

§2 The Forcing Construction

In the following let $\omega < \kappa < \beta$, where κ is regular, β is a cardinal, $2^\kappa = \kappa$ and $2^\beta = \beta$. Set $M = L_\beta^A = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set $N = \langle H_{\beta^+}, M, <, \dots \rangle$, where $<$ is a well ordering of N . (Then N is a ZFC⁻ model.) Let \mathcal{L} be a language on N satisfying the above conditions (*) of §1. Suppose moreover that it has two constants $\dot{m}, \dot{\pi}$ and contains the additional axioms:

(**) • $\underline{\kappa} = \omega_1$

• $\dot{M} = \langle \dot{M}_i \mid i \leq \underline{\kappa} \rangle$ where $\dot{M}_i = L_{\beta_i}^{A_i}$ is countable for $i < \underline{\kappa}$ and $\dot{M}_{\underline{\kappa}} = \underline{M}$

• $\dot{\pi} = \langle \pi_{ij} \mid i \leq j \leq \underline{\kappa} \rangle$ is a commutative, continuous system of elementary embeddings $\pi_{ij} : \dot{M}_i \hookrightarrow \dot{M}_j$ ("continuous" means that $\dot{M}_\lambda = \bigcup_{i < \lambda} \text{rng}(\pi_{i\lambda})$ for limit λ .)

• $\beta_i < \kappa_{i+1}$ for $i < \underline{\kappa}$ (where $\kappa_i = \text{crit}(\pi_{i,i+1})$)

We shall show that if \mathcal{L} satisfies two further conditions, we can form a set generic extension of the universe containing $\langle \dot{M}_i \mid i \leq \underline{\kappa} \rangle, \langle \pi_{ij} \mid i \leq j \leq \underline{\kappa} \rangle$ satisfying (**) (and also some other

statements which are provable in \mathcal{L}). The first of these conditions is consistency. The second is resurrectionability:

Def Let \mathcal{L} be a consistent language on N satisfying $(*)$, $(**)$. \mathcal{L} is resurrectionable iff the following holds:

In the generic collapse $V[G]$ of β^+ to ω we have:

Let \mathcal{M} be a solid model of \mathcal{L} . Let $\bar{N}, \bar{\mathcal{L}}, \sigma$ be s.t. $\bar{N}, \bar{\mathcal{L}} \in H_\kappa^{\mathcal{M}}$, $\sigma \in \text{wfc}(\mathcal{M})$ and

$\sigma: \langle \bar{N}, \bar{\mathcal{L}} \rangle \prec \langle N, \mathcal{L} \rangle$ s.t., letting $\alpha = \sigma^{-1}(\kappa)$

we have: $\bar{M} = \dot{M}_\alpha^{\mathcal{M}}$ and $\sigma \upharpoonright \bar{M} = \dot{\pi}_\alpha^{\mathcal{M}}$,

where $\sigma(\bar{M}) = M$. Let $\bar{\mathcal{M}} \in H_\kappa^{\mathcal{M}}$ be a solid model of $\bar{\mathcal{L}}$. Define:

$\tilde{M} = \langle \tilde{M}_i \mid i \leq \kappa \rangle$, $\tilde{\pi} = \langle \tilde{\pi}_i \mid i \leq \kappa \rangle$ by:

$$\tilde{M}_i = \begin{cases} \dot{M}_i^{\bar{\mathcal{M}}} & \text{if } i \leq \alpha \\ \dot{M}_i^{\mathcal{M}} & \text{if } i \geq \alpha \end{cases}; \quad \tilde{\pi}_i = \begin{cases} \dot{\pi}_i^{\bar{\mathcal{M}}} & \text{if } i \leq \alpha \\ \dot{\pi}_\alpha^{\bar{\mathcal{M}}} \dot{\pi}_\alpha^{\mathcal{M}} & \text{if } \alpha \leq i \leq \kappa \\ \dot{\pi}_i^{\mathcal{M}} & \text{if } \alpha \leq i \leq \kappa \end{cases}$$

Form $\tilde{\mathcal{M}}$ by reinterpreting $\dot{M}, \dot{\pi}$ as $\tilde{M}, \tilde{\pi}$.

Then $\tilde{\mathcal{M}}$ is a solid model of \mathcal{L} .

(Note Taken literally this requires that \mathcal{L} have no non logical predicates, constants, or function symbols other than \dot{E}, \underline{x} ($x \in N$) and $\dot{M}, \dot{\pi}$. If, in fact, \mathcal{L} has other non logical symbols, we form the reduced language \mathcal{L}' with just these symbols, taking

as its axioms the statements of the reduced language which are provable in \mathcal{L} .)

In the following let \mathcal{L} be consistent and resessionable.

We define forcing conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as follows:

Def $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_{\mathcal{L}}$ is the set of $\langle p_0, p_1, p_2 \rangle$ s.t.

- (a) p_0 is a finite partial map from κ to κ
- (b) $\text{dom}(p_1) = \text{dom}(p_0)$ and each $p_1(i)$ is a finite partial map from $p_0(i)$ to β for $i \in \text{dom}(p_1)$
- (c) $\text{dom}(p_2) = \text{dom}(p_0)$ and $p_2(i) \in M$.

At $p \in \tilde{\mathbb{P}}$ we set: $D(p) = \text{dom}(p_0)$, $\beta_i^p = p_0(i)$,
 $\pi_i^p = p_1(i)$, $\alpha_i^p = p_2(i)$ for $i \in D(p)$. We also
 set: $D'(p) = \{i \in D(p) \mid \alpha_i^p \neq \emptyset\}$.

Def Let $p \in \tilde{\mathbb{P}}$, $\mathcal{L}(p)$ is \mathcal{L} augmented by
 the further axioms:

- (a) $\beta_i^p = \text{om} \cap \dot{M}_i$; $\pi_i^p \in \pi_{i, \kappa}$ for $i \in D(p)$
- (b) $\forall \bar{a} \in \pi_{i, \kappa} : \langle \dot{M}_i, \bar{a} \rangle \prec \langle \underline{M}, \underline{a}_i \rangle$ for $i \in D(p)$

In the following we write: π_i^p for $\pi_{i, \kappa}^p$.

We also set: $\alpha_i^p(m) = \{x \mid \langle x, m \rangle \in \alpha_i^p\}$ for $m \in \omega$.

As mentioned above we also write:

$$\beta_i = \text{Onn } M_i, \quad \kappa_i = \pi_i^{-1}(\kappa).$$

We are now ready to define $IP = IP_{\mathcal{L}}$ as a subset of \tilde{IP} . We first define:

Def Let $p \in \tilde{IP}$, p is good iff $\mathcal{L}(p)$ is consistent.

Def Let $p \in \tilde{IP}$, $i, j \in D(p)$, $i < j$. i is neat in j iff a_i^p is $\langle M, a_j^p(m) \rangle$ definable in parameters from $\text{rng}(\pi_j^p)$.

p is neat iff i is neat in j whenever $i, j \in D(p)$, $i < j$, and $R_i^p \neq \emptyset$.

Def Let \mathcal{L} be consistent and resectionable.

$IP = IP_{\mathcal{L}} = IP_{\mathcal{L}}^{\sim}$ the set of $p \in \tilde{IP}$ which are good and neat.

$p \leq q$ in IP iff $p, q \in IP$, $p \supseteq q$, and

$$\pi_i^p = \pi_i^q, \quad a_i^p = a_i^q \quad \text{for } i \in D(q).$$

Before proceeding further we prove a number of lemmas on the extendability of conditions in IP .

Lemma 1.1 Let $p \in IP$. Let u be finite s.t. $D(p) \subset u \subset \kappa$. There is $p' \leq p$ s.t. $D(p') \supseteq u$.

proof of Lemma 1.1

Work in a generic collapse $V[G]$ of β^t to ω .

Let M be a solid model of $\mathcal{L}(p)$. Set:

$$P'_0 = \langle \beta_i \mid i \in U \rangle \text{ where } \beta_i = \text{Om} \cap \dot{M}_i^{\text{str}}. \text{ We}$$

$$\text{then set: } P'_h(i) = \begin{cases} P_h(i) & \text{if } i \in D(p) \\ \emptyset & \text{if } i \in U \setminus D(p) \end{cases}$$

for $h = 1, 2$. Then $p' \leq p$ has the desired property. QED (Lemma 1.1)

In similar fashion:

Lemma 1.2 Let $p \in P$, $i \in D(p)$ and let

$u \subset \beta_i^p$ be finite. There is $p' \leq p$ s.t.

$$u \subset \text{dom}(\pi_i^{p'})$$

prf. (Assume w.l.o.g. $\text{dom}(\pi_i^p) \subset u$)

We again let $M \in V[G]$ be a solid model of $\mathcal{L}(p)$. Set $p'_0 = p$. p'_1 is defined

$$\text{by: } p'_1(j) = \pi_j^{p'} = \pi_j^p \text{ if } j \in D(p), j \neq i,$$

$$p'_1(i) = \pi_i^{p'} = \pi_i^p \upharpoonright u; \quad p'_2 = p'_1$$

Then $p' \leq p$ has the desired property

QED (1.2)

Lemma 1.3 Let $p \in P$, $u \subset \beta$

where u is finite. There is $p' \leq p$

s.t. $u \subset \text{rng}(\pi_j^{p'})$ for some j .

prf. of Lemma 1.3.

Let $M \in V[G]$ be a solid model of $\mathcal{L}(P)$.
Pick $j < \omega$ s.t. $D(P) \subset j$ and $u \in \text{rng}(\pi_{i,\omega}^M)$.

Set: $P'_0 = P_0 \cup \{ \langle \beta_{i,j}^M, i \rangle \}$ ($\beta_{i,j} = 0$ on M_j)

$$P'_1(h) = \begin{cases} \pi_h^P & \text{for } h \in D(P) \\ \pi_{j,k}^M \upharpoonright (\pi_{i,j}^{M-1} u) & \text{for } h=i \end{cases}$$

$P'_2 = P_2 \cup \{ \langle \emptyset, j \rangle \}$. Then $P' \leq P$ has the desired property. QED(1.3)

Lemma 1.4 Let $p \in IP$, $u \in \text{rng}(\pi_\lambda^P)$, where λ is a limit ordinal. There is $p' \leq p$ s.t. $u \in \text{rng}(\pi_i^{P'})$ for a $i < \lambda$.

prf.

Let $M \in V[G]$ be a solid model of $\mathcal{L}(P)$.

Pick $j < \lambda$ s.t. $u \in \text{rng}(\pi_{j,\lambda}^M)$ and $j \notin D(P)$. Set:

$P'_0 = P_0 \cup \{ \langle \beta_{j,i}^M, i \rangle \}$ and

$$P'_1(i) = \begin{cases} \pi_i^P & \text{for } i \in D(P) \\ \pi_{i,\lambda}^M \upharpoonright (\pi_{i,\lambda}^{M-1} u) & \text{for } i=j \end{cases}$$

$P'_2 = P_2 \cup \{ \langle \emptyset, i \rangle \}$. Then $P' \leq P$ has the desired property. QED(1.4)

Lemma 1.5 Let $p \in IP$. There is $p' \leq p$ s.t. $\text{rng}(\pi_i^{P'}) \subset \text{rng}(\pi_j^{P'})$ whenever $i, j \in D(P')$, $i < j$.

proof.

Set $P'_0 = P_0$, $P'_2 = P_2$. We define P'_1 as follows: Set $u_i = \bigcup_{h \in D(P) \cap i} \text{rng}(\pi_h)$ for $i \in D(P)$.

Let \mathcal{M} be a solid model of $\mathcal{L}(P)$. Set:
 $\pi_i^{P'} = \pi_{i\kappa}^{\mathcal{M}}$ for $i \in D(P)$. Then P' has the
 desired property. QED (1.5)

Lemma 1.6 Let $P \in \mathcal{P}$, $\vec{\zeta}_1, \dots, \vec{\zeta}_m \in \text{rng}(\pi_i^P)$
 s.t. $M \models \forall \gamma \varphi(\gamma, \vec{\zeta})$. There is $P' \leq P$ s.t.
 $\forall \gamma \in \text{rng}(\pi_i^{P'}) M \models \varphi(\gamma, \vec{\zeta})$.

prf.
 Let $\mathcal{M} \in \mathcal{V}[G]$ be a solid model of $\mathcal{L}(P)$.
 Since $M \models \forall \gamma \varphi(\gamma, \vec{\zeta})$ and $\pi_{i\kappa}^{\mathcal{M}}: M_i^{\mathcal{M}} \prec M$,
 there is $\gamma \in \text{rng}(\pi_{i\kappa}^{\mathcal{M}})$ s.t. $M \models \varphi(\gamma, \vec{\zeta})$.

Let $\pi_{i\kappa}^{\mathcal{M}}(\vec{\gamma}) = \gamma$. Set $P' = \langle P_0, P_1, P_2 \rangle$
 where $\pi_h^{P'} = \begin{cases} \pi_h^P & \text{for } h \neq i \\ \pi_h^P \cup \{ \langle \gamma, \vec{\gamma} \rangle \} & \text{for } h = i \end{cases}$

Then $P' \leq P$ has the desired property.

Note The last proof also gives:

Let $\langle M, a_i^P \rangle \models \forall \gamma \varphi(\gamma, \vec{\zeta})$, where

$\vec{\zeta}_1, \dots, \vec{\zeta}_m \in \text{rng}(\pi_i^P)$, then there is

$P' \leq P$ s.t. $\forall \gamma \in \text{rng}(\pi_i^{P'}) \langle M, a_i^P \rangle \models \varphi(\gamma, \vec{\zeta})$.

Lemma 2 Let G be IP-generic.

(A) $\cup \{P_\alpha \mid P \in G\} = \langle \beta_i \mid i < \kappa \rangle$, where $\beta_i < \kappa$ for $i < \kappa$.

(set: $\beta_\kappa = \beta$)

(B) Let $i < \kappa$. Set $\bar{\pi}_i = \cup \{\pi_i^P \mid P \in G \wedge i \in D(P)\}$

Then $\bar{\pi}_i : \beta_i \rightarrow \beta$ is monotone with

$\kappa_i = \text{crit}(\bar{\pi}_i)$ where $\kappa_i = \bar{\pi}_i^{-1}(\kappa)$

(C) $i \leq j \leq \kappa \rightarrow \text{rng}(\bar{\pi}_i) \subset \text{rng}(\bar{\pi}_j)$

(letting $\bar{\pi}_\kappa = \text{id} \upharpoonright \beta$)

(D) If $\lambda \leq \kappa$ is a limit ordinal, then

$$\text{rng}(\bar{\pi}_\lambda) = \bigcup_{i < \lambda} \text{rng}(\bar{\pi}_i)$$

(E) Let $i \leq \kappa$. Set $X_i =$ the smallest $X \prec M$

s.t. $\text{rng}(\bar{\pi}_i) \subset X$. Then $X \cap \beta = \text{rng}(\bar{\pi}_i)$

(F) Set $\pi_i : M_i \xrightarrow{\cong} M \upharpoonright X_i$ ($i \leq \kappa$).

Then $\bar{\pi}_i : M_i \prec M$, $\bar{\pi}_i \upharpoonright \beta_i = \bar{\pi}_i$, $\beta_i = \text{On} \cap M_i$,

$\bar{\pi}_i \upharpoonright \kappa_i = \text{id}$, where $\kappa_i = \bar{\pi}_i^{-1}(\kappa)$.

(G) Set $\pi_{ij} = \pi_j^{-1} \pi_i$ ($i \leq j \leq \kappa$). Then

$\pi_{ij} : M_i \prec M_j$ and $\langle \pi_{ij} \mid i \leq j \leq \kappa \rangle$ is a

commutative continuous system of embeddings.

Prf.

We prove (A)-(E), since (F), (G) then follow trivially.

(A) follows by 1.1. By 1.2 it follows that $\bar{\pi}_i: \beta_i \rightarrow \beta$. $\bar{\pi}_i$ is monotone however, since each π_i^p is monotone for $p \in G$, since $\mathcal{L}(p)$ is consistent. This gives (B), (C) follows by Lemma 1.5, (D) follows by Lemma 1.3 for $\lambda = \kappa$ and Lemma 1.4 for $\lambda < \kappa$.

We prove (E), using Lemma 1.6. Let $f: \beta \xrightarrow{\text{onto}} M$ be M -definable. Set

$$X = f'' \text{rng}(\bar{\pi}_i). \text{ Then}$$

Claim 1 $X < M$

prf.

$$\text{Let } x_1, \dots, x_n \in X, M \models \forall y \varphi(y, \vec{x}).$$

It suffices to show:

$$\text{Claim } \forall y \in X \quad M \models \varphi(y, \vec{x})$$

$$\text{Let } x_i = f(\bar{z}_i), \bar{z}_i \in \text{rng}(\bar{\pi}_i) \quad (i=1, \dots, n).$$

$$\text{Then } M \models \forall y \varphi(f(y), f(\bar{z}_1), \dots, f(\bar{z}_n)).$$

Hence there is such an $y \in \text{rng}(\bar{\pi}_i)$,

and we take $y = f(y)$. QED (Claim 1)

Claim 2 $\text{rng}(\bar{\pi}_i) \subset X$.

prf.

$$\text{Let } \bar{z} \in \text{rng}(\bar{\pi}_i). \text{ Then } M \models \forall y \bar{z} = f(y).$$

Hence there is such an $y \in \text{rng}(\bar{\pi}_i)$.

Hence $\bar{z} = f(y) \in X$. QED (Claim 2)

X is then the smallest $X \prec M$ s.t., $\text{rng}(\bar{\pi}_i) \subset X$, since any such X is closed under f . It remains only to note:

Claim 3 $X \cap \beta \subset \text{rng}(\bar{\pi}_i)$.

prf.

Let $\gamma \in X$, $\gamma = f(\bar{z})$, $\bar{z} \in \text{rng}(\bar{\pi}_i)$.

Then $M \models \forall \gamma \gamma = f(\bar{z})$. Hence $\gamma \in \text{rng}(\bar{\pi}_i)$ by Lemma 1.6. QED (Lemma 2)

Lemma 3 Let G be \mathbb{P} -generic. Then κ is regular in $V[G]$.

prf.

Let $p \Vdash \dot{f}: \delta \rightarrow \kappa$, where $\delta < \kappa$.

Claim There is $p' \leq p$, $\alpha < \kappa$ s.t., $p' \Vdash \text{rng}(\dot{f}) \subset \check{\alpha}$.

Set $N^+ = \langle H_{\beta^{++}}, N, <, \mathcal{L} \rangle$, where $<$ is a wellordering of $H_{\beta^{++}}$.

We can assume w.l.o.g. that $\dot{f} \in N$ (e.g. we can take

$$\dot{f} = \{ \langle \langle \check{s}, \check{x} \rangle, \check{r} \rangle \mid p \geq r \Vdash \dot{f}(\check{x}) = \check{r} \}$$

Then $\dot{f} \in N^+$. Let $\langle a(m) \mid m < \omega \rangle$ enumerate the $a \in M$ which are N^+ -definable from the parameters $\dot{f}, p, \delta, \kappa, M$.

Set $a = \{ \langle x, n \rangle \mid n < \omega \wedge x \in a(n) \}$,

If \mathcal{M} is a model and X a set, we write $X \prec \mathcal{M}$ to mean: $X \subset \mathcal{M}$ and $\mathcal{M} \upharpoonright X \prec \mathcal{M}$,

Fact For any $X \subset \mathcal{M}$ the following are equivalent:

(a) $X \prec \langle \mathcal{M}, a(n) \rangle$ for $n < \omega$

(b) Let $Y =$ the smallest $Y \prec N^+$ s.t.

$X \cup \{f, p, \delta, n\} \subset Y$. Then $Y \cap \mathcal{M} = X$,

(b) \rightarrow (a) is trivial. (a) \rightarrow (b) follows from the fact that each $z \in Y$ is N^+ -definable in parameters from $X \cup \{f, p, \delta, n\}$.

Now collapse β^{++} generically to ω . Work in the resulting model $V[G]$. Let \mathcal{M} be a solid model of $\mathcal{L}(p)$. Then $\langle \mathcal{M}, a \rangle \subset N \subset \mathcal{M}$. Since κ is regular in \mathcal{M} , there must be an $d < \kappa$ s.t. $d = \text{crit}(\pi_{d, \kappa}^{\mathcal{M}}) = \kappa_d^{\mathcal{M}} > D(p)$

and there is $\bar{a} \subset M_d^{\mathcal{M}}$ s.t. $\pi_{d, \kappa}^{\mathcal{M}} : \langle M_d^{\mathcal{M}}, \bar{a} \rangle \prec \langle \mathcal{M}, a \rangle$. We define

$p' \leq p$ as follows: $p' = \langle p'_0, p'_1, p'_2 \rangle$,

where: $D(p') = D(p) \cup \{d\}$, $p'_h \upharpoonright d = p_h$ ($h=0,1,2$),

$p'_0(d) = \beta_d^{\mathcal{M}}$, $\pi_d^{p'} = \{ \langle n, d \rangle \}$, $a_d^{p'} = a$

Then p' is good, since \mathcal{M} models $\mathcal{L}(p')$.

p' is neat, since each a_i^p ($i \in D(p)$) is $\langle M, a(m) \rangle$ -definable in no parameters for some m (in fact $a_i^p = a(m)$). Thus $p' \leq p$ and it suffices to prove:

Claim $p' \Vdash \text{rng}(f) \subset \check{\alpha}$.

Suppose not.

Then there are $q \leq p'$, $\bar{\delta} < \delta$ s.t.

(1) $q \Vdash f(\check{\bar{\delta}}) \not\subset \check{\alpha}$.

Working in $V[G]$, where G collapses β^+ , pick a solid \mathcal{M} modelling $\mathcal{L}(q)$. Then

(2) $\pi_{\delta, \kappa}^{\mathcal{M}} : \langle M_{\delta}^{\mathcal{M}}, \bar{a} \rangle \prec \langle M, a \rangle$ for

for $\bar{a} = \pi_{\delta, \kappa}^{\mathcal{M}-1} a$ and $a = a_{\delta}^q$.

Let $X = \text{rng}(\pi_{\delta, \kappa}^{\mathcal{M}})$ and $Y =$ the smallest $Y \prec N^+$ s.t. $X \cup \{f, p, \delta, \kappa\} \subset Y$. Then

$Y \cap M = X$ by the above fact and (2).

Set $\pi : \bar{N}^+ \xrightarrow{\sim} N^+ \upharpoonright Y$, where \bar{N}^+ is transitive. Then $\pi \supset \pi_{\delta, \kappa}^{\mathcal{M}}$ and $\pi(M_{\delta}^{\mathcal{M}}) = M$.

(3) $\bar{N}^+ \in \mathcal{M}$

proof. Let $\tilde{Y} =$ the smallest $Y \prec N^+$ s.t. $M \cup \{f, p, \delta, \kappa, B\} \subset Y$. Let $\tilde{\pi} : \tilde{N}^+ \xrightarrow{\sim} N \upharpoonright \tilde{Y}$. Then $\tilde{N}^+ \in N \subset \mathcal{M}$, since \tilde{N}^+ is transitive and of cardinality $\leq \beta$. Let

Note that $\tilde{\pi} \upharpoonright M = \text{id}$, Let

$$\tilde{\pi}(\tilde{f}, \tilde{p}, \tilde{\delta}, \tilde{\alpha}, \tilde{\beta}) = f, p, \delta, \alpha, \beta \dots$$

Let Y^* = the smallest $Y^* \prec \tilde{N}^+$ s.t.

$$X \cup \{\tilde{f}, \tilde{p}, \dots, \tilde{\beta}\} \subset Y^*$$

Since $X \in \mathcal{D}_3$, it follows that $Y^* \in \mathcal{D}_3$.

But \tilde{N}^+ is the transitive closure of $\tilde{N}^+ \upharpoonright Y^*$.

Hence $\tilde{N}^+ \in \mathcal{D}_3$ by Mostowski's isomorphism theorem in \mathcal{D}_3 . QED(3)

Now let $\pi(\bar{N}, \bar{\mathcal{L}}, \bar{P}, \bar{p}, \bar{f}) = N, \mathcal{L}, P, p, f$.

Then $\bar{\mathcal{L}}$ is a consistent infinitary language on \bar{N} and $\bar{P} = P_{\bar{\mathcal{L}}}$ in \bar{N}^+ .

Note that $\pi \upharpoonright \alpha = \text{id}$, $\pi \upharpoonright \alpha = \text{id}$. Set:

$$q' = \langle q \upharpoonright \alpha, q \upharpoonright \alpha, q \upharpoonright \alpha \rangle$$

Then $q' \in P$ and $q \leq q'$

(4) $q' \in Y$.

proof.

$$\bullet i \in D(q') \rightarrow \beta_i^{q'} = \beta_i^{i \upharpoonright \alpha} \in \alpha \subset Y$$

Hence $q' \upharpoonright \alpha \subset Y$ is finite. Hence $q \in Y$

$$\bullet i \in D(q') \rightarrow \text{rng}(\pi_i^{q'}) \subset \text{rng}(\pi_i^{i \upharpoonright \alpha}) \subset \text{rng}(\pi_{\alpha}^{i \upharpoonright \alpha}) \subset X, \text{ Hence } q' \upharpoonright \alpha \times Y \subset Y \text{ is finite.}$$

• $i \in D(q')$ $\rightarrow a_i^{q'}$ is $\langle M, a(M) \rangle$ - definable in parameters from $\text{rng}(\pi_d^q) \subset \text{rng}(\pi_d^{q'}) = X$. Hence $a_i^{q'} \in Y$, since $a(M) \in Y$. Hence $q'_2 \subset Y \times d \subset Y$ is finite. QED(4)

Set: $\bar{q}' = \pi^{-1}(q')$. Then $\bar{q}' \leq \bar{p}$ in \bar{P} , since $q' \leq p$ in P . Since $\Vdash_{\bar{P}}^{\bar{N}^+} \bar{f} : \check{\alpha} \rightarrow \check{\alpha}$, there is $\bar{\alpha} \leq \bar{q}'$ in \bar{P} s.t.

(5) $\bar{\alpha} \Vdash_{\bar{P}}^{\bar{N}^+} \bar{f}(\check{\alpha}) = \check{\alpha}$ for a $\check{\alpha} < \check{d}$.

Set $\alpha = \pi(\bar{\alpha})$. Then $\alpha \in P$ and

(6) $\alpha \Vdash_P^{\bar{N}^+} f(\check{\alpha}) = \check{\alpha}$. Hence:

(7) α is incompatible with q in P , since $q \Vdash f(\check{\alpha}) \geq \check{\alpha}$.

We derive a contradiction by showing that α is, in fact, compatible with q .

Note that $\bar{L}(\bar{\alpha})$ is consistent, since $\bar{\alpha} \in \bar{P}$.

Since \bar{N} is countable in M , there is an $\bar{M} \in \bar{M}$ which is a solid model of $\bar{L}(\bar{\alpha})$. By resurrectionability we can then define a new model \tilde{M} , interpreting $\bar{M}, \bar{\pi}$ by

$\tilde{M} = \langle \tilde{M}_i : i \leq \kappa \rangle$, $\tilde{\pi} = \langle \tilde{\pi}_i : i \leq i \leq \kappa \rangle$,

where:

$$\tilde{M}_i = \begin{cases} M_{\nu}^{\bar{\alpha}} & \text{for } \nu \leq \alpha \\ M_{\nu}^{\bar{\beta}} & \text{for } \nu \geq \alpha \end{cases}$$

$$\tilde{\pi}_{\nu\tau} = \begin{cases} \frac{\cdot}{\pi} \frac{\bar{\alpha}}{\nu\tau} & \text{for } \nu \leq \tau \leq \alpha \\ \frac{\cdot}{\pi} \frac{\bar{\beta}}{\alpha\tau} \frac{\cdot}{\pi} \frac{\bar{\alpha}}{\nu\tau} & \text{for } \nu \leq \alpha \leq \tau \\ \frac{\cdot}{\pi} \frac{\bar{\beta}}{\nu\tau} & \text{for } \alpha \leq \nu \leq \tau. \end{cases}$$

\tilde{M} is then a solid model of \mathcal{L} . But \tilde{M} also models $\mathcal{L}(\bar{\alpha})$, since $\beta_i^{\bar{\alpha}} = \beta_i^{\bar{\beta}}$ for $i \in D(\bar{\alpha})$, $\pi_i^{\bar{\alpha}} \in \tilde{\pi}_{i\kappa}$ for $i \in D(\bar{\alpha})$ and $\tilde{\pi}_{i\kappa} : \langle M_i^{\bar{\alpha}}, \bar{a} \rangle \prec \langle M_{\kappa}^{\bar{\alpha}}, a_i^{\bar{\alpha}}(m) \rangle$ for $i \in D(\bar{\alpha})$, $\kappa < \omega$, where $\pi_{i\alpha}^{\bar{\beta}} : \langle M_i^{\bar{\beta}}, \bar{a} \rangle \prec \langle M_{\alpha}^{\bar{\beta}}, a_i^{\bar{\beta}}(m) \rangle$. Since $\bar{\alpha} \leq q'$ in IP, it follows that \tilde{M} models $\mathcal{L}(q')$. But since in \tilde{M} nothing was changed $\geq \alpha$, we have: \tilde{M} models $\mathcal{L}(q \upharpoonright (\kappa \setminus \alpha))$. Hence \tilde{M} models $\mathcal{L}(q)$, since $q = q' \cup q \upharpoonright (\kappa \setminus \alpha)$. Hence \tilde{M} models $\mathcal{L}(\bar{\alpha} \cup q)$ and $\bar{\alpha} \cup q$ is good, so we increase its middle component to obtain an $\bar{\alpha} \leq \bar{\alpha}, q$. Note that $\bar{\alpha} = \pi(\bar{\alpha}) \in Y = \pi''N^+$. But then $\bar{\alpha}$ is N^+ -definable in a parameter $w \in X = \text{rng}(\frac{\cdot}{\pi}_{\alpha\kappa}^{\bar{\beta}}) = \text{rng}(\frac{\cdot}{\pi}_{\alpha\kappa}^{\bar{\alpha}})$. Thus every

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$a_i^{\aleph_1}$ ($i \in D(\alpha)$) is $\langle M, a(m) \rangle$ definable in w .

Let $\pi_{h/\alpha}^{\aleph_1}(w_h) = w$ for $h \in D(\beta)$, $h \geq \alpha$.

Set $\alpha_0 = \alpha_0 \cup \beta_0$, $\alpha_2 = \alpha_2 \cup \beta_2$, and

$$\pi_i^{\aleph_1} = \begin{cases} \pi_i^{\aleph_1} & \text{for } i \in \dots \\ \pi_i^{\aleph_1} \cup \{ \langle w, w_i \rangle \} & \text{for } \dots \end{cases}$$

where $i \in D(\alpha) \cup D(\beta)$.

It follows easily that \tilde{M} models $\mathcal{L}(\aleph_1)$ and \aleph_1 is neat. Hence $\aleph_1 \leq \alpha, \beta$.

Contra! QED (Lemma 3)

We had promised to define a forcing IP s.t. if G is IP-generic, then $V[G]$ contains $\langle M_i : i \leq \aleph_1 \rangle$, $\langle \pi_i^{\aleph_1} : i \leq \aleph_1 \rangle$ satisfying (**). The only clause in (***) not yet verified is: $\aleph_1 = \omega_1$.

We have shown, however, that \aleph_1 is regular in $V[G]$, so if necessary we can do an additional forcing which makes all $\alpha < \aleph_1$ countable. (In many cases this step is unnecessary.)