§ 5 Semi-subproper Forcing

When dealing with semi-subproper BA's, we must replace the notion "full" by "almost full" as defined in [SPSC] §3.1. Recall that a (possibly ill-founded) ZFC-model $M$ is called solid iff its well-founded core $wfc(M)$ is transitive and $E_{M} \cap wfc(M)^{2} = E \cap wfc(M)^{2}$.

A transitive ZFC-model $W$ is called almost full iff there is a solid model $M$ of ZFC-s.t. $W \in wfc(M)$, $M = V = L(W)$, and $W$ is regular in $M$ (i.e., if $f : X \rightarrow W$, $X \in W$, and $f \in M$, then $f \in W$).

If it is clear from [SPSC] §3 that the arguments used to verify that specific algebras are subcomplete or subproper would work equally well if, in the definition of those concepts, we replaced "full" by "almost full". There seemed no reason to do so, however, since the
modified definitions seem harder to fulfill, and the iteration theorem were provable for the concepts as given. In the case of semi-subproper forcing, however, we unfortunately do not know how to prove iterability without making the change; (This means that Lemma 7 of [SPSC]§2 is unproven as stated — and possibly unprovable.)

To facilitate the treatment of semi-subproper forcing, we also define:

**Def.** Let $W$ be a transitive $\mathbf{ZFC}$-model. Let $IB \subseteq W$ be a complete BA in the sense of $W$. $G \subseteq IB$ is weakly $IB$-generic over $W$ if the following hold:

- $G$ is an ultrafilter on $IB$.
- Whenever $\Delta \subseteq W$ is a net predicate in $IB$ (i.e., $U\Delta = 1$ in $IB$) s.t. $W \Vdash \exists \bar{\Delta} \in \Delta \cap G \neq \emptyset$.

We then define:
Def A is semi-subproper iff it satisfies the definition of "subproper" with two changes:

- "full" is replaced by "almost full"
- Clause (1d) is replaced by:
  \( (d) \quad G = G' \preceq \sigma^{-1}(t) \quad \text{G is weakly } \overline{A} \text{ generic over } \overline{W}. \)

(Note (1d) can be equivalently be replaced by:
Whenever \( t \in \overline{W} \overline{A} \) and \( \sigma(t)^G < \omega_1 \), then
\( \sigma(t)^G < \omega_1 \).

We leave this to the reader.)

We now consider the implications of \((a)-(d)\).

Recall that \( W = \mathcal{L}_A = \langle L_\mathcal{E}(A), \mathcal{E}, A \rangle \). Since G is generic we can form the ZFC model
\( W[G] = \langle W[G], A, W \rangle \)
whose language has the predicates \( \mathcal{E}, A, W \).

By a \( \overline{W} \overline{A} \) - sentence let us mean an
sentence \( \phi(t_1, \ldots, t_n) \), where \( \mathcal{E} \) is a formula of
this language and \( t_1, \ldots, t_n \in \overline{W} \overline{A} \).

If we assign each \( \overline{W} \overline{A} \) - sentence \( \psi \) a
truth value \( \llbracket \psi \rrbracket = \llbracket \psi \rrbracket ^{W[A]}, \) we have:
(1) \( \llbracket W[G] = \phi \rrbracket ^{W[A]} \iff \llbracket \phi(t^G) \rrbracket \in G \)
for \( t_1, \ldots, t_n \in \overline{W} \overline{A} \). Now assign \( \llbracket \psi \rrbracket ^{W} \)
for \( \overline{W} \overline{A} \) sentences the same way.
Clearly:
(21) \( \sigma_0(\lceil \varphi(t') \rceil \vec{w}) = \lceil \varphi(\sigma(t')) \rceil \vec{w} \).

Define a structure:
\[ \vec{w}'' = \langle \vec{w}'', I, E, A', \vec{w}' \rangle \] by:
\[ \vec{w}'' = \vec{w}^{\vec{w}'}, \quad t \in t' \iff \lceil t = t' \rceil \in \vec{G}, \]
\[ t \in t' \iff \lceil t \in t' \rceil \in \vec{G}, \quad A' \iff \lceil A_t \rceil \in \vec{G}, \]
\[ W' \iff \lceil W' \rceil \in \vec{G}. \]

Noting that for any \( \vec{w}^{\vec{w}'} \) - sentence \( \forall v \varphi(v, t') \), there is \( \vec{w} \in \vec{w}^{\vec{w}'} \) such that:
\[ \lceil \forall v \varphi(v, t') \rceil = \lceil \varphi(v, t') \rceil, \] we see that:
(31) \( \lceil \forall v \varphi(v, t') \rceil \in \vec{G} \iff \forall v \in \vec{w}^{\vec{w}'} \setminus \lceil \varphi(v, t') \rceil \in \vec{G} \).

Using this we get:
(4) \( \vec{w}'' \models \varphi \iff \lceil \varphi \rceil \in \vec{G} \) for \( \vec{w}^{\vec{w}'} \) - sentences \( \varphi \).

Clearly \( E \) is well-founded, since
\[ t \in t' \iff \sigma_0(t) \vec{G} \in \sigma_0(t') \vec{G}. \]

By (21), \( \vec{w}'' / I \) models the axiom of extensionality. Hence there is:
\[ \vec{g} : \vec{w}'' / I \cong \vec{w}^{**} = \langle \vec{w}'', A', \vec{w}' \rangle, \]
where \( \vec{w}^{**} \) is transitive. Set:
\[ t^G = \vec{g}(t / I). \]
Then:
(5) \( t^G = \{ l^G \mid \lceil l \in t \rceil \vec{w} \in \vec{G} \} \),
(6) \( \vec{w}^{**} \models \varphi(t^G) \iff \lceil \varphi(t'') \rceil \in \vec{G} \)
for \( t, \ldots, t_m \in \vec{w}^{\vec{w}'} \).
By (1), (21), (6), we then have:

(7) $\overline{W}** \models \varphi[\overline{G}] \iff W[G] \models \varphi[\overline{\sigma}(G)]$.

Hence there is $\overline{\sigma} : \overline{W}** \rightarrow W[G]$ defined by:

$$\overline{\sigma}^*(t^G) = \overline{\sigma}(t)^G.$$ 

Since $G$ is generic over $W$, we have $\overline{x}^G = x$ for $x \in W$. $G$, however, may not be fully generic, so we may not have $\overline{x}^G = x$ for $x \in \overline{W}$. However we do have:

(8) $\overline{\sigma}^*(\overline{x}) = \overline{x}$ for $\overline{x} \in \overline{W}$

by weak genericity. Thus $[t \leq \overline{x}] = \bigcup_{\nu < \overline{x}} [t = \nu]$.

At $\overline{G}$ is the canonical $\overline{B}_\lambda^*$-generic name,

Then $\overline{\sigma}^*(\overline{G}) = G$; hence, setting

$\overline{G}^* = \overline{\sigma}^*(\overline{G})$, we have:

(9) $\overline{W}** = \overline{W}^* [\overline{G}^*]$ and $\overline{G}^* = \overline{\sigma}^{-1}(G) \upharpoonright \overline{B}_\lambda^*$ - generic over $\overline{W}^*$.

Hence $\overline{W}^*, \overline{\sigma}^*$ to some extent play the role of $\overline{W}, \overline{\sigma}$ in the case of
subproper forcing $\overline{\sigma}$. We refer to $\overline{W}^*, \overline{\sigma}^*$ as the canonical extension of $\overline{W}, \overline{\sigma}$.
Set: $k(x) = x^G$ for $x \in \bar{w}$. Then

$$\text{101} \quad k: \bar{w} < \bar{w}^*$$

By:

$$\text{111} \quad \sigma_o^* k = \sigma_o^* 1$$

Set: $k(x_0) = \text{om} \cap \bar{w}^*$. Then:

$$\text{121} \quad k(x_i) = \text{sup} k^i \bar{x}^i \quad (i = 0, m)$

Let $t \in \bar{w}^*$. Then $t^G < k(\bar{x})$.

Claim: $t^G < k(\bar{x})$ for $a < \bar{x}$.

We have $[t < \bar{x}] \in G$, where

$$[t < \bar{x}] = \left\{ [t = \bar{x}] \right\}$$

Let $X = \left\{ 3 < \bar{x} : [t = \bar{x}] \neq 0 \right\}$. Then $X$ is bounded in $\bar{x}$, since $\text{card}(\bar{w}^*) < \bar{x}$ in $\bar{w}$. Let $X < \bar{x} < \bar{x}$. Then

$$[t < \bar{x}] \in G$$

Here we took $[t < \bar{x}] = 1$. Q.E.D. (12)}
But then $k \mid \overline{W} \prec W^*$ cofinally. Hence $\overline{W}^*$ is almost full by [SPSC] §3.1 Fact 9.

But then

(13) $\overline{W}^{**} = \overline{W}^* [\overline{\sigma}^*]$ is almost full.

Proof: Let $\Omega$ verify the almost fullness of $\overline{W}^*$. Then $\overline{\sigma}^* \in k(\mathbb{B}_{\lambda}^\lambda)$ generic over $\Omega$,

since $\overline{W}^*$ is regular in $\Omega$. Hence $\Omega [\overline{\sigma}^*]

verifies the almost completeness of $\overline{W}^{**}$.

Q.E.D. (13)
Using this we prove a technical lemma which will save time later.

**Lemma 6.** Let $A \subseteq IB$ s.t.

$\forall^* (IB / \mathcal{G} \text{ is semi-subproper}).$

Let $\theta \succ IB$ s.t.

$\forall^* (\theta \text{ verifies the semi-subproperness of } IB / \mathcal{G})$

Let $\lambda_1, \ldots, \lambda_m < \theta$ s.t. $\lambda_i \succ IB$ is regular for $i = 1, \ldots, m$. Let $W = L^A_\theta$ where $\tau > \theta$ is regular and $H_\theta \subseteq W$. Let $\bar{W} = L^A_\bar{\theta}$ be countable and almost full. Let

$\bar{\theta}, \bar{A}, \bar{IB}, \bar{\lambda}_1, \ldots, \bar{\lambda}_m, c \in \bar{W}; \tau \in V^A$, where $c \in \bar{IB}$. Let $a \in A, \sigma_0 \in V^A$ s.t. $a \neq 0$ and whenever $G \ni a \in A$ - generic and $\sigma = h_{A}(G)$, then

(a) $\sigma_0 \upharpoonright \bar{W} \leq W$

(b) $\sigma_0 (\bar{\theta}, \bar{A}, \bar{IB}, \bar{\lambda}_i) = \bar{\theta}, \bar{A}, \bar{IB}, \bar{\lambda}_i$ for $i = 1, \ldots, m$

(c) $\tau \upharpoonright \sigma_0 \in \bar{W}$

(d) $\bar{G}_0 = \sigma_0^{-1} \upharpoonright \bar{G}$ is weakly $A$-generic over $\bar{W}$ and $h_{A}(\bar{c}) \in \bar{G}_0$.

Then there are $b \in IB, \sigma \in V^IB$ s.t.

$a = h_{A}(b)$ and whenever $G \ni b \in IB - generic$, $\sigma = h_{A}(G), G_0 = G \cap IA$,

$\sigma_0 = h_{A}(\sigma) = \sigma \upharpoonright G_0$, Then:
(e) \( \sigma : \overline{W} \leq W \)

(f) \( \sigma (\overline{\theta}, \overline{A_1}, \overline{IB}, \overline{\lambda_c}) = \theta_1 A_1 IB \lambda_c \) \( (c = 1, \ldots, m) \)

(g) \( \sigma (t \overline{\varepsilon}) = \sigma (t \overline{\varepsilon}) \)

(h) \( \operatorname{up} \sigma^{-1} \overline{\lambda_c} = \operatorname{up} \sigma^{-1} \overline{\lambda_c} \) \( (c = 0, \ldots, m) \)

where \( \overline{\lambda}_0 \) = f \( \bigcup \lambda \overline{W} \)

(i) \( \overline{G} = \sigma^{-1} \) \( \overline{G} \) is weakly \( \overline{IB} \)-generic over \( \overline{W} \) and \( \varepsilon \in \overline{G} \).

**Proof.**

Let \( G_0 \in A \) be \( A \)-generic, \( \overline{G_0} = \sigma_0 \overline{G_0} \). Our assumptions justify the analysis in (11)-(13) above. Let \( \overline{W}^*, \sigma_0^* \) be the canonical extension of \( \overline{W}, \sigma_0 \). Then

\[ \sigma_0^* : \overline{W}^* [\overline{G_0^*}] \leq W [G_0] , \sigma_0^* (\overline{G_0^*}) = G_0 , \]

where \( \overline{W}^* [\overline{G_0^*}] \) is full in \( W [G_0] \). If we set \( k(x) = x \overline{G} \) for \( x \in \overline{W} \), we get

\[ k : \overline{W} \leq \overline{W}^* \] and \( \sigma_0^* k = \sigma_0 \). Moreover

\[ k(\overline{\lambda}_c) = \operatorname{up} \kappa(\overline{\lambda}_c) \] for \( c = 0, \ldots, m \) (where \( k(\overline{\lambda}_0) = f \bigcup \overline{W}^* \)). Since

\[ \sigma_0^* (h^A_\varepsilon (e)) = h^A_\varepsilon (\sigma_0 (e)) \in G_0 , \] we have \( \sigma_0 (e) / G_0 \neq 0 \).

Since \( \overline{IB} = \overline{IB} / G_0 \) is semiproper in \( W [G_0] \),

we conclude that there \( \varepsilon \in \overline{b} \in \overline{IB} / \overline{G_0} \).
Let $b' \in \sigma_0(e)/G_0$ and whenever $G' \supseteq b'$ is $\mathfrak{B}$-generic over $W[G_0]$, there is $\tau \in W[G_0][G']$ s.t.

(i) $\tau : \overline{W}^* [\mathcal{E}^*_0] < W[G_0], \sigma^*(\mathcal{E}^*_0) = G_0$

(ii) $\sigma^*(k(x)) = \sigma^*_0(k(x)) = \sigma^*_0(x)$ for $x = \overline{e}, \overline{A}, \overline{B}, e, \overline{x}_i (i = 1, \ldots, m)$

(iii) $\sup_{i \neq 0} \sigma^*_0(k(x_i)) = \sup_{i \neq 0} \sigma^*_0(k(x_i)) = \sup_{i \neq 0} \sigma^*_0(x_i)$

(iv) $\overline{G}' = \sigma^* -1 '' G' \upharpoonright \text{weakly } k(\overline{B})/G_0^*$

- generic over $\overline{W}^* [\mathcal{E}^*_0]$.

Set: $\sigma = \sigma^*_0 k$.

Then $\sigma$ satisfies (e), (f), (g), (h) by (i) - (iii).

Set $G = G_0 \times G' = \{ b \in \mathcal{B} \mid b/G_0 \in G' \}$. Then $G$ is $\mathcal{B}$-generic and we prove:

(14) $\sigma$ satisfies (i') for $G = G_0 \times G'$. i.e.

$\overline{G} = \sigma^* -1 '' G \upharpoonright \text{weakly } \overline{\mathcal{B}}$-generic over $\overline{W}$ and $e \in \overline{G}$.

Proof:

$\overline{G}$ is obviously an ultrafilter, since $G$ is an ultrafilter, $\sigma : \overline{W} < W$, and $\sigma(\overline{B}) = \mathcal{B}$. Moreover $\sigma(e) = \sigma_0(e) \in G$ since $\sigma_0(e)/G_0 \in G'$. Hence $e \in \overline{G}$. Now let $\Delta$ be predense in $\overline{B}$ s.t. $\overline{\Delta} \subseteq \omega_1$ in $\overline{W}$.

Claim $G \cap \Delta \neq \emptyset$

$k(\Delta) = k '' \Delta$ since $k(1)(\omega_1 + 1) = id$. But $k(\Delta)$ is predense in $k(\overline{B})$; hence
\(-11\)
\[ \Delta' = \{ b / \bar{\sigma}_0^* \mid b \in k(\Delta) \} \] \[ \text{is proclaimed in } k(\bar{\Gamma})/\bar{\sigma}_0^*. \]

But \[ \Delta' = \{ b / \bar{\sigma}_0^* \mid b \in \Delta \} \]. Hence \[ k(b) / \bar{\sigma}_0^* \in \bar{\sigma}_0' \]
for some \( b \in \Delta \). Hence \[ \bar{\sigma}_0^* (k(b) / \bar{\sigma}_0^*) = \sigma_0 (b) / \sigma_0 \in G' \]. Hence \[ \sigma_0 (b) \in G. \]
Hence \( b \in \bar{G}. \)

Q.E.D. (14)

We have seen that for every \( A \)-generic \( G \exists a \) there is a \( b \in 1B / \bar{G} \) with certain properties. Hence it is forced that there is such a \( b \). Hence there is \( b' \in V^A \) s.t.

the above holds of \( b' = b' / \bar{G} \). Whenever \( G \exists a \) is \( 1A \)-generic, we may also assume w.l.o.g. that \[ \ll \ll b' \neq 0 \ll \] = \( a \).
Hence \[ \ll \ll b' \in 1B / \bar{G} \ll \]. But then there is a unique \( b \in 1B \) s.t. \[ \ll \ll b / \bar{G} = b' \ll \].
Hence \[ h_A (b) = \ll \ll b / \bar{G} \neq 0 \ll \] = \( a \).

Now let \( G \exists b \) be \( 1B \)-generic. Set \( G_0 = G \cap IA \), \( \bar{G}_0 = \bar{\sigma}_0 G = \sigma_0 \bar{G}_0 \), \( 1B' = 1B / \bar{G}_0 \).

Then \( b' = b / \bar{G}_0 \in 1B' \) and \( G = G_0 * G' \),

where \( G' = G / \bar{G}_0 = \{ b / \bar{G}_0 \mid b \in G \} \]. Hence there is \( \sigma \in \Gamma [G_0] [G'] = \Gamma [G] \)

satisfying \( (e) - (g) \). (g) follows by (4).
Since this holds for every $1B$-generic $G \ni b$, then in a $\sigma \in \mathcal{V}_{1B}$ s.t. (8) - (11) hold for $\sigma = \sigma_G$ whenever $G \ni b \in \mathcal{V}_{1B}$-generic.

\textbf{QED (Lemma 6)}

Using this we prove the two-step theorem for semi-subproper forcing:

\textbf{Thm 7} Let $A \subseteq 1B$ s.t. $A$ is semi-subproper and \[ \text{\vdash}_{A} \langle 1B/G \in \text{semi-subproper} \rangle. \]

Then $1B$ is semi-subproper.

\textbf{pf.}

Let $\theta$ be a cardinal big enough to verify the semi-subproperness of $A$ and s.t. \[ \text{\vdash}_{A} \langle \theta \text{ verifies the semi-subproperness of } 1B/G \rangle. \]

Let $\tau > \theta$ be regular, $W = L^A_\tau$, where $H_\theta \subseteq W$, \[ \text{\vdash } \tau \in W, \text{ where } W \text{ is transitive, countable and almost full. Let } \]

$\sigma(\theta, A, 1B, \bar{\alpha}, \bar{\lambda}, i) = \theta, A, 1B, \lambda, \lambda_i \quad (i = 1, \ldots, n)$

where $\bar{1B} < \lambda_i$ and $\lambda_i < \theta$ is regular for $i = 1, \ldots, n$.

Let $\sigma(\bar{G}) = G \subseteq 1B \setminus \{0\}$. 

\[ \text{\vdash}_{\sigma} \langle 1B/G \in \text{semi-subproper} \rangle. \]
Claim There is \( b \in \mathcal{B} \setminus \mathcal{E}_0 \) s.t. \( b \subseteq E \) and whenever \( G \ni b \in \mathcal{B} \) is \( \mathcal{B} \)-generic, then there is \( \sigma \in V[G] \) s.t.,

(c) \[ \sigma \upharpoonright \mathcal{B} \models \bar{\sigma} \models \mathcal{B} \text{-generic, over } \bar{W} \]

(d) \[ G = \sigma \uparrow \mathcal{B} \text{ is weakly } \mathcal{B} \text{-generic over } \bar{W} \]

Proof:
Since \( \Theta \) verifies the semi-subs property of \( \mathcal{B} \), there are \( a, \sigma \) s.t. \( a \in \mathcal{A} \setminus \mathcal{E}_0 \), \( a \models \mathcal{A} \), and whenever \( G_0 \ni a \in \mathcal{A} \)-generic and \( \sigma_0 = \sigma_0 \uparrow G_0 \), then

- (a1)-(c1) hold with \( \sigma_0 \) in place of \( \sigma \)
- \( G_0 = \sigma_0 \uparrow \mathcal{B} \) is weakly \( \mathcal{A} \)-generic over \( \bar{W} \)

By Lemma 6 there is then \( b \in \mathcal{B} \) with the derived properties. (Take \( \bar{e} = \bar{e}, t = \bar{t} \))

QED (Thm 7)
We are now ready to prove:

**Thm 8** Thm 1 holds with "semi-subproper" in place of "subcomplete".

**Proof (sketch)**

The proof is very much like that of Thm 5. We are given an RCS iteration $IB = \langle IB_i \mid i < \lambda \rangle$ satisfying (a1)-(c) of Thm 1 with "semi-subproper" in place of "subcomplete".

By induction on $i$ we prove:

**Claim** Let $h < i$, let $G$ be $IB_h$-generic. Then $IB_i/G$ is semi-subproper in $V[G]$.

The cases $h = 0$, $i = 0$, and $i = 1 + i$ are as before. Let $i = \lambda$ with $\lambda$ a limit ordinal. We consider the same two cases:

**Case 1** of $(\lambda) \leq IB_i$ for an $i < \lambda$.

Just as before it suffices to prove:

**Claim** Assume $(\lambda) \leq \omega_1$ in $V$.

Then $IB_\lambda$ is semi-subproper.
We again fix \( f : \omega_1 \to \lambda \) s.t., \( \sup f'' \omega_1 = \lambda \).

Let \( \theta > \lambda \) as before be a cardinal s.t.,

\( I^\lambda \mathop{<} \theta \) a and \( H_i \) (\( \theta \) witnesses the semi-subproperness of \( I^\lambda_1 / G \)) for \( i \leq j < \lambda \). Let \( \lambda = L_\gamma^\omega \), where \( \gamma > \theta \) is regular and \( H_{\mathop{<} \gamma} \subseteq W \). Let \( \sigma : \overline{W} \prec \lambda \) s.t., \( \overline{W} \) is countable, transitive, and almost full. Suppose moreover that:\n
\[ \sigma ( f, \theta, \lambda, I^\lambda, \lambda, \lambda_0, (\lambda_i : i = 1, \ldots, m) \text{ where a code an } \alpha \in I^\lambda \backslash \theta \text{ and } \lambda_0 < \theta \text{ is regular with } I^\lambda_\alpha < \lambda_0 \text{ for } i = 1, \ldots, m. \]

At this point the show:

Claim: There \( \alpha \in I^\lambda \backslash \theta \) s.t., \( \alpha, \lambda_0 \) and whenever \( G \in \alpha \) is \( I^\lambda_\alpha \) - generic, then there \( \alpha \in \sigma \in V[G] \) s.t.,

(a) \( \sigma : \overline{W} \prec \lambda \)

(b) \( \sigma ( f, \theta, \lambda, I^\lambda, \lambda, \lambda_0, (\lambda_i : i = 1, \ldots, m) \)

(c) \( \sup \sigma ' \lambda_i = \lambda_i \) \( (i = 0, \ldots, m) \)

where \( \lambda_0 = \text{omn } \overline{W} \), \( \lambda_i = \sup \sigma ' \lambda_i \) \( (i = 0, \ldots, m) \)

(d) \( G = \sigma '^{-1} (\sigma \in \text{weakly } I^\lambda_0 \text{ - generic over } \overline{W} \))
Making use of the fact we again define \( \langle \bar{\mathfrak{s}}_i, i < \omega \rangle \) monotone and cofinal in \( \bar{X} \). We again let \( \bar{\mathfrak{s}}_i = \sigma(\bar{\mathfrak{s}}_{i-1}) \) and conclude:

1. \( \langle \bar{\mathfrak{s}}_i, i < \omega \rangle \in \bar{W} \) if \( \sigma'(11) = \omega \)

2. \( \sigma'(\bar{\mathfrak{s}}_i) = \bar{\mathfrak{s}}_{i+1} \) whenever \( \sigma'; \bar{W} \subset \bar{W}, \sigma'(1) = f \).

As before, our strategy is to define sequences \( c_i \in \bar{B}_{\bar{\mathfrak{s}}_i} \), \( \sigma_i \in \bar{V}^{\bar{B}_{\bar{\mathfrak{s}}_i}} \) such that \( \langle c_i, i < \omega \rangle \) is a thread in \( \langle \bar{B}_{\bar{\mathfrak{s}}_i}, i < \omega \rangle \) and \( c_i \) forms that \( \sigma_i ; \bar{W} \subset \bar{W} \) and \( \bar{c}_i = \sigma_{i+1}^{-1} \sigma_i \) is weakly \( \bar{B}_{\bar{\mathfrak{s}}_i} \)-generic over \( \bar{W} \) whenever \( \bar{c}_i \in \bar{c}_i \in \bar{B}_{\bar{\mathfrak{s}}_i} \)-generic and \( \bar{c}_i = \bar{c}_i \).

We then set \( c = \bigcap \bar{c}_i \) and will have built enough pointwise correspondence between the \( \bar{c}_i \) that we can define:

\[ \sigma'(x) = _{11} \sigma_i(x) \] for \( x \) large enough

where \( \sigma_i = \sigma_i^i \) and \( \bar{G} \in \bar{c} \in \bar{B}_{\bar{\mathfrak{s}}_i} \)-generic.

The pointwise correspondence will also guarantee that (b), (c) of the claim hold and

1. \( \bar{G}_i = \sigma_{i+1}^{-1} \sigma_i \) is weakly \( \bar{B}_{\bar{\mathfrak{s}}_i} \)-generic over \( \bar{W} \) (\( i < \omega \)).
As before we define a dense subset $X$ of $\overline{B}_A^-$ in two cases:

**Case A**  \( \text{cf}(\lambda) = \omega \). Then \( \text{cf}(\lambda^-) = \omega \) in \( \overline{W} \) and we have \( \langle \overline{\delta}_i^\omega, 1 < \omega \rangle \subseteq \overline{W} \). We let \( X = \bigcup_{i < \omega} b_i \) s.t. \( \langle b_i, 1 < \omega \rangle \subseteq \overline{W} \) is a thread in \( \langle \overline{B}_A^-, 1 < \omega \rangle \). (Hence \( b = \bigcap_{i < \omega} h_{\delta_i}^\omega (b_i) \) in \( \overline{W} \)).

**Case B**  \( \text{cf}(\lambda) > \omega \). Then \( \text{cf}(\lambda^-) > \omega \) in \( \overline{W} \) and we let \( X = \bigcup_{i < \lambda^-} \overline{B}_A^- \setminus \{ \overline{0} \} \). (Hence \( b = h_{\delta_i}^\omega (b_i) \) for some \( i \) if \( b \in X \)).

We then face the problem of getting:

(c1) \( \mathcal{G} = \mathcal{J}^{-1, \omega} \) is weakly \( \overline{B}_A^- \) - generic over \( \overline{W} \) from (c1') above. We again solve this by guaranteeing that a "master sequence" lies in \( \mathcal{G} \):

**Def**  By a master sequence we mean a sequence \( \langle b_i, 1 < \omega \rangle \) s.t.:

(a) \( b_i \in X \), \( b_i < b_h \) and \( h_{\delta_i}^\omega (b_i) = h_{\delta_h}^\omega (b_h) \) for \( h < i \).

(b) Whenever \( G \subseteq \overline{B}_A^- \) is an ultrafilter on \( \overline{X} \) and \( G \cap \overline{B}_A^- \subseteq \overline{G} \), \( 3 \delta_i, 1 < \omega \) such that \( G \cap \overline{B}_A^- \subseteq \overline{G} \) is weakly \( \overline{B}_A^- \) - generic over \( \overline{W} \) for \( i < \omega \), then \( G \) is weakly \( \overline{B}_A^- \) - generic over \( \overline{W} \).

We prove...
(3) There is a master sequence \( \langle b_i \mid i < \omega \rangle \) s.t. \( b_0 \subseteq \bar{a} \).

Let \( b_0 \subseteq \bar{a} \) s.t. \( b_0 \subseteq X \). Let \( \langle \Delta_i \mid i < \omega \rangle \) enumerate the \( \Delta_i \in \bar{W} \) s.t. \( \Delta_i \) is predense in \( \bar{B}_x \) (i.e. \( \omega_{\Delta_i} = 1 \)) and \( \bar{\Delta}_i \leq \omega_1 \) in \( \bar{W} \).

Let \( b_i \) be given. By (4) of the proof of Thm 5 Case 1, there is \( b \in b_i \) s.t. \( h_{\bar{\Delta}_i}^{-}(b_1) = h_{\bar{\Delta}_i}^{-}(b_i) \) and \( \bar{\Delta}_i = \{ a \in \bar{B}_{\bar{\Delta}_i}^{-} \mid \forall d \in \bar{\Delta}_i \text{ and } b \subseteq d \bar{\Delta}_i \} \) a strongly dense below \( h_{\bar{\Delta}_i}^{-}(b_1) \) in \( \bar{B}_{\bar{\Delta}_i}^{-} \).

Set \( b_{i+1} = b \). This defines \( \langle b_i \mid i < \omega \rangle \).

Now let \( G \) be an ultrafilter on \( \bar{B}_x \) s.t. \( G \cap \bar{B}_{\bar{\Delta}_i}^{-} \) is weakly generic over \( W \) for \( i < \omega \) and \( \exists b_i \mid i < \omega \bar{\Delta}_i \subseteq G \). Let \( \Delta \) be predense in \( \bar{B}_x \) s.t. \( \bar{\Delta} = \omega_1 \) in \( \bar{W} \). Then \( \bar{\Delta}_i = \bar{\Delta} \).

For \( d \in \Delta \) s.t. \( a_d = \bigcup \{ a \in \bar{\Delta} \mid a \subseteq b_{i+1} \bar{\Delta}_i \} \). Then \( \exists d \mid d \in \bar{\Delta}_i \) is predense above \( h_{\bar{\Delta}_i}^{-}(b_i) \) and has cardinality \( \leq \omega_1 \) in \( \bar{W} \).

Hence there is \( d \) with \( a_d \in G \). Hence \( a_d \cap b_i \subseteq d_i \). Hence \( d \subseteq \Delta \cap G \). GED.
From now on let \( \langle b_i \mid i < \omega \rangle \) be a fixed master sequence r.t. \( b_0 \in a \), where \( s(a) = a \), and let \( \langle x_i \mid i < \omega \rangle \) be a fixed enumeration of \( \bar{W} \) with infinite repetitions.

As before we construct \( c_k \in (B_{\bar{S}_k})^V \) r.t. \( c_k \subseteq \bar{S}_k \) (\( k < \omega \)) r.t. \( \langle c_i \mid i < \omega \rangle \) is a thread in \( \langle B_{\bar{S}_i} \mid i < \omega \rangle \) and (*) holds as in Case 1 of the proof of Thm 5 - except that (**)(d) is now changed to:

\[
(d) \quad \bar{G}_k = s^{-1} \circ \bar{G}_k \text{ is weakly } \bar{B}_{\bar{S}_k} \text{ - generic over } \bar{W},
\]

The proof that (*) implies the claim is exactly as in Thm 5 with minor verbal changes (e.g., "weakly generic" instead of "generic").
At remain only to construct $c_i', \bar{\sigma}_i$ and verify (\*1). We set: $c_0 = 1, \bar{\sigma}_0 = \bar{\sigma}$. Given $c_j, \bar{\sigma}_j$ with $k = j + 1$, we use Lemma 6 to get $c_k, \bar{\sigma}_k$ (with $a = c_j, A = IB_{\bar{\sigma}_j}, b = c_k, B = IB_{\bar{\sigma}_k}, \bar{\sigma} = \bar{\sigma}_j, \bar{\sigma} = \bar{\sigma}_k, e = h_{\bar{\sigma}_k}(b_k)$) and appropriately defined $1$. QED (Case 1)

Case 2 Case 1 fails.

Then $\lambda$ is regular and $IB_{\bar{\sigma}} < \lambda$ for $\bar{\sigma} < \lambda$. We closely follow the proof of Thm 5. We again let $\bar{W}, W, \bar{\sigma}, \bar{\sigma}$ be as before with

$$\sigma'(\bar{\theta}, IB, \bar{\lambda}, \bar{\lambda}, \bar{\lambda}, \bar{\lambda}, \bar{\lambda}, \bar{\lambda}, (i = 1, \ldots, n))$$

where $a \in IB_{\lambda}, a_{\bar{\sigma}}$ and $\lambda_{\bar{\sigma}1}, \lambda_{\bar{\sigma}n}$ are as before.

We set: $\lambda_{m+1} = \lambda, \bar{\lambda}_{m+1} = \lambda, \bar{\lambda}_0 = \mathrm{On} \times \bar{W}$ and

$$\bar{\lambda}_i = \sup \sigma'' \bar{\lambda}_{F^i} \quad (i = 0, \ldots, m+1)$$

as before. (We also write: $\bar{\lambda} = \bar{\lambda}_{m+1}$.) We again fix an enumeration $\langle x_i, i < \omega \rangle$ of $\bar{W}$ with infinite repetitions and a master sequence $\langle b_i, i < \omega \rangle$ with $b_0 < \bar{a}$.

Here Case B holds, so our scheme set $X$ is in $IB_{\lambda}$.

Claim There is $\zeta \in IB_{\lambda}$, and $c \neq a$, $c \neq 0$, and whenever $G \ni c \in IB_{\lambda}$, generic there is $\sigma' \in V[\zeta]$ s.t.

\[ (a) \ \sigma' : \bar{W} \to \mathbb{W} \]

\[ (b) \ \sigma'(\bar{\theta}, IB, \bar{\lambda}, \bar{\lambda}, \bar{\lambda}) = \Theta_{ib, \bar{\lambda}} \]

\[ (c) \ \sup \sigma'' \bar{\lambda}_i = \bar{\lambda}_i \quad (i = 0, \ldots, m+1) \]

\[ (d) \ \bar{G} = \sigma'' \bar{G} \ni \text{weakly } IB_{\lambda} - \text{generic over } \bar{W}. \]
As before we choose \( \langle \overline{z}_i^j \mid 1 \leq i \leq n \rangle \) monotone and cofinal in \( \overline{z}_i^j \) for \( j = 0, 1, \ldots, m+n \) and set \( \overline{z}_i^j = \delta^{-1}(\overline{z}_i^{j-1}) \).

We also write \( \overline{z}_i^j = \overline{z}_i^{m+n} \), \( \overline{z}_i^j = \overline{z}_i^{m+n} \).

We inductively construct \( c_k \in B_{\overline{z}_i^j} \), \( \sigma_k \in V_{\overline{z}_i^j} \) \( n+t \) \( I \) of Case 2 in the proof of Thm 5 holds and \( II \) of Case 2 in the proof of Thm 5 holds except that \( II(d) \) is replaced by \( II(d) \) let \( c_k \in B_{\overline{z}_i^j} \), \( \sigma_k \in V_{\overline{z}_i^j} \), \( n+t \) Then

\[ \overline{G} = \sigma_k^{-1} \sigma_k(G) \] is weakly \( B_{\overline{z}_i^j} \)-generic over \( W \).

The proof that this implies the claim is exactly as before. It remains to construct \( c_k, \sigma_k \) and verify \( I, II \).

As before we shall in fact construct \( b_k, \sigma_k \) with \( b_k \in B_{\overline{z}_i^j} \) in an intermediate step before constructing \( c_k \in b_k \).

We inductively verify \( I-IV \), where \( III, IV \) are exactly as before. Suppose now that \( I-IV \) hold below \( k \), and that \( b_k, \sigma_k \) are given satisfying \( III(a1-\gamma) \) and \( IV, \) We must construct \( c_k \) and verify \( I, II \) and \( III(d) \). We proceed exactly as before. We define

\[ \alpha_i^m (\nu \leq \overline{z}_i^j, \mu \leq \overline{z}_i^j, \sup_{\overline{z}_i^j} < \nu) \] and \( A_k \) exactly as before. \( IV \) then gives

\[ \delta^e_a (a \in A_k) \]
(4) \( \sigma_a' = \sigma'_a \) for \( a = a^{v} \in A_k \) whenever
\( G \ni a^{v} \subseteq B_{v} \) - generic and \( G' = G \cap B_{v} \).

As before, we conclude:

(5) \( \mathcal{A} \vdash G \ni a^{v} \subseteq B_{v} \) - generic, \( a = a^{v} \in A_k \),
then (a) - (g) hold with \( \sigma_a = \sigma_a' \) in place of
\( \sigma_a^{v} = \sigma_a', G = \sigma'_a G_{v}^{v} \) for \( i < k \),
where \( G_{v}^{v} = G \cap B_{v} \) for \( v \leq v \).

We then apply Lemma 6 to get:

(6) Let \( a \in A_k \), \( a = a^{v} \in A_k \).
There are \( \tilde{a} \in B_{v} \),
\( \bar{a}' \in \mathcal{V} B_{v} \) s.t. \( h_{v}(\tilde{a}) = a \) and whenever
\( G \ni a \subseteq B_{v} \) - generic, \( \sigma_{a} = \sigma_{a}^{v} \), \( \sigma_{a}' = \sigma_{a}' \) and
\( \sigma_{i} = \sigma_{i}^{v} G_{v}^{v} \) for \( i < k \) (\( G_{v}^{v} = G \cap B_{v} \)),

Then:

(a) \( \sigma_{a}' : \bar{W} \rightarrow \mathcal{W} \)
(b) \( \sigma_{a}' (\bar{a}, B_{v}, \bar{a}', \bar{X}_{i}) = \Theta (B_{v}, \bar{a}', \bar{X}_{i}) \) \( (i = 1, \ldots, m + n) \)
(c) \( \sup \sigma_{a}' = \bar{X}_{i} = \bar{X}_{i} \) \( (i = 1, \ldots, m + n) \)
(d) \( \tilde{G} = \sigma_{a}^{-1} G \) is weakly \( B_{v}^{v} \) - generic over \( \bar{W} \)
(e) Let \( r \) be least s.t. \( r \leq \bar{X}_{i} \). Then
\( \sigma_{a}' (x_{i}^{v}, b_{i}, d_{i}) = \sigma_{a}(x_{i}^{v}, b_{i}, d_{i}) \) for \( i < r \)

where \( d_{i} \) is as before for \( i < k \) and
\( d_{i}^{v} = \left\{ \begin{array}{ll}
\text{the } \bar{W} - \text{least } d \in X_{i} \text{ s.t. } \sigma_{a}(d) = G_{v}
\text{ if such exists,}
0 \text{ if not}
\end{array} \right. \)

for \( k \leq i < r \).
(f) Let \( r \) be as above. Let \( i = 0, \ldots, m+1 \). Let \( \sigma_a (\tilde{\sigma}_{m+i}) \leq \tilde{\sigma}_a (\tilde{\sigma}_{m+1}) \). Then

\[
\sigma_a'(\tilde{\sigma}_d') = \sigma_a(\tilde{\sigma}_d') \quad \text{for } d \leq m+1
\]

Let \( \sigma_a'(h_{\frac{r}{3}_{i+1}}(\tilde{b}_{i+1})) \in G \).

This follows by Lemma 6, taking

\[
A = IB_x, \quad A = IB_{\tilde{x}}, \quad IB = IB_y, \quad IB = IB_{\tilde{y}},
\]

\[
a = a', \quad b = a', \quad \sigma = \sigma_a, \quad \sigma = \sigma_a', \quad e = h_{\frac{r}{3}_{i+1}}(\tilde{b}_{i+1})
\]

and defining \( t \) appropriately.

The rest of the proof is virtually identical to that of Thm 5.

QED (Thm 8)

It is not difficult to reformulate and reprove Lemma 2 - Lemma 4 for "semi-subproper" in place of "subcomplete."