§2 Subcomplete Forcing

We recall the definition:

Def. Let $IB$ be a complete $BA$. $IB$ is sub-
complete iff for sufficiently large card-
inals $\theta$ the following hold:

Let $IB \in H_\theta$. Let $\theta > \theta$ be regular s.t.
$H_{\theta} \subset W = L^A_\mathcal{E} (= \langle L_\mathcal{E}[\mathcal{A}], \epsilon, A \rangle)$. Let
\( \sigma : \overline{W} < W \) where $\overline{W}$ is countable, transitive,
and full. * Let $\sigma (\overline{\theta}, \overline{IB}, \overline{x}, \overline{x}_1, \ldots, \overline{x}_m) = \overline{\theta}, \overline{IB}, \overline{x}, \overline{x}_1, \ldots, \overline{x}_m$, where $\overline{x}_i$ is regular
s.t. $\overline{IB} < \lambda_i$ for $i = 1, \ldots, m$. Let $G$ be $\overline{IB}$-
generic over $\overline{W}$. There is $a \in IB \setminus \subseteq \theta$ s.t.,
whenever $G \ni a \in IB$ - generic, then there
is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \overline{W} < W$

(b) $\sigma_0 (\overline{\theta}, \overline{IB}, \overline{x}, \overline{x}_i) = \theta, \overline{IB}, \overline{x}, \lambda_i$ \( (i = 1, \ldots, m) \)

(c) $\sup_{\sigma_0} \overline{x}_i = \sup_{\sigma} \overline{x}_i$ \( (i = 0, \ldots, m) \),
where \( \overline{x}_0 = \text{On} \cap \overline{W} \)

(d) $\sigma_0 " G \subset G $,

* $\overline{W}$ is called full iff, letting $\mathcal{P}$ be least
s.t. $L_{\mathcal{P}}[\overline{W}] = \mathcal{F}C^-$, $\overline{W}$ is regular in
$L_{\mathcal{P}}[\overline{W}]$ (i.e. if $f \in L_{\mathcal{P}}[\overline{W}]$, $x \in \overline{W}$ and $f(x) \to \overline{W}$,
then $f \in \overline{W}$).
We call $IB$ weakly subcomplete iff there is an $x$ s.t. for sufficiently large $\theta$ the above holds whenever $x \in \text{rng}(\sigma)$. It turns out that weak subcompleteness implies subcompleteness.

The two-step iteration theorem says that if $IB$ is subcomplete and

$\models_{IB} C$ is subcomplete, then $IB \ast C$ is subcomplete. (In other words, if $IB \subseteq C$ and $\models_{IB} (\check{C}/C \ast \check{C})$ is subcomplete, then $C$ is subcomplete, $\check{C}$ being the canonical generic name.)

We shall now prove the following iteration theorem:

(We, of course, write $IB_{\check{C}}$ for $IB_{\check{C}}$.)
Thm 1. Let \( 
abla B_i = \{ B_i : i < \alpha \} \) be an RCS -

iteration set, for all \( i+1 < \alpha \):

(a) \( B_i \neq B_i+1 \)

(b) \( \nabla B, (B_i+1 \in \nabla \text{ complete}) \)

(c) \( \nabla B, \{ B_i \} \text{ has cardinality } \leq \omega_1 \)

Then every \( B_i \in \nabla \text{ complete} \).

proof

By induction we prove:

Claim: Let \( h \leq i \). Let \( G \) be \( B_h \) - generic. Then

\( B_i \) is complete in \( V[G] \).

The case \( h = i \) is trivial, since then

\( B_i \) is \( \emptyset \), \( 0 \), \( 1 \). Hence \( i = 0 \) is trivial.

Now let \( i = j + 1 \). Then \( B_j \) is \( B_i \) - generic over \( V[G] \). Then \( G' = G \) - generic over \( V \).

But then \( (B_i \in \nabla) = (B_i \in \nabla \text{ complete in } V[G'] = V[G][G] \). Hence

\( B_i \) is complete in \( V[G] \).

The two-step theorem, (Since \( B_j \in \nabla \) is complete and \( B_{i+1} \in \nabla \) is complete, and \( B_i \in \nabla \) is complete,)

\( B_i \in \nabla \) is complete.
There remain the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis, $IB_i | G_h$ is subcomplete in $V[G_h]$ whenever $h < i < \lambda$.

But then $\langle IB_{i+h} | G_h \mid i < \lambda - h \rangle$ satisfies the same induction hypothesis, since if $i' \leq i < \lambda - h$ and $\tilde{G} \in IB_{i+h} | G_h$ — generic over $V[G_h]$, then $G = G_h \times \tilde{G} \in IB_{i+h} | G_h$ — generic over $V$ and $(IB_{i+h} | G_h) / \tilde{G} \cong IB_{i+h} | G \in$ subcomplete in $V[G_h] = V[G_h][\tilde{G}]$. We consider two cases:

Case 1 of $(\lambda) \leq IB_i$ for an $i < \lambda$.

Then $(\lambda) \leq \omega_1$ in $V[G_i]$ whenever $i < j < \lambda$ and $G_i \in IB_j$ — generic. It suffices to prove the claim for such $j$ since if $h < j$ and $G_h \in IB_j$ — generic, we can use the two-step lemma to show — exactly as in the successor case — that $IB_j / G_h$ is subcomplete in $V[G_h]$.

Thus it will suffice to prove:

Claim Assume $(\lambda) \leq \omega_1$ in $V$. Then $IB_\lambda \in$ subcomplete.
since the same proof can then be carried out in $V[G']$ to show that $IB_\lambda / G'$ in
an $\omega$be complete. Fix $f: \omega_1 \to \lambda$ s.t. $\sup f'' \omega_1 = \lambda$.

Let $\theta > \lambda$ be a cardinal s.t. $\bar{B} < \theta$ and
$\theta$ is big enough that $\exists I_i$ (with inputs the sub-
completeness of $IB_{\lambda_i} / G$) for $i \leq \lambda$. Let $W = L^\alpha_x$,
where $\alpha$ is regular, $\alpha > \theta$ and $H_\theta \subseteq W$.

Let $\sigma : \bar{W} \subseteq W$ s.t. $\bar{W}$ is countable, transitive,
and full. Suppose moreover that:

$\sigma(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{I}_B, \bar{\alpha}, \bar{\lambda}_i) = f, \theta, \lambda, IB, \alpha, \lambda_i \ (i = 1, \ldots, n)$,

where $\lambda_i < \theta$ is regular s.t. $\bar{I}_B < \lambda_i$ for
$i = 1, \ldots, n$. (Note that $\sigma(\bar{I}_B^-) = IB_{\lambda_i}$.) Let
$\bar{G}$ be $\bar{I}_B -$ generic over $\bar{W}$. It suffices to
show:

Claim: There is a $G \in \mathbb{G}$ s.t. whenever $G \in C$,
$\exists IB_\lambda -$ generic, then there is $\sigma' \in V[C]$ s.t.

(a) $\sigma'' \bar{W} \subseteq W$
(b) $\sigma''(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{I}_B, \bar{\alpha}, \bar{\lambda}_i) = f, \theta, \lambda, IB, \alpha, \lambda_i \ (i = 1, \ldots, n)$
(c) $\sup \sigma'' \bar{\lambda}_i = \bar{\lambda}_i \ \ (i = 0, \ldots, n)$,

where $\bar{\lambda}_0 = \text{On} \cap \bar{W}$, $\bar{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$.

(d) $\sigma'' \bar{G} \subseteq G$. 

Note The use of the parameter $f, B$ is justified by the fact that weak subcompleteness implies subcompleteness.

Set: $\bar{x} = \sup \sigma^{\bar{w}} \bar{x}$; $\bar{x}_i = \sup \sigma^{\bar{w}} x_i$ ($i = 0, \ldots, \infty$).

At $\bar{x}$ easily verified that there is a sequence $\langle y_i | i < \omega \rangle$ in $\bar{w}$ s.t.

setting $\bar{y}_i = \sigma(y_i)$, we have: $\bar{y}_0 = 0$ and $\langle \bar{y}_i | i < \omega \rangle$

is monotone and cofinal in $\bar{x}$. Set:

$\bar{z}_i = \sigma(\bar{y}_i)$. Then $\bar{z}_i = \sigma(\bar{z}_i)$ and $\langle \bar{z}_i | i < \omega \rangle$

is monotone and cofinal in $\bar{x}$.

Moreover:

(1) $\sigma'(\bar{z}_i) = \bar{z}_i$ whenever $\sigma' : \bar{w} \to \bar{w}$ s.t.

$\sigma'(\bar{y}) = \bar{y}$.

For each $i = 0, \ldots, \infty$ choose $\langle \bar{z}_i^i | i < \omega \rangle$ which

is monotone and cofinal in $x_i$. Set

$\bar{z}_i^i = \sigma(\bar{z}_i^i)$. Then $\langle \bar{z}_i^i | i < \omega \rangle$ is monotone and cofinal in $\bar{x}_i$.

(However, we have nothing corresponding to (1).)

Our strategy will be to construct by

induction on $k < \omega$ a $c_k \in B_{\bar{z}_k}$ and a $\sigma \in V\bar{z}_k$

s.t. $\langle c_k | k < \omega \rangle$ is a thread through $\langle \bar{z}_k | k < \omega \rangle$ and the following holds:

\( 
\)
(1) Let \( G_k \subset C_k \) be \( \text{IB}^{\frac{1}{2}}_k \) - generic. Set:
\[ \sigma_i = \sigma_j^G_k = \sigma_j^{G_j} \text{ for } j \leq k, \text{ where } G_j = G_k \cap \text{IB}^{\frac{1}{2}}_j. \]

Then:
(a) \( \sigma_0 = \sigma \)
(b) \( \sigma_k : \overline{W} \rightarrow W \)
(c) \( \sigma_k(\overline{f}, \overline{\theta}_i, \overline{\lambda}, \overline{\text{IB}}, \overline{\lambda}, \overline{\lambda}_i) = \overline{f}_i \theta_i \lambda_i \text{IB}_i \lambda_i (i = 0, \ldots, n) \)
(d) \( \sup \sigma_k \overline{\lambda}_i = \overline{\lambda}_i \text{ for } i = 0, \ldots, n \)
(e) \( \sigma_k \overline{G} \subset G_k \) where \( \overline{G}_k = \overline{G} \cap \text{IB}^{\frac{1}{2}}_k \)
(f) \( \sigma_k (x_l) = \sigma_j^G (x_l) \) for \( j \leq l \leq k, \) where \( \langle x_k | l < \omega \rangle \) is a fixed enumeration of \( \overline{W} \).
(g) \( \exists i = 0, \ldots, m \) and \( k = j + 1 \) such
\[ \sigma_j (\overline{x}_m^i) = \overline{x}_m^i < \sigma_j (\overline{x}_{m+1}^i). \]
Then
\[ \sigma_k (\overline{x}_m^i) = \sigma_j (\overline{x}_m^i) \text{ for } l \leq m+1. \]

**\textbf{Note:}\** As in [ESPSC] we assume that the "natural injection" of \( V^{\text{IB}}_k \) into \( V^{\text{IB}}_d \) is the identity if \( k \leq d. \) Then \( ^{G_k}T = ^{G_k}T \) if \( t \in V^{\text{IB}}_k. \) This also implies:
\[ [\Phi(t_1, \ldots, t_m)]_{\text{IB}^k} = [\Phi(t_1, \ldots, t_m)]_{\text{IB}^k}. \]

**\textbf{Note:}\** We have \( \sigma_k (\overline{x}_i^j) = \overline{x}_i^j \) for \( i < \omega, \) since \( \sigma_k (\overline{f}). \) We do not necessarily have \( \sigma_k (\overline{x}_i^j) = \overline{x}_i^j \) even though \( \sigma_k \) takes \( \overline{x}_i^j \)
cofinally to \( \overline{x}_i^j. \)
Before constructing $c_i$, $\omega_i$ ($i < \omega$) and verifying (*1), we show that (*1) will prove the claim. Clearly $c(\tilde{x}) = \omega$ in $V_i$. Hence we can set $c = \bigcap_i \omega_i$; getting $c \neq 0$ and $c \in IB_\omega \subset IB_\lambda$. Let $G \ni c$ be $IB_\lambda$-generic. Set $G_h = G \cap IB_{\tilde{3}_h}$ for $h < \omega$, $\delta_h = \delta_h (G) = \delta_h (G_h)$, clearly, (*1) holds for all $h < \omega$, since $c_h \in G_h$.

But by (f) we can define a new $\tilde{\omega} \in \omega$ by: $\tilde{\omega}(x) = y$ if $\tilde{\omega}(x) = \gamma$ for sufficiently large $i$, $\tilde{\omega}(x) = y$ if $\tilde{\omega}(x) = \gamma$ for sufficiently large $i$. Then obviously that (a1), (b1) of the Claim hold. To prove (c1) we note first that $\tilde{\omega}(\tilde{x}) < \tilde{\omega}(\tilde{x})$, since if $y < \tilde{x}$, there is $k < \omega$.

$\tilde{\omega}(\tilde{x}) = \delta_1 (\tilde{\omega}) < \tilde{\omega}(\tilde{x})$. To see that $\tilde{\omega}(\tilde{x})$ is unbounded in $\tilde{x}$, let $\tilde{3}_i$ be given. Since $\sup \tilde{\omega}(\tilde{x}) = \tilde{x}$, there is $k < \omega$.

$\delta_i (\tilde{3}_m) < \tilde{3}_k < \delta_i (\tilde{3}_m)$ where $k = i + 1$.

Hence $\delta_i (\tilde{3}_m) = \delta_i (\tilde{3}_m)$ for $h \geq k$.

Hence $\delta_i (\tilde{3}_m) > \tilde{3}_k > \tilde{3}_j$, QED (c1)
We now prove (d1). We first note that \( \sigma' \cap G_h \subseteq G \) for \( h < \omega \), since if \( a \in G_h \), then \( \sigma'(a) = \sigma(h) \in G \) for sufficiently large \( h \).

If \( \chi = \omega_1 \), then \( \chi = \omega_1 \) in \( \bar{W} \) and \( \bigcup \bar{B}_x \) is dense in \( \bar{B}_x \). Hence \( \bigcup \bar{G}_x \) is dense in \( \bar{G} \) - i.e. if \( a \in \bar{G} \) there is a' \in a \) closed in \( \bar{G} \) - i.e. if \( a \in \bar{G} \) there is a' \in a \) closed in \( \bar{G} \). Hence \( \sigma(a) \supseteq \sigma(a') \in G \).

Now let \( \chi = \omega \). Then \( \chi = \omega_1 \) in \( \bar{W} \).

Let \( \langle v_i : i < \omega \rangle \in \bar{W} \) be monotone and cofinal in \( \bar{G} \). For any \( a \in G \) there is a thread \( \langle a_i : i < \omega \rangle \in \bar{W} \) through \( \langle \bar{B}_{v_i} : i < \omega \rangle \) and \( a_i \subseteq a \). Hence \( \sigma(a_i) \in G \) and \( \bigcap a_i \subseteq a \). Hence \( \sigma(a_i) \in G \) and \( \bigcap \sigma(a_i) = \sigma(\bigcap a_i) \subseteq \sigma(a) \).

where \( \bigcap \sigma(a_i) \subseteq G \). Q.E.D. (d1)
All that remains now is to define $c_k, \sigma^*_k$ for $k < \omega$ and verify (*)]. We proceed by induction on $k$.

For $k = 0$ we set: $c_0 = 1$, $\sigma^*_0 = \sigma$. All verifications are trivial. Now let $k = f + 1$.

The construction of $c_k, \sigma^*_k$ is essentially a repeat of the two-step iteration lemma. We give the details. Let $G_j \ni c_j$ be $\Pi_3^1_{\mathfrak{g}_j}$ -

- generic. Set: $\sigma^*_j = \sigma_j^{G_j}$. Then $\sigma^*_j$ extends uniquely to a $\sigma^*_j: \mathcal{W}[G_j] \to \mathcal{W}[G_j]$ such that $\sigma^*_j(G_j) = G_j$ be (*)]. But then the sub-completeness of $\Pi_3^1_{\mathfrak{g}_j} = \Pi_3^1_{\mathfrak{g}_j} / G_j$ in $\mathcal{W}[G_j]$.

Hence there is a $\in \mathcal{P} \setminus \mathfrak{g}_j$ not whenever $G_j \ni c_j$ in $\Pi_3^1_{\mathfrak{g}_j}$ - generic, there is $\sigma^* \in \mathcal{W}[G_j]$ such with the following properties:

(a) $\sigma^*: \mathcal{W}[G_j] \leq \mathcal{W}[G_j]$, $\sigma^*(G_j) = G_j$.

(b) $\sigma^*(f_i, \bar{G}_i, \bar{x}_i, \bar{R}_i, \bar{\xi}_i) = f_i, \bar{G}_i, \bar{x}_i$ $\forall i < m$.

(c) $\cup_i \sigma^*(\bar{x}_i) = \bar{x}_i$ $\forall i \leq 0, \ldots, m$.

(d) $\sigma^*: \mathcal{G} \leq \mathcal{G}$ where $\mathcal{G} = \mathcal{G}_h / \mathcal{G}_h = \{ b/\bar{c} : \bar{b} \in \mathcal{G}_h \}$.

$\mathcal{G}$ is easily seen to be $\Pi_3^1_{\mathfrak{g}_j}$ - generic over $\mathcal{W}$.

Since we can force any finite amount of pointwise coherence between $\sigma_i$ and $\sigma^*$, we can also impose the requirements:
(e) $T'(x_i) = \sigma_f(x_i)$ for $i \leq k$

(f) Let $i = 0, \ldots, m$ and $\sigma_1(\overrightarrow{x}_m) \leq \overrightarrow{x}_k < \sigma_1'(\overrightarrow{x}_{m+1})$. Then,

$$\sigma_k(\overrightarrow{x}_k) = \sigma_1(\overrightarrow{x}_1) \text{ for } k \leq m+1.$$

Note that $G = G_1 \times G' = \{ \overrightarrow{b} \in (\overrightarrow{B}, \{ b \in G \} : b \in G' \} \in \overline{\mathcal{B}}_k$. Moreover, $\overline{G}_k = \overline{G}_1 \times G' = \{ \overrightarrow{b} \in (\overrightarrow{B}, \{ b \in G \} : b \in G' \} \in \overline{C}_k$. Hence $\overrightarrow{x} \in \overline{G}_k$. If we set $\overrightarrow{\tau} = \overrightarrow{x} \in \overline{\mathcal{N}}$, it is clear that $\tau_k$ satisfies (**) (11)-(14) with $G = G_1 \times G'$. But $\tau_k \in \overline{V}[G_1][G'] = \overline{V}[G]$.

We may assume without loss of generality that $c' = (\overrightarrow{c}, G')$, where all of the above is forced by $c_j$ — i.e., the above holds if $c' = (\overrightarrow{c}, G')$ whenever $c_j \in G_j$ and $G_j \in \overline{\mathcal{B}}_j$. — generic. We may also suppose without loss of generality that $\models_{\overrightarrow{G}_j}(c_j \& G_j \rightarrow c' = 0)$, $\overrightarrow{G}_j$ being the canonical generic name. Thus $\models_{\overrightarrow{G}_j}(c' \in \overrightarrow{B}_j)$. Hence there is $c_k \in \overrightarrow{B}_k$ such that $\models_{\overrightarrow{G}_j}(c_k / G_j = c')$. But then

$$h_{\overrightarrow{G}_j}(c_k) = \left[ c_k / G_j \neq 0 \right]_{\overrightarrow{G}_j} = \left[ c' \neq 0 \right]_{\overrightarrow{G}_j} = c_j.$$

Thus if $G_j \in \overrightarrow{G}_j$ is $\overrightarrow{G}_j$-generic, then $c_j \in G_j$ and $c_k / G_j = c'$. But $c_j \in \overrightarrow{G}_j$, which means that there is $\overrightarrow{c} \in \overline{V}[G]$. 


Satifying (a)(b)-(g). But then we may assume \[ T_k = \frac{c_k}{\bar{k}} G_k \], where this fact is forced by \( c_k \). Thus \( c_k \) have the desired properties.

Q.E.D. (Case 1)

Case 2 Case 1 fails.

Then \( \lambda \) is regular and \( \frac{\mu^i}{\sigma} < \lambda \) for all \( i < \lambda \).

Let \( \bar{w}, W, \theta, \sigma \) be as before with \( \sigma'(\bar{w}, \bar{G}, \bar{z}, \bar{\lambda}, \bar{x}) = \theta, G, \lambda, \lambda', \lambda, \lambda' \) and \( \lambda_1, \ldots, \lambda_m \) are as before. (However, there is nothing corresponding to the function \( f \).

As before set: \( \bar{X}_0 = \bigcap \bar{w} \cap \bar{X} \); \( \bar{X}_i = \sup \sigma'\bar{x}_i \) for \( i = 0, \ldots, m \). We extend the sequence \( \lambda_1, \ldots, \lambda_m \) by setting: \( \lambda_{m+1} = \lambda \). We also set: \( \bar{x}_{m+1} = \bar{X}_i \); \( \bar{x}_{m+1} = \bar{X} = \sup \sigma'\lambda \).

Clearly it will suffice to show:

Claim. There is \( c \in \bar{B}_i \) s.t. whenever \( G \neq c \in \bar{B}_i \) - generic, there is \( \sigma' \in \mathcal{V}[G] \) s.t.

(a) \( \sigma'\bar{w} \subseteq \bar{w} \)

(b) \( \sigma'(\bar{w}, G, \bar{z}, \bar{\lambda}, \bar{x}) = \theta, G, \lambda, \lambda', \lambda, \lambda' \) \( (i = 1, \ldots, m+1) \)

(c) \( \sup \sigma'\bar{x}_i = \bar{X}_i \) \( (i = 0, \ldots, m+1) \)

(d) \( \sigma'\bar{G} \subset \bar{G} \)
Just as before we choose for \( j = 0, \ldots, m+1 \) a sequence \( \langle \bar{f}_c^i \mid i < \omega \rangle \) which is monotone and cofinal in \( \bar{S}^i_j \). We then set \( \bar{f}_c^i = \sigma(\bar{f}_c^i) \).

Hence \( \langle \bar{f}_c^i \mid i < \omega \rangle \) is cofinal in \( \bar{S}^i_c \).

We also set \( \bar{f}_c^0 = \bar{f}_c^{m+1} \), \( \bar{f}_c^i = \bar{f}_c^{m+1} \).

As before, our strategy is to construct \( c_k \upharpoonright \bar{f}_c^i \) \( (k < \omega) \) s.t. \( \langle c_k \mid k < \omega \rangle \) is a thread in \( \langle \mathcal{B}_k \mid k < \omega \rangle \) and \( c_k \) force \( \bar{f}_c^i \upharpoonright \bar{W} \downarrow \bar{W} \), The intention is again that if \( \bar{c} = \bigcap c_k \in \mathcal{G} \)

and \( \mathcal{G} \in \{\mathcal{B}_k \mid k \in \omega \} \) generic, then we will be able to define the desired embedding in \( V[G] \) from \( \langle \bar{c}_k \mid k < \omega \rangle \), where \( \bar{c}_k = \bar{c}_k \upharpoonright \bar{f}_c^i \).

However, we no longer have the function \( \bar{f} \) available in defining

\( \langle \bar{f}_c^i \mid i < \omega \rangle \). As a result, we will no longer be able to enforce \( \sigma_k(\bar{f}_c^i) = \bar{f}_c^i \). However, we can still enforce \( \sup_k \bar{c} = \bar{c} \).

We shall have to make do with

\( \bar{c}_i \). Note that, although \( \bar{B}_\lambda \geq \bar{c} \), we have \( \bar{B}_i \mathrel{<} \lambda \) for \( i < \lambda \). Thus \( \lambda = \lambda_{m+1} \) in like the other \( \lambda \)’s with \( \langle \mathcal{B}_i \mid i < \lambda \rangle \).
We inductively construct $c_k \in B_{\frac{\delta}{k}}$, $\delta \in V_{\frac{\delta}{k}}$. 

(I) (a) $c_0 = 1$, $\delta_0 = \delta$

(b) $h_{\frac{\delta}{k}}(c_k) = c_j$ for $k = j + 1$

(II) Let $G \subseteq B_{\frac{\delta}{k}}$ be $B_{\frac{\delta}{k}}$-generic. Set:

$G_\gamma = G \cap B_{\frac{\delta}{k}} (\gamma \leq \frac{\delta}{k})$, $G_{\frac{\delta}{k}} = G \cap B_{\frac{\delta}{k}} (\frac{\delta}{k} \leq \frac{\delta}{k})$ and

$G_{\frac{\delta}{k}} = G_{\frac{\delta}{k}} (\frac{\delta}{k} \leq \frac{\delta}{k})$ for $1 \leq k$. Then:

(a) $G_{\frac{\delta}{k}} = \bigcap_{i=1}^{\infty} G_{\frac{\delta}{k}^i}$

(b) $\sigma_{\frac{\delta}{k}}(\delta, 1B_{\frac{\delta}{k}}, \delta, \lambda_i) = \Theta_{\frac{\delta}{k}} 1B_{\frac{\delta}{k}}, \lambda_i (i = 1, \ldots, m+1)$

(c) $\sup_{\frac{\delta}{k}^i} \sigma_{\frac{\delta}{k}^i}(\delta_i) = \delta_i$ (i = 0, \ldots, m+1)

(d) Let $\bar{\delta}_{\frac{\delta}{k}}(\frac{\delta}{k}) \leq \frac{\delta}{k} < \sigma_{\frac{\delta}{k}}(\frac{\delta}{k} m+1)$. Then

$G_{\frac{\delta}{k}} \subseteq \frac{\delta}{k} \subseteq G$

(e) Let $h = j + 1$. Then $\sigma_{\frac{\delta}{k}}(x_i) = \sigma_{\frac{\delta}{k}}(y_i)$ for $i \leq j$

(f) Let $k = j + 1$, $i = 0, \ldots, m+1$. Then $\sigma_{\frac{\delta}{k}}(\frac{\delta}{k}^i) \leq \frac{\delta}{k}^i < \sigma_{\frac{\delta}{k}}(\frac{\delta}{k})$, then $\sigma_{\frac{\delta}{k}}(\frac{\delta}{k}^i) = \sigma_{\frac{\delta}{k}}(\frac{\delta}{k}^i)$ for $i \leq m+1$.

Note: By (e) it follows that $\sigma_{\frac{\delta}{k}}(x_i) = \sigma_{\frac{\delta}{k}}(y_i)$ for $i \leq k$.

We shall, in fact, arrange that if $\sigma_{\frac{\delta}{k}}(\frac{\delta}{k}^i) \leq \frac{\delta}{k}^i < \frac{\delta}{k} < \sigma_{\frac{\delta}{k}}(\frac{\delta}{k})$, then $\sigma_{\frac{\delta}{k}} = \sigma_{\frac{\delta}{k}}$. We now show that I, II imply the claim.
Set \( c = \bigcap_{k} c_k \). Then \( c \in IB_{\lambda} \subset IB_{\lambda} \). Let \( G \ni c \) be \( IB_{\lambda} \)-generic. Set:
\[
\tilde{\sigma}_k = \frac{1}{\lambda} \sigma_k = \frac{1}{\lambda} \tilde{\sigma}_k.
\]
By (II) there is \( \tilde{\sigma} \in V[G] \) s.t. \( \tilde{\sigma} \upharpoonright \tilde{\mathbb{W}} \leq \tilde{\mathbb{W}} \)
defined by \( \tilde{\sigma}(x) = \tilde{y} \) s.t. \( \tilde{\sigma}_k(x) = y \) for sufficiently large \( k \).
(a), (b) of the Claim are clearly satisfied.
We prove (c):
\( \tilde{\sigma} \upharpoonright \tilde{\mathbb{X}} \subset \tilde{\mathbb{X}} \).
Since \( \tilde{\sigma}_k(x) = \tilde{y} \) for some \( k \), where \( \tilde{y} \in \tilde{\mathbb{X}} \).

We now prove (d). Let \( p \in G_{\tilde{\mathbb{X}}} \). Let \( l > k \)
be large enough that \( \tilde{\sigma}_k(p) = \tilde{\sigma}_k(p) \).

All that remains is to inductively define \( c_k, \tilde{\sigma}_k \) and verify (I), (II). This will
be somewhat trickier than the corresponding step in Case 1. We shall, in fact, have to add some further inductive hypotheses. Before defining \( c_k \) we shall define a \( b_k \in IB_{\tilde{\mathbb{X}}} \).
III

(a) \( b_0 = 1 \)

(b) \( h_{\frac{x_i}{j}}(b_k) = c_j \) if \( k = d + 1 \)

(c) \( \Pi \) (a1)-(a3) hold whenever \( b_k \in G \).

(d) \( c_k \leq b_k \).

\( \sigma^i_k \) will be defined simultaneously with \( b_k \), before defining \( c_k \). Our next induction hypothesis states an important property of \( b_k \):

Definition. Let \( \nu \leq \frac{x_i}{k} < \mu \leq \frac{x}{k} \) a. t. \( \frac{x_i}{k} < \nu \) for \( i < k \),

\[
\alpha^i \mu = b_k \cap \prod \sigma^i_k (\frac{x_i}{k}) = \nu \cap \sigma^i_k (\frac{x_i}{k+1}) = \nu \cap \frac{x}{k}.
\]

It is easily seen that:

1. \( \alpha^i \mu \cap \alpha^i \nu' = 0 \) if \( \langle \alpha^i \mu, \alpha^i \nu' \rangle \neq \langle \alpha^i \mu', \alpha^i \nu' \rangle \).

Proof:
Suppose \( \alpha^i \mu \cap \alpha^i \nu' \subseteq G \), where \( G \in \mathcal{B}_k \) g- generic. Then \( i = i' \) since if \( e.g. \), \( i < i' \), then \( \mu = \sigma^i_k (\frac{x_i}{k+1}) \leq \sigma^i_k (\frac{x_i}{k'}) = \nu' \leq \frac{x_{k'}}{k} \).

Contradiction! But then \( \nu = \sigma^i_k (\frac{x_i}{k}) = \nu' \) and \( \mu = \sigma^i_k (\frac{x_i}{k+1}) = \mu' \). Contradiction! QED (1).

We shall inductively verify:

IV \( \alpha^i \mu \cap \left[ \sigma^i_k (\frac{x}{k}) = \frac{y}{d} \right]_{\frac{x}{k}} \subseteq \mathcal{B}_k \)

for \( \sup \frac{x_i}{k} \leq \nu \leq \frac{x}{k} \).

(Hence \( \alpha^i \mu = \alpha^i \mu \cap \left[ \sigma^i_k (\frac{y}{d}) = \frac{y}{d} \right] \subseteq \mathcal{B}_k \))
Set $A = A_k = \text{the set of } a_i^{<\kappa} \neq 0$ not.

$\sup_{i<k} \beta_i < \gamma \leq \beta_k$. By IV we see that for each $a = a_i^{<\kappa} \in A$ there is $\dot{\alpha} \in \mathcal{V}^{IB, \gamma}$ not.

(21) $\dot{\alpha}_G = \dot{\alpha}_G^k$ for $IB_{\beta_k} - \text{generic } G \exists \alpha$.

But:

But:

(3) $\mathcal{H} \exists a \in IB_{\beta_k} - \text{generic, then } G \vDash \exists \alpha$ to a $IB_{\beta_k} - \text{generic } G' \exists \alpha$. Hence

$\dot{\alpha}_G = \dot{\alpha}_G' = \dot{\alpha}_G^k$.

Hence:

(4) Let $G \exists a$ be $IB_{\beta_k} - \text{generic, where } a = a_i^{<\kappa} \in A_k$.

Then II holds with $\dot{\alpha}_G = \dot{\alpha}_G^k$ in place of $\dot{\alpha}_G$.

For $j < k$, where $G_j = G \cap IB_{\gamma}$ for $\gamma \leq \nu$, $G_\gamma = G \cap IB_{\gamma}$ for $\gamma \leq \beta_k$.

(An particular we have for all that

$\dot{\alpha}_G(x) = \dot{\alpha}_G(x)_{\nu}$ for $x \leq \gamma$ where $k = \gamma + 1$.)

Whenever $\nu < \kappa \leq \lambda$ and $G \in IB_{\beta_k} - \text{generic, we know that } IB_{\gamma} \mathcal{H} \text{ is subcomplete in } \mathcal{V}^{IB, \gamma}$.

Thus, using (41) and repeating the construction of $C_{j+1}, \dot{\alpha}_1, \dot{\alpha}_j$ from $C_{j+1}, \dot{\alpha}_j$ in Case 1, we get:
15) Let $a \in A_k$, $a = a_i^i$. There are $\tilde{a} \in IB$, $	ilde{\sigma}_a' \in V^{IB}$ s.t. $h_\nu(\tilde{a}) = a$. and whenever $G \ni \tilde{a} \cap IB$ - generic, $\sigma_a = \tilde{\sigma}_a'$ and $\sigma_a = \tilde{\sigma}_a'$, Then:

(a) $\sigma_a' \cap W \subset W$

(b) $\sigma_a'(\tilde{\theta}, IB, \tilde{x}_i, \tilde{\lambda}_i) = \Theta, IB, x_i, \lambda_i (i = 1, \ldots, m+1)$

(c) $\sup \sigma_a' \cap \tilde{x}_i = \tilde{x}_i (i = 0, \ldots, m+1)$

(d) $\sigma_a' \cap G \subset \tilde{G}$ (recall $\sigma_a(\tilde{x}_i') = \mu^{i+1}$)

(e) Let $r$ be least s.t. $\mu \leq \tilde{x}_r$. Then $\sigma_a' (x_i) = \sigma_a (x_i)$ for $i < r$.

(f) Let $r$ be as above. Let $i = 0, m+1$ and let $\sigma_a(\tilde{x}_i') \leq \tilde{x}_r < \sigma_a(\tilde{x}_i'')$. Then $\sigma_a' (\tilde{x}_i) = \sigma_a (\tilde{x}_i')$ for $r \leq m+1$.

For each $a \in A_k$, we fix such a pair $\tilde{a}, \tilde{\sigma}_a'$, which can be regarded as an instruction to be used later in forming $b_\mu$, where $\mu$ is least s.t. $\mu \leq \tilde{x}_r$. And $G \ni IB$ - generic
and an \( b \in G \), we want:

\[ \tilde{a} \in G \quad \text{and} \quad \tilde{a} \tilde{b} = \tilde{a} \quad \text{(where} \quad \tilde{a} = \tilde{a} \tilde{b} \text{)} \]

In particular, we need: \( \tilde{a} \# b = \tilde{a} \). But we shall also require \( h_{\tilde{b}} (b) = c \). Hence we need:

\[ a \# c = h_{\tilde{a}} (a \# b) = h_{\tilde{a}} (\tilde{a}) \]

This is why \( b \) must be "shrink" to \( c \). Accordingly, we define \( c \) as follows:

**Def.** Let \( b \) be given. Set \( \tilde{b} = b \setminus \bigcup A_k \).

Then \( c = \tilde{b} \cup \bigcup_{a \in A_k} h_{\tilde{a}} (\tilde{a}) \).

We are working by induction on \( k \).

We assume \( \text{II} \) to hold for all \( j < k \) and \( \text{III} (a) \), \( \text{IV} \) to hold at \( k \).

We must now verify \( \text{I}, \text{II}, \text{III}(d) \) at \( k \).

**III(d)** is immediate; hence \( a \in A_k \). \( \text{I}(b) \)

holds since:

\[ h_{\tilde{a}} (\tilde{a}) = h_{\tilde{a}} (\tilde{a}) = h_{\tilde{a}} (\tilde{a}) = h_{\tilde{a}} (\tilde{a}) = h_{\tilde{a}} (\tilde{a}) \]

for \( a = a \in A_k \). Hence:

\[ h_{\tilde{a}} (c) = h_{\tilde{a}} (\tilde{a}) \cup \bigcup_{a \in A_k} h_{\tilde{a}} (\tilde{a}) = h_{\tilde{a}} (\tilde{a}) \]

where \( c = c \).
For $I(a)$ note that $A_0 = \{a_1^2 \}$ where $a = a^0, \tilde{a}_1 = 1$. Since $\sigma_0 = \tilde{\epsilon}$ by III (c), hence $\epsilon_0 = h_0(\tilde{\epsilon}) = 1$.

This completes the proof that $I-IV$ hold at $k$, assuming III (a)-(c) and IV to hold at $k$ and $I-IV$ to hold below.

Now assume $I-IV$ to hold below $k$.

We must define $b_k, \sigma^0_k, \sigma_k$ and verify III (a)-(c) and IV at $k$.

For $k = 0$ set $b_0 = 1, \sigma_k = \sigma$. The verifications are trivial.

Now let $k = j + 1$. Note that $A_0, \{a_1 \alpha \in A_0 \}$ have been defined for all $l \leq j$. Set:

Def $A_j = \{a = a^\mu \in \bigcup_{l \leq j} A_l \}$

s.t. $\bar{a}_l < \mu$.

Note that:

(6) Let $a = a^\mu \in A_0, l \leq j$. Then

$\sigma^h (\bar{a}_l) = \sigma^h (\bar{a}_l)$ for $h \leq i + 1$.

Proof: By induction on $j$ we prove this for all $j$ s.t. every $l \leq j$ satisfies $I-IV$,

At $l = j$ there is nothing to prove,

so let
\[ \ell \leq m \text{ where } j' = m+1. \] Then

\[ \overline{m} \left( \overline{s}_i \right) = \nu \leq \overline{s}_j < \mu = \overline{m} \left( \overline{s}_{i+1} \right). \]

The conclusion follows by \( \text{II} \) (f).

\[ \text{Q.E.D. (6)} \]

Using this we can repeat the proof of (11) to get:

\[ (7) a^{\nu \nu} \wedge a^{\nu' \nu'} = 0 \text{ if } \langle i, i, \mu \rangle \neq \langle i', i', \mu' \rangle \]

and \( a^{\nu \nu}, a^{\nu' \nu'} \in \hat{A}_j \).

We now define:

\[ \text{Def: } b_k = \bigcup \{ h_{\overline{s}_k} (\alpha) \mid \alpha \in \hat{A}_j, \overline{s}_k \}. \]

for \( k = j+1 \).

To define \( \hat{\sigma}_k \) we set:

\[ \hat{A} = \text{the set of } a^{\nu \nu} \in \hat{A}_j \text{ s.t. } \mu \leq \overline{s}_k, \]

\[ \hat{\sigma}_k \in \nu \setminus \hat{A} \text{ is then a name s.t. } \]

\[ \llbracket \hat{\sigma}_k \rrbracket = \overline{s}_j \]

\[ \llbracket a \rrbracket = \hat{\sigma}_j \]

\[ \cap b_k = b_k' = b_k \setminus \nu \hat{A}. \]

It is straightforward to see that

\[ (8) \text{ III (c) holds at } k. \]

The proof is left to the reader.
III (a) holds vacuously at \( k = j+1 \). We prove:

(9) III (b) holds at \( k \).

proof:

Clearly \( h_{j+1}^a(b_k) = \bigcup_{a \in \hat{A}_j} h_{j+1}^a(\hat{a}) \). Hence we need:

Claim \( c_j = \bigcup_{a \in \hat{A}_j} h_{j+1}^a(\hat{a}) \).

For \( j = 0 \) this is trivial. So let \( j = l+1 \).
Recall that \( c_j = \overline{b} \cup \bigcup_{a \in \hat{A}_j} h_{j+1}^a(\hat{a}) \), where

\( \overline{b} = b_j \setminus \bigcup_{a \in \hat{A}_j} a \), so it suffices to

show:

Claim \( \overline{b} = \bigcup_{a \in \hat{A}_j} h_{j+1}^a(\hat{a}) \) where \( \hat{A}' = \hat{A}_j \setminus \hat{A}_j' \).

(C) Let \( a' \in \hat{A}' \). Then \( a' \notin \hat{A}_j' \). Hence

\( h_{j+1}^a(\hat{a}') \subset b_j \). Thus it suffices to note

\( h_{j+1}^a(\hat{a}') \cap h_{j+1}^a(\hat{a}) = \emptyset \) for all \( a \in \hat{A}_j' \),

since, in fact, \( a' \cap a = \emptyset \) by (7).

(C) Suppose not. By the definition of \( b_j \) there is \( a \in \hat{A}_j \setminus \hat{A}_j' \) s.t. \( h_{j+1}^a(\hat{a}) \cap \overline{b} \neq \emptyset \), hence \( a' \cap \overline{b} \neq \emptyset \).

We derive a contradiction. Let \( \emptyset \neq a' \cap \overline{b} \)

be \( B_j \)-generic. Note that

\( a = a' \cap b' \) with \( r \leq \overline{e} < \mu \leq \overline{3}_j \).
since otherwise \( a \in A' \). But then
\[
\sigma_i (\bar{s}_i) = \nu \leq \bar{s}_i < \sigma_j (\bar{s}_{i+1}) = \mu .
\]
Hence
\[
\sigma_i (\bar{s}_i) = \sigma_i (\bar{s}_2) \text{ for } i \leq i+1 .
\]
But then
\[
\sigma_i (\bar{s}_{i+1}) \leq \bar{s}_i \text{ and there must be an } m > i \text{ s.t. } \sigma_i (\bar{s}_m) = \bar{s}_i < \sigma (\bar{s}_{m+1}),
\]
Set \( \nu^* = \sigma_i (\bar{s}_m) \), \( \mu^* = \sigma_i (\bar{s}_{m+1}) \).

Then \( a^* = a^m \nu^*, \mu^* \in G \). Hence
\( a^* \in A' \). But \( a^* \bar{b} \neq 0 \) since \( \bar{b} \in G \), contradicting the definition of \( \bar{b} \).

\( \text{QED} (9) \)

It remains only to show:

(10) IV holder at \( k \).

As a preliminary we show:

(11) Let \( a = a^{i'} \nu' \in A_k \), \( a' = a^{i'} \nu' \in A_i \)

s.t. \( \nu' > \nu \). Then \( a \wedge a' = 0 \).

Proof:

Suppose not. Let \( a \wedge a' \in G \) where
\( G \) is \( \beta \)-generic. Then
\[
\sigma_i (\bar{s}_i) = \nu \leq \bar{s}_i < \sigma_j (\bar{s}_{i+1})
\]
Hence $\sigma_k(\tilde{x}_i) = \sigma_j(\tilde{x}_i)$ for $\ell \leq i' + 1$. But then

$\sigma_k(\tilde{x}_i) = \nu < \mu = \sigma_k(\tilde{x}_{i' + 1})$, Hence $i \leq i'$. Thus

$\nu = \sigma_k(\tilde{x}_i) \leq \sigma_k(\tilde{x}_i) = \nu' \leq \tilde{x}_i$. Contradiction!

since $a^{v''} \mu \in A_k$. $\text{QED (11)}$

We can now prove (10). Let $a = a^{v''} \mu \in A_k$. Set $A' = \text{the set of } e = a^{v''} \mu' \in \hat{A}$, s.t. $\mu' \leq \nu$.

Since $b_k = \bigcup_{e \in \hat{A}} h_{\tilde{e}}(\tilde{e})$ and $h_{\tilde{e}}(\tilde{e}) = \tilde{e}$ for $e \in A'$, we conclude by (10) that:

$a = a \cap b = \bigcup_{k} a \cap \tilde{e}$. But for $e = a^{v''} \mu' \in A'$

we have: $[\dot{\sigma}_k = \dot{\tilde{e}}'] \in \tilde{B}_\mu'$. Moreover,

$[\dot{\sigma}_e(x) = \tilde{y}] \in \tilde{B}$, since $\dot{\sigma}_e(x), \tilde{y} \in \bigcap \tilde{B}_\mu'$

and "$\dot{\sigma}_e(x) = \tilde{y}$" in $\Sigma_0$. (cf. the note following *(x) in Case 1.) Since $a = \bigcap_{k} [\dot{\sigma}_k(\tilde{x}_i) = \tilde{y}] \cap [\dot{\tilde{e}}' \tilde{x}_i = \tilde{y}]$, we conclude:

$a \cap [\dot{\tilde{e}}'(\tilde{x}_i) = \tilde{y}] \cap \tilde{e} = \tilde{e} \cap [\dot{\tilde{e}}'(\tilde{x}_i) = \tilde{y}] \cap [\dot{\tilde{e}}(\tilde{x}_{i+n}) = \mu] \in \tilde{B}_\mu \cap \tilde{B}$.

Hence $a \cap [\dot{\tilde{e}}(\tilde{x}_i) = \tilde{y}] = \bigcup_{e \in A'} a \cap [\dot{\tilde{e}}(\tilde{x}_i) = \tilde{y}] \cap \tilde{e} \in \tilde{B}$

$\text{QED (10)}$

$\text{QED (Theorem 1)}$