§6 Another Look

We now return to Example 1 and consider the case $\beta = \omega_2$ (hence we assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$). Our forcing then adds a cofinal ω-sequence in ω_2 without adding new reals. This is, of course, what Namba forcing was designed to accomplish. When we first developed the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of Example 1 we were enormously proud of having accomplished the same task in a "completely different" way. Indeed the motivation of the construction and the combinatorics of the proof are different, but the forcing turns out to be the same, for we have: $BA(\mathbb{P}) \simeq BA(\mathbb{N})$, where $\mathbb{N}$ is the set of Namba conditions. We shall now prove this. Until further notice we assume: $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. (We also let β denote ω_2.) We first define the set $\mathbb{N}$ of Namba conditions:

Def $\mathbb{N}$ = the set of subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ s.t. $\overline{\{t \mid s \leq_T t\}} = \omega_2$ for all $s \in T$.

$\mathbb{N}$ is partially ordered by: $T \leq T' \longleftrightarrow T \subset T'$.

Note Call T a strict Namba tree iff

- $\bigwedge s \in t \bigvee t \, (s \leq_T t \wedge t$ is a split pt. of $T)$
- If s is a split point of T, then s has ω_2 many immediate successors in T.

$\mathbb{N}$ is sometimes defined as the set of strict Namba trees. This is equivalent, however, since the strict Namba trees are dense in the Namba trees.

Call $f : \omega \to \omega_2$ a branch in T iff $f \upharpoonright n \in T$ for $n < \omega$. If H is $\mathbb{N}$-generic over V, then $h = \bigcup \bigcap H$ is a cofinal map of ω to ω_2. Moreover, $H = \{T \in \mathbb{N} \mid h$ is a branch in $T\}$. We then say that h is a Namba-generic sequence.

Let $\mathbb{P} = \mathbb{P}$ be as in Example 1 with $\beta = \omega_2$. (Thus we assume $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_2$.)

Lemma 1 Let G be $\mathbb{P}$-generic. Let $h \in V[G]$ s.t. $h : \omega \to \omega_2^V$ cofinally. Then h is Namba-generic.

proof of Lemma 1

Let $\overset{\circ}{h}^G = h$. Assume w.l.o.g. that $\Vdash \overset{\circ}{h} : \check{\omega} \to \check{\omega}_2$ cofinally.

<u>Claim 1</u> $\Delta_{\overset{\circ}{h}} = \{ p \mid |p| = \omega_1^{M^p_{|p|}} \wedge$

$\wedge \bigvee \bar{h} \subset M^p_{|p|} \; p \Vdash \overset{\circ}{h} = \overset{\circ}{\pi}_{|p|, \omega_1} \circ \check{\bar{h}} \}$

is dense in $\mathbb{P}$.

proof.

Let $r \in \mathbb{P}$. We seek $p \leq r$ s.t. $p \in \Delta_{\overset{\circ}{h}}$. Let $G \ni r$ be $\mathbb{P}$-generic, $h = \overset{\circ}{h}^G$. Then there must be $\alpha < \omega_1$ s.t. $\mathrm{rng}(h) \subset \mathrm{rng}(\pi^G_{\alpha \omega_1})$. Hence $h = \pi^G_{\alpha, \omega_1} \circ \bar{h}$ for an $\bar{h} \subset M^G_\alpha$. But then there must be a $p \in G$ s.t. $|p| \geq \alpha$, $|p| = \omega_1^{M^p_{|p|}}$, and $p \Vdash \overset{\circ}{h} = \overset{\circ}{\pi}_{\check{\alpha} \check{\omega}_1} \circ \check{\bar{h}}$. We may the assume $|p| = \alpha$ (otherwise replace $\bar{h}$ by $\pi^p_{\alpha, |p|} \circ \bar{h}$), QED (Claim 1)

<u>Def</u> Let $p \in \Delta_{\overset{\circ}{h}}$. Set

$T^p =$ the set of $s \in \omega_2^{<\omega}$ s.t. $[\![\varphi_s]\!] \neq \emptyset$ in $BA(\mathbb{P})$, where $\varphi_s =$

$(\check{p} \in \overset{\circ}{G} \wedge \bigwedge_{i < |s|} \overset{\circ}{h}(\check{i}) = \check{s}_i)$. ($|s| =_{\mathrm{df}}$ length(s))

<u>Claim 2</u> $T^p \in IN$ for $p \in \Delta_h^\circ$.

prf. Suppose not. Let $T = T^p$.
Then there is $s \in T$ s.t. $\overline{\overline{T_{(s)}}} < \omega_2$,
where $T_{(s)} = \{t \in T \mid s \leq_T t \vee t \leq_T s\}$.
Hence $\{t(i) \mid t \in T_{(s)} \wedge i < |t|\} \subset \gamma < \omega_2$
for some γ. Let G be $\mathbb{P}$-generic s.t. $G \cap [\![\varphi_s]\!] \neq \emptyset$. Then
$p \in G$ and $\mathring{h}^G(i) = s_i$ for $i < |s|$.
But $\sup \mathring{h}''\omega_1 = \omega_2^V$; hence there is
j s.t. $\mathring{h}^G(j) \geq \gamma$. But then
$V[G] \models \varphi_t$, where $t = \mathring{h}^G \upharpoonright (j+1)$,
Hence $t \in T_{(s)}$ and $t(j) \geq \gamma$.
Contr! QED (Claim 2)

In the following let $p \in \Delta_h^\circ$ and
let $\bar{M} = M_\alpha^p$, $\alpha = |p|$. Let
$p \Vdash \mathring{\pi}_{\alpha\omega_1}^{\vee} \circ \check{\bar{h}} = \mathring{h}$.

Def For $r \in M$ set $M^{(r)} = L_r^A$, where
$M = L_\beta^A$. Similarly, for $r \in \bar{M} = M_\alpha^p =$
$= L_{\beta_\alpha}^{A_\alpha}$ we set $\bar{M}^{(r)} = L_r^{A_\alpha}$.

<u>Claim 3</u> Let $\bar{\nu} = \bar{h}(i)$ and let $\nu = s(i)$ for an $s \in T = T^p$. Set:

f = the M-least $f : \omega_1 \xrightarrow{\text{onto}} M^{(\nu)}$

$\bar{f}$ = " $\bar{M}$- " $f : \omega_1 \xrightarrow{\text{onto}} \bar{M}^{(\bar{\nu})}$.

Set $\pi = \pi^{(\bar{\nu}, \nu)} = \{\langle \bar{f}(\xi), f(\xi) \rangle \mid \xi < \alpha\}$.

Let $\langle a, \bar{a} \rangle \in F^p$. Then

$\pi : \langle \bar{M}^{(\bar{\nu})}, \bar{a} \cap \bar{M}^{(\bar{\nu})} \rangle \prec \langle M^{(\nu)}, a \cap M^{(\nu)} \rangle$.

proof.

Let G be $\mathbb{P}$-generic with $[\![\varphi_s]\!] \cap G \neq \emptyset$.

Then $\pi^G_{\alpha\omega_1}(\bar{f}) = f$. Hence

$\pi^G_{\alpha\omega_1}(\bar{f}(\xi)) = f(\xi)$ for $\xi < \alpha$ + hence

$\pi = \pi^G_{\alpha\omega_1} \upharpoonright \bar{M}^{(\bar{\nu})}$, where

$\pi^G_{\alpha\omega_1}(\langle \bar{M}^{(\bar{\nu})}, \bar{a} \cap \bar{M}^{(\bar{\nu})} \rangle) = \langle M^{(\nu)}, a \cap M^{(\nu)} \rangle$,

since $\pi^G_{\alpha\omega_1}(\bar{\nu}) = \nu$ and

$\pi^G_{\alpha\omega_1} : \langle \bar{M}, \bar{a} \rangle \prec \langle M, a \rangle$. QED (Claim 3)

<u>Claim 4</u> Let $T \leq T^p$ in N. There is $q \leq p$ s.t. $q \Vdash \mathring{h}$ is a branch in $\check{T}$.

proof of Claim 4.

Let $N^* = \langle H_\delta, M, <, p, T, \mathbb{P}, \mathbb{N}, \ldots \rangle$ where $\delta > \beta^+$. Let $p' \leq p$ conform to N^*. Let $\bar{N}^* = \bar{N}^*(p'; N^*) = \langle \tilde{N}, \tilde{M}, <, \tilde{p}, \tilde{T}, \tilde{\mathbb{P}}, \tilde{\mathbb{N}}, \ldots \rangle$. Let $\bar{H}$ be $\tilde{\mathbb{N}}$-generic over $\bar{N}^*$ s.t. $\tilde{T} \in \bar{H}$. Set $\tilde{h} = \bigcup \bar{H}$. Then $\tilde{h} : \omega \to \tilde{\beta}$ cofinally, where $\tilde{\beta} = On \cap \tilde{M}$. (Note that $\tilde{M} = M^p_{|p|}$.) Obviously Claim 3 holds relativized to $\bar{N}^*$. Thus, for $\bar{\nu} = \bar{h}(i)$, $\tilde{\nu} = \tilde{h}(i)$ we can define $\pi^{(\bar{\nu}, \nu)}$ as in Claim 3. It is easily seen that if $\bar{h}(i) < \bar{h}(j)$, then

$$\pi^{(\bar{h}(i), \tilde{h}(i))} \subset \pi^{(\bar{h}(j), \tilde{h}(j))}.$$

Thus, setting $\pi = \bigcup_{i<\omega} \pi^{(\bar{h}(i), \tilde{h}(i))}$,

we have:

$$\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \tilde{M}, \tilde{a} \rangle \text{ cofinally}$$

for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\tilde{p}}$. Since $\langle \tilde{M}, \tilde{a} \rangle$ is a ZFC^- model, we conclude:

(1) $\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \tilde{M}, \tilde{a} \rangle$ for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\hat{p}}$.

Moreover:

(2) $\mathrm{rng}(\pi) =$ the smallest $X \prec \tilde{M}$ s.t. $\mathrm{rng}(\tilde{h}) \cup \alpha \subset X$ $(\alpha =_{\bar{p}_f} |p|)$

proof of (2)

($\supset$) is trivial. But if $\alpha \cup rng(\tilde{h}) \subset X \prec \tilde{M}$, then $f_{\tilde{h}(i)} \in X$ for all $i < \omega$. Hence

$$rng(\pi^{(\tilde{h}(i), \tilde{h}(i))}) = f_{\tilde{h}(i)}{}'' \alpha \subset X.$$

QED (2)

Now let $\tilde{\alpha} = |p'|$ (hence $\tilde{\alpha} = \omega_1^{\bar{N}^*}$). Since $\bar{H}$ is $\bar{\mathbb{N}}^*$-generic over $\bar{N}^*$, $\bar{N}^*[\tilde{h}]$ is a ZFC$^-$ model. In $\bar{N}^*[\tilde{h}]$ we define $\langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle, \langle \tilde{\pi}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$ as follows:

For $\xi < \tilde{\alpha}$ let X_ξ = the smallest $X \prec \tilde{M}$ s.t. $\xi \cup rng(\tilde{h}) \subset X$. Set:

$C = \{\xi \geq \alpha \mid \xi = (X_\xi \cap \tilde{\alpha})\}$. Then $\tilde{\alpha} \in C$ and $C \cap \tilde{\alpha}$ is club in $\tilde{\alpha}$. Set:

$$\tilde{C} = C \cup \{\omega_1^{M_i^p} \mid i \leq \alpha\} \quad (\alpha = |p|).$$

For $\alpha \leq i \leq \tilde{\alpha}$ set: $\tilde{\pi}_i : \tilde{M}_i \stackrel{\sim}{\longleftrightarrow} X_{\tilde{\alpha}_i}$, where $\tilde{M}_i$ is transitive. For $i \leq \alpha$ set: $\tilde{M}_i = M_i^p$, $\tilde{\tilde{\pi}}_i = \tilde{\pi}_\alpha \pi_{i\alpha}^p$. We then set:

$$\tilde{\tilde{\pi}}_{ij} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i \text{ for } i \leq j \leq \tilde{\alpha}.$$

Define q by: $M^q = \langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle$, $\pi^q = \langle \tilde{\tilde{\pi}}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$, $F^q = F^{p'}$.

But then:

(3) $q \in \mathbb{P}$

proof.

Let $\mathfrak{A}$ model $\mathcal{L}(p')$. Change $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$ by replacing $\dot{M}_i^{\mathfrak{A}}, \dot{\pi}_{ij}^{\mathfrak{A}}$ by $\tilde{M}_i, \tilde{\pi}_{ij}$ for $i \leq j \leq \tilde{\alpha}$ — i.e

$$\dot{M}_i^{\tilde{\mathfrak{A}}} = \begin{cases} \dot{M}_i^{\mathfrak{A}} & \text{for } i \geq \tilde{\alpha} \\ \tilde{M}_i & \text{for } i \leq \tilde{\alpha} \end{cases}$$

$$\dot{\pi}_{ij}^{\tilde{\mathfrak{A}}} = \begin{cases} \dot{\pi}_{ij}^{\mathfrak{A}} & \text{for } \tilde{\alpha} \leq i \leq j \\ \dot{\pi}_{\tilde{\alpha} j}^{\mathfrak{A}} \circ \tilde{\pi}_{i\tilde{\alpha}} & \text{for } i \leq \tilde{\alpha} \leq j \\ \tilde{\pi}_{ij} & \text{for } i \leq j \leq \tilde{\alpha} \end{cases}$$

Then $\tilde{\mathfrak{A}}$ models $\mathcal{L}(q)$. QED (3)

But then:

(4) $\leq$.

prf.

$M^p = M^q \upharpoonright (|p|+1)$, $\pi^p = \pi^q \upharpoonright (|p|+1)^2$ by the construction of q. But if $\langle a, \bar{a} \rangle \in F^p$, then, since $p \leq p'$, there is a' s.t. $\langle a, a' \rangle \in F^{p'}$ and

$\pi^{p'}_{|p|,|p'|} : \langle M^p_{|p|}, \bar{a} \rangle \prec \langle M^{p'}_{|p'|}, a' \rangle$.

But then, whenever $G \ni p'$ is $\mathbb{P}$-generic, we have:

$\pi^G_{|p'||p|} : \langle M^{p'}_{|p'|}, a' \rangle \prec \langle M, a \rangle$.

Since $\pi^G_{|p'|,|p|}$ extends uniquely to $\sigma : \bar{N}^* \prec N^*$ with $\pi^{G}_{|p'|,|p|} \cup F^{p'} \subset \sigma$, it follows that $F^{p'}(a') = \sigma(a') = a$, hence $a' = \tilde{a} = \sigma^{-1}(a)$. Thus we have $\pi^{q}_{|p|,|q|} : \langle M^{p}_{|p|}, \bar{a} \rangle \prec \langle M^{p'}_{|p'|}, \tilde{a} \rangle$ by (1), where $\langle a, \tilde{a} \rangle \in F^{q}$.

QED(4)

It remains only to show:

(5) $q \Vdash \overset{\circ}{h}$ is a branch in $\check{T}'$.

Let $G \ni q$ be $\mathbb{P}$-generic. Let $h = \overset{\circ}{h}^{G}$. Then $\pi^{G}_{\tilde{\alpha}\omega_1}$ extends to a unique $\sigma : \bar{N}^* \prec N^*$ s.t. $F^{q} \subset \sigma$. Since $p \in G$, we know that $h = \pi^{G}_{\alpha\omega_1} \circ \bar{h}$. But $\tilde{h} = \pi^{G}_{\alpha\tilde{\alpha}} \circ \bar{h}$. Hence $h = \pi^{G}_{\tilde{\alpha}\omega_1} \circ \tilde{h}$. Since $\tilde{h}$ is a branch in $\tilde{T}'$, h must be a branch in $T' = \sigma(\tilde{T}')$.

QED (Claim 4)

We now prove Lemma 1. Let G be $\mathbb{P}$-generic, $h = \overset{\circ}{h}^{G}$. We must show that

$$H = \{T \mid h \text{ is a branch in } T\}$$

is $\mathbb{N}$-generic.

Let Δ be dense in $\mathbb{N}$. It suffices to show that Δ^* is dense in $\mathbb{P}$, where:
$\Delta^* = \{p \mid \forall T \in \Delta \; p \Vdash \mathring{h}$ is a branch in $\check{T}\}$
is dense in $\mathbb{P}$. Let $r \in \mathbb{P}$. By Claim 1 there is $p \leq r$ s.t. $p \in \Delta^\circ_h$. Let $T \leq T_p$ s.t. $T \in \Delta$. By Claim 4 there is $p' \leq p$ s.t. $p' \Vdash \mathring{h}$ is a branch in $\check{T}$.

QED (Lemma 1)

Let $\mathbb{A} = BA(\mathbb{N})$, $\mathbb{B} = BA(\mathbb{P})$. We wish to show that $\mathbb{A} \simeq \mathbb{B}$. For $a \in \mathbb{A}$ let $\mathbb{A}|a$ be the restriction of $\mathbb{A}$ to $\{a' \mid a' \subset a\}$. Then $\mathbb{A}|a$ is a complete BA and $1\!\!1_{\mathbb{A}|a} = a$. We also have: $\mathbb{A}|a = BA(\mathbb{P} \cap a)$. Similarly for $\mathbb{B}$. As a corollary of Claim 4 in the foregoing proof we have:

<u>Lemma 2.1</u> Let $p \in \Delta^\circ_h$. Then
$\mathbb{B} | [p]_{\mathbb{P}} = \mathbb{A} | [T_p]_{\mathbb{N}}$.
(Here $[p]_{\mathbb{P}}$ = the smallest $b \in \mathbb{B}$ s.t. $p \in b$. Similarly for $[T]_{\mathbb{N}}$.)

pf. of Lemma 2.1

Let $G \ni p$ be $\mathbb{P}$-generic. Set:
$H_G = \{T \in \mathbb{N} \mid \mathring{h}^G$ is a branch in $T\}$. Then H_G is $\mathbb{N}$-generic by Claim 4. But $\mathring{h}^G = \bigcup \bigcap H_G \in V[H_G]$ and $\langle M_i^G \mid i \leq \omega_1 \rangle$, $\langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle$ are uniformly $V[H_G]$-definable in $M, p, \mathring{h}^G$. But then G is uniformly $V[H_G]$-definable in M, p, H_G, since

$$q \in G \longleftrightarrow (M^q = M^G \upharpoonright (|q|+1) \wedge \pi^q = \pi^G \upharpoonright (|q|+1)^2 \wedge$$
$$\wedge \bigwedge \langle a, \bar{a} \rangle \in F^q (\pi^G_{|q|,\omega_1} : \langle M^G_{|q|}, \bar{a} \rangle \prec \langle M, a \rangle).$$

Hence there is a canonical $\ddot{G} \in V^{\mathbb{N}}$ s.t. $\ddot{G}^{H_G} = G$, whenever $G \ni p$ is $\mathbb{P}$-generic. For $a \in \mathbb{B} | [p]_{\mathbb{P}}$ set

$$\sigma(a) = \sigma([\![\check{a} \cap \mathring{G} \neq \emptyset]\!]_{\mathbb{P}}) = [\![\check{a} \cap \ddot{G} \neq \emptyset]\!]_{\mathbb{N}}$$

σ is easily seen to be a homomorphism of $\mathbb{B} | [p]$ into $\mathbb{A} | [T_p]$. But σ is injective, since if $\sigma(a) = \emptyset$, then $a \cap G = \emptyset$ for all $\mathbb{P}$-generic $G \ni p$. Hence $a = \emptyset$. It remains only to show that σ is onto. It is enough to show that $[T] \in \mathrm{rng}(\sigma)$ for each $T \leq T_p$ in $\mathbb{N}$. Let $a = [\![\check{T} \in \mathring{H}_G]\!]_{\mathbb{P}}$.

We claim: $\sigma(a) = [T]$, or in other words:

$\sigma(a) \cap H \neq \emptyset \leftrightarrow T \in H$ for $\mathbb{N}$-generic T with $T_p \in H$

If not there is $T' \leq T_p$ which forces the negation of this equivalence.
Let $G \ni p$ be $\mathbb{P}$-generic s.t. $T' \in H_G$.
Then $G = \tilde{G}^H$, where $H = H_G$. Hence
$\sigma(a) \cap H \neq \emptyset \leftrightarrow a \cap \tilde{G}^H = a \cap G \neq \emptyset \leftrightarrow T \in H$,
Contr! QED (Lemma 2.1)

Using this we prove:

<u>Lemma 2.2</u> $\mathbb{A} \simeq \mathbb{B}$ ($\mathbb{A} = BA(\mathbb{N})$, $\mathbb{B} = BA(\mathbb{P})$)

prf.
We first note some facts about $\mathbb{N}$. We recall that the strict Namba trees are dense in $\mathbb{N}$.

(1) Let $T \in \mathbb{N}$ be strict. Then

$$\mathbb{A} | [T] \simeq \mathbb{A}$$

proof.
$\mathbb{A} | [T] \simeq BA(\{T' | T' \leq T\})$. But forcing with subtrees of T is the same as forcing with subtrees of the set S of split points in T.
But $S \simeq 2^{<\omega_1}$. QED (1)

(2) Let $a \in \mathbb{A} \setminus \{\emptyset\}$. Then $\mathbb{A} | a \simeq \mathbb{A}$

prf. of (2)

$\mathbb{A}|a = BA(\{T | T \in a\})$. But $\{T | T \in a\}$ then collapses 2^{ω_2}, hence cannot satisfy the 2^{ω_2}-chain condition. Thus there is an max antichain $\langle T_\nu | \nu < 2^{\omega_2} \rangle$ in $\{T | T \in a\}$. We may w.l.o.g assume that each T_ν is strict. Similarly there is such a maximal antichain $\langle T'_\nu | \nu < 2^{\omega_2} \rangle$ in $\mathbb{N}$. Let $\sigma_i : \mathbb{A}|[T_\nu] \overset{\sim}{\longleftrightarrow} \mathbb{A}|[T'_\nu]$. Then $a = \bigcup_\nu [T_\nu]$, $1\!\!1 = \bigcup_\nu [T'_\nu]$ and we can define $\sigma : \mathbb{A}|a \overset{\sim}{\longleftrightarrow} \mathbb{A}$ by $\sigma(b) = \bigcup_{\nu < 2^{\omega_2}} \sigma_\nu(b \cap [T_\nu])$. QED(2)

Since Δ_h° is dense in $\mathbb{P}$, there is a max. antichain $\langle p_\nu | \nu < 2^{\omega_2} \rangle$ in $\mathbb{P}$ s.t. $p_\nu \in \Delta_h^\circ$ for all ν. Hence we may pick $\sigma_\nu : \mathbb{B}|[p_\nu] \overset{\sim}{\longleftrightarrow} \mathbb{A}|[T'_{p_\nu}]$ by Lemma 2.1. We then define $\sigma : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{A}$ by $\sigma(b) = \bigcup_\nu \sigma_\nu(b \cap [p_\nu])$.

QED (Lemma 2.2)

We now show that, even if $\beta > \omega_2$, the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of Example 1 is equivalent to a variant of Namba forcing. We define:

<u>Def</u> Let $\beta > \omega_1$ be regular. By a <u>Namba amoeba</u> on β we mean a subtree T of $\mathcal{P}_{\omega_2}(\beta)^{<\omega}$ s.t. if $s \in T$, then

(a) $s(i) \subset s(j)$ for $i \leq j < |s|$

(b) If $u \in \mathcal{P}_{\omega_2}(\beta)$, then

$$\{t \mid s \leq_T t \wedge \bigvee i \; u \subset t(i)\} \neq \emptyset.$$

From now on we let $\mathbb{N}$ be the set of Namba amoebas ordered by:

$$T \leq T' \longleftrightarrow T \subset T' \quad \text{for } T, T' \in \mathbb{N}.$$

We develop the main properties of Namba amoebas with a view to proving Lemma 3 below.

<u>Def</u> In any forcing extension of V we call h a <u>neat sequence</u> iff $h : \omega \longrightarrow \mathcal{P}_{\omega_2}(\beta)^V$, $h(i) \subset h(j)$ for $i \leq j < \omega$, and for every $u \in \mathcal{P}_{\omega_2}(\beta)^V$ there is i s.t. $u \subset h(i)$.

It is easily seen that if G is $\mathbb{N}$-generic and $h = \bigcup \bigcap G$, then h is a neat sequence. We shall also show that $\mathbb{N}$ does not add new reals.

We shall prove:

Lemma 3 Let G be $\mathbb{P}$-generic over V.

(a) $V[G] = V[h]$, where h is a neat sequence.

(b) If $h \in V[G]$ is a neat sequence, then h is $\mathbb{N}$-generic over V and $V[G] = V[h]$

Note We do not know whether $BA(\mathbb{P}) \simeq BA(\mathbb{N})$.

Def Let $T \in \mathbb{N}$. $s \in T$ is a big split point in T iff $\operatorname{card}(\{u \mid s^\frown\langle u\rangle \in T\}) \geq \beta$.

(1) Let $s \in T$. Then there is a big split point $t \geq s$ in T.

prf. Suppose not.

By ind. on n there are fewer than β $t \in T$ s.t. $s \leq_T t$ and $|t| \leq n$. Hence $\overline{\overline{T_{(s)}}} < \beta$, which contradicts (b) in the definition of Namba amoeba.

(Here $T_{(s)} = T \restriction \{t \mid s \leq_T t \vee t \leq_T s\}$)

As in the case of ordinary Namba trees we get an amalgamation lemma for Namba amoebas:

<u>Def</u> By an <u>amalgamation</u> <u>sequence</u> we mean a sequence $\langle\langle T_u, s_u\rangle \mid u \in \beta^{<\omega}\rangle$ s.t.

(a) $T_u \in \mathbb{N}$ and $s_u \in T_u$ is a big split pt. in T_u s.t. $T_{u^\frown\langle i\rangle} \subset T_{u(s_u)}$

(b) $s_u \subsetneqq s_v$ if $u \subsetneqq v$

(c) There is a 1-1 enumeration $\langle s_u^i \mid i<\beta\rangle$ of the immediate successors of s_u s.t. $s_u^i \leq_{T_u} s_{u^\frown\langle i\rangle}$ for $i<\beta$.

(d) If $v \in \mathcal{P}_{\omega_2}(\beta)$, then $\forall i \, \forall n \; v \subset s_{u^\frown\langle i\rangle}^{(n)}$.

(2) Let $\langle\langle T_u, s_u\rangle \mid u \in \beta^{<\omega}\rangle$ be an amalgamation sequence. Then

$$\bigcap_{n<\omega} \bigcup_{|u|=n} T_u = \bigcup_{h:\omega\to\beta} \bigcap_{n<\omega} T_{h\restriction n}$$

is a Namba amoeba.

<u>Note</u> If such a sequence is defined for $|u|<n$, it can be extended to $|u|\leq n$,

<u>Note</u> If $T^* = \bigcap_{n<\omega} \bigcup_{|u|=n} T_u$, then the s_u $(u\in\beta^{<\omega})$ are exactly the split points of T^*. Hence every split pt. of T^* is a big split pt.

Using this we get:

(3) Let G be $\mathbb{N}$-generic over V. Then $P(\omega)$ is absolute in $V[G]$.

prf.

Let $\Vdash \mathring{f} : \check{\omega} \to \check{2}$. It suffices to show:

<u>Claim</u> $\Delta = \{T \mid \exists f\ T \Vdash \mathring{f} = \check{f}\}$ is dense in $\mathbb{N}$

Let $T \in \mathbb{N}$. We first construct an amalgamation sequence $\langle T_u, s_u \rangle$ $(u \in \beta^{<\omega})$ s.t. $T_u \leq T$, $T_u \Vdash \mathring{f}(|\check{u}|) = \check{m}$ for some m. (We construct $\langle\langle T_u, s_u \rangle \mid u \in \beta^n \rangle$ by induction on n.) Let $T^* = \bigcap_n \bigcup_{|u|=} T_u$.

Then $T^*_{(s_u)} \Vdash \mathring{f}(|\check{u}|) = \check{m}$ for some $m < 2$ for all $u \in \beta^{<\omega}$. For each $f : \omega \to 2$ define a game $\mathbb{G}_f$ by:

In the i-th step I chooses $v_i \in P_{\omega_2}(\beta)$ s.t. $v_i \supset v_h$ for all $h < i$.

II then chooses $\xi_i < \beta$ s.t. $v_i \subset s_{\xi_0 \cdots \xi_i}(h)$ for an $h < |s_{\xi_0 \cdots \xi_i}|$.

II wins iff $T^*_{(s_{\xi_0 \cdots \xi_{i-1}})} \Vdash \mathring{f}(\check{i}) = \check{f(i)}$

for all $i < \omega$.

Clearly, I can only win at a finite stage. Hence one player has a winning strategy.