\sup \tilde{\sigma}^m_{d+i} \leq \tilde{\sigma}^d_{d+i} \leq \tilde{\sigma}^d_{d+i+1} \leq \nu$$

By (4) we conclude:

$$\sup \tilde{\sigma}^m_{m+i} \leq \tilde{\sigma}^d_1 < \tilde{\sigma}^d_{d+i} < \nu = \tilde{\sigma}^m_{m+i}$$

QED (Lemma 6.2.1)

Now fix $T \in N^\circ$, $u \in N^d$ as in Lemma 6.2.1 and set:

$A$ = the set of $\tilde{\sigma}$ s.t. for some $\tilde{\sigma}^{\circ}$,
the pair $(\tilde{G}, \tilde{\sigma})$ satisfies (i')-(iv') of Lemma 6.2.1,
Then \( A \subseteq A_{i:m} \) and \( \overline{e} \subseteq (\delta^i, 1 < \omega) \).

We prove:

**Lemma 6.2.2** \( A \) is dispersed at \( m \) (where \( m = |\text{atm} (\overline{n})| \)).

**Proof.**

Let \( n = |\text{atm} (\overline{n})| \). Let \( \delta < \omega \). Pick \( \nu \) s.t. \( \lambda \nu > \overline{n} \), \( \nu > \delta^\nu \), and there is \( j < \omega \)

s.t. \( \lambda \uparrow \nu \leq \delta^\nu \) and \( \delta^\nu < \delta^j < \nu \), \( \nu \leq \delta^\nu \) or \( \lambda \nu > \nu \leq \nu \),

\( \delta^\nu < \delta^\nu \) or \( \lambda \nu > \nu \leq \nu \),

Apply Lemma 6.2.1 to \( \overline{n} \), getting

\( (\overline{n}, \overline{\nu}) \). Then \( \overline{\delta}^\nu > \delta^\nu \) and

\( \lambda \uparrow \nu \leq \delta^\nu < \delta^\nu < \delta^\nu \), where \( \delta^\nu \leq \delta^\nu \) (and \( C_{\overline{n}} = (\delta^i, 1 < \omega) \)).

Since \( \text{atm} (\overline{n}) = m + 1 \), it follows

readily that \( \overline{n} \subseteq A \) with \( \nu > \delta^\nu \). \( \text{Q.E.D. (6.2.2)} \)
But then by Lemma 5.7 there is $\tilde{G} \in A$, $T \in N'$ r.t.

(a) $\text{st}(T) = \sigma^d(\sigma \wedge m)$

(b) If $G \ni T$ is $\mathcal{N}'$-generic over $N$, then

$\tilde{\sigma} \in \mathcal{N} \{ \sigma \}$ r.t. $\sigma'' \leq N$, $\sigma'' \upharpoonright \mathcal{N}'$ $= \tilde{\sigma} \upharpoonright \mathcal{N}'$

$\sigma''(\theta^d, \mathcal{N}'^d) = \theta \upharpoonright \mathcal{N}'$, $C_\mathcal{N}'(\sigma'' \upharpoonright \mathcal{N}') = C_\mathcal{N}'(\sigma'' \upharpoonright \mathcal{N}')$, and $\sigma'' \in \tilde{G} \subseteq G$.

Let $G \ni T$ be $\mathcal{N}'$-generic over $N$. Let $\tilde{G} \subseteq A$ be as above. By the definition of $A$ there are $\tilde{G} \subseteq A \subseteq \mathcal{N}$ and $\tilde{\sigma}$ r.t. (i) - (v) of Lemma 6.2.1 hold. Define $\tilde{\sigma} \in \mathcal{N} \{ \sigma \}$ by $\tilde{\sigma}' = \begin{cases} \tilde{\sigma}'' & \text{if } \nu = a \\ \tilde{\sigma}'' \upharpoonright \mathcal{N}' & \text{if } \nu < a \end{cases}$. Then $\tilde{\sigma}'$ verifies Lemma 6.2.

Q.E.D. (Thm 6)