Appendix to §5

In Lemma 1.2 it appeared to be essential that $\beta > \omega_1$ be regular. We now show that we can modify this condition, letting $\beta > \omega_1$ be any cardinal. In particular, if $\text{cf} (\beta) = \omega_1$, we shall construct a separable IP which gives every regular $\tau \in (\omega_1, \beta)$ cofinality $\omega$ while preserving $\beta^+$. Hence forcing with this IP is certainly different from collapsing each regular $\tau \in (\omega_1, \beta)$ to $\omega_2$ and then applying Namba forcing, since $\beta^+$ would be $\leq$ The $\omega_3$ of the first extension $+$ would therefore certainly be collapsed by the second (by §4 Lemma 4.1). We shall also see that this IP is subcomplete in the sense of our later paper "Subproper and subcomplete forcing".

Note The question whether it is possible — for some $\beta$ with $\text{cf} (\beta) = \omega_1$ — to make every regular $\tau \in (\omega_1, \beta)$ $\omega$-cofinal while preserving $\omega_1$ and $\beta^+$ was first formulated by Moti Gitik.
Gitik also answered the question positively for a $\beta$ in a special model which he obtained by collapsing to make a supercompact cardinal become $\omega_1$.

The more general result which we present here is implicit in §5, but we failed to notice it until we saw Gitik's work. (Gitik also used his special model to answer positively.

The harder question: Is it possible for some strongly inaccessible $\beta$ to make every regular $\tau \in (\omega_1, \beta)$ $\omega$-cofinal while preserving $\omega_1$ and $\beta^+$? A more general form of that result is as yet unknown.)
We first note that Facts 1-5 hold in an appropriate reformulation for a wider class of structures than the transitive $\mathcal{ZFC}$-models. Call $M$ smooth if $M$ is transitive and is either a model of $\mathcal{ZFC}$- or a model of Zermelo set theory not $M = \bigcup \{ H^M_{\alpha} \mid \alpha \text{ is regular in } M \}$. As before, we call a map $\sigma : M \to M'$ cofinal if $M' = \bigcup \{ \sigma(u) \mid u \in M \}$. At first glance, we see that if $M$ is smooth and $\sigma : M \to M'$ is a cofinal $\mathcal{E}_0$-preserving map, then $M'$ is smooth. Conversely, if $M'$ is smooth, then $\sigma \in M$. Write $\sigma : M \to^* M'$ to mean that $\sigma$ is an $\mathcal{E}_0$-preserving map.

We then have:

Fact $1^\prime$. Let $\tilde{\pi} : M \to \tilde{M}'$ where $M$ is smooth.

Set $\tilde{M} = M' \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \in \mathcal{E}_0$, $M'$ and $\tilde{M} \to^* \tilde{M}'$ cofinally.

If $\tau > \omega$ is regular in $M$, $M$ is smooth, and $\tau : M \to \tilde{M}'$, we define the notion: "$\tau$ is $\tau$-cofinal" exactly as before.
Fact 2'. Let \( z > w \) be regular in \( M \), where \( M \) is smooth. Let \( \tilde{\pi} : M \to M' \). Set:
\[
\tilde{H} = H_z^M, \quad \tilde{H} = \bigcup_{u \in H} \tilde{\pi}(u), \quad \tilde{\pi} = \tilde{\pi} \circ \tilde{H}.
\]
Then \( \tilde{\pi} : \tilde{H} \times \tilde{H} \). Now let \( z > w \) be regular in \( M \) where \( M \) is smooth. Let \( \tilde{H} = H_z^M + \tilde{\pi} : \tilde{H} \times \tilde{H} \). cofinally. By a
\underline{lifitup} of \( \langle M, \tilde{\pi} \rangle \) we mean a pair \( \langle M', \tilde{\pi}' \rangle \)
\( \tilde{\pi} \) is transitive, \( \pi \tilde{H} = \tilde{\pi} \), and \( \tilde{\pi} : M \to M \). cofinally. Then \( \tilde{\pi} \) is
\( \tilde{\pi} \) smooth.) Exactly as before!

Fact 3'. Let \( \langle M, \tilde{\pi} \rangle \) be as above. Then there is
at most one lifitup.

Fact 4'. The lifitup of \( \langle M, \tilde{\pi} \rangle \) exists if \( E \)
is well founded, where \( E \subset \tilde{\pi} \). It exists by \( i \)
\( \langle x, f \rangle E \langle x, g \rangle \iff \langle x, f \rangle E \langle x, g \rangle \implies \langle x, f \rangle E \langle x, g \rangle \}
Hence the interpolation lemma:

Fact 5'. Let \( \tilde{\pi} : M \to M' \). Let \( z > w \) be regular
in \( M \), where \( M \) is smooth. Let \( \tilde{H} = H_z^M, \tilde{\pi} = \tilde{\pi}_H, \)
\( \tilde{H} = \bigcup_{u \in H} \tilde{\pi}(u) \). Then the lifitup

of \( \langle M, \tilde{\pi} \rangle \) exists. Moreover, there is a unique
\( \sigma : \tilde{M} \to M' \). i.t. \( \sigma \tilde{\pi} = \tilde{\pi} \) and \( \sigma \tilde{H} = \tilde{id} \).
Note that if \( \beta > \omega \) and \( 2^\beta = \beta \), then \( N = \langle H_\beta, \mathcal{E}, \in \rangle \) is always smooth, since either \( \beta \) is regular, in which case \( N \) is a 
\( \mathbb{ZFC} \)-model, or else \( 2^\delta < \beta \) for \( \delta < \beta \) 
(since otherwise, letting \( \varepsilon < \beta \) be 
regular s.t. \( cf(\delta) \leq \varepsilon \), we have \( 2^\varepsilon = \beta \) + hence \( cf(2^{\varepsilon}) \leq \varepsilon \). Contradiction.).
Let $2^\beta = \beta$, $\beta > \omega_1$. Set $M = L^\beta = \langle L^\beta[A], A \rangle$, where $H^\beta = L^\beta[A]$. Set $N = \langle H^\beta+, M, <, \ldots \rangle$ where, as before, $<$ is a well ordering of $N$. Let $L$ be the language on $N$ containing the basic axioms of §3 together with:

- $\beta = \emptyset$
- Let $i < \omega_1$, $\beta_i = \sup \{ \omega_1 \beta \mid i \leq \beta \}$, and $M_i = L^\beta_i$. Then $\langle M, M_i, \omega_1 \rangle$ is the lift up of $M_i$ by $\pi_i$, $\omega_1$, $\pi_i$, $\omega_2$.
- $i \in \omega$ if $i = cf(\beta) < \beta$.
- $\tilde{\beta}_0 = \beta$ if $\beta$ is regular.

By the last two clauses, $L \vdash \tilde{\beta}_0 = \beta$ whenever $cf(\beta) \neq \omega_1$. For $cf(\beta) = \omega_1$, however, we have $i < \omega_1$, $L \vdash \tilde{\beta}_0 < \beta$ for $i < \omega_1$, since otherwise $\omega_1$ would be collapsed in a model of $L$. For $\beta$ regular this is like the $L$ of §5 except that we have omitted the axioms which say that the entire tower

$\langle M_i \mid i \leq \omega_1 \rangle$, $\langle \pi_i \mid i < i \leq \omega_1 \rangle$ is determined by $\pi_0$, $\omega_1$. (In particular,
We first show:

Lemma 2 is consistent.

The proof is much like that of §5 Lemma 1.

Let $N^+ = \langle H_{(2^\aleph_1)^+}, N, <, \ldots \rangle$.

Let $\sigma : \bar{N}^+ < N^+$, where $\bar{N}^+$ is countable and transitive. Let $\bar{\sigma} : \bar{N}^+ < \bar{N}^+$ be the lift up of $N^+$ by $\sigma \uparrow H_{\bar{\omega}_2}$. Then there is $h : \bar{N}^+ < N^+$ s.t. $h \bar{\sigma} = \sigma$. At

an Hicer to show:

Claim $\tilde{L}$ is consistent, where $k(\tilde{L}) = L$.

We construct a solid model $M_L$ of $\tilde{L}$ as follows: For $x \leq \omega_1$ set:

$X_x =$ the smallest $X < \tilde{N}^+$ s.t.

$\sigma \uparrow \text{rng}(\sigma) \subseteq X$.

For $i \leq \omega_1$ set:

$a_i =$ the $i$-th $a$ s.t. $a = \omega_1 \cap X_{a_0}$.

(Hence $a_0 = \omega_1^{\bar{N}^+}, X_{a_0} = \text{rng} \bar{\sigma}$).

$a_{\omega_1} = \omega_1, X_{\omega_1} = \bar{N}^+$.

Set: $\tilde{\sigma}_i : \bar{N}^+_i \leftrightarrow X_{a_i}, \tilde{\sigma}_i^{-1} =$ the lift up of $\bar{N}^+_i$ by $\tilde{\sigma}_i$. Then $\tilde{\sigma}_i : \bar{N}^+_i < \bar{N}^+_i$ s.t.

for $i \leq j \leq \omega_1$. Then $\tilde{\sigma}_i$ has the

structure $\langle \text{rng} \tilde{\sigma}_i, < \rangle$.
Since the embedding in $w_2\tilde{M}^+$ is cofinal, set $M_i = \tilde{\omega}_{i\omega}^\omega(\tilde{M})$ where $k(\tilde{M}) = M$;

$\tilde{\pi}_{i+}^i = \tilde{\omega}_{i\omega}^\omega \cap M_i$.

Then $N_\beta = \langle H_{(2^\beta)^{++}}, \langle M, \omega_1 \leq \omega_\beta \rangle, \langle \tilde{\pi}_{i+}^i, 1 \leq i \leq \omega_\beta \rangle \rangle$

modeled in $\tilde{L}$, QED (Lemma 1).

Now let $IP = IP_{\tilde{L}}$.

**Lemma 2** IP is revivable

**proof.**

This follows mutatis mutandis by the proof of §5 Lemma 2, we omit the details.

**Lemma 3** Assume $cf(\beta) = \omega_1$. Let $G$ be $IP$-generic. Then $\beta^+ \in$ absolute in $V[G]$.

**proof.** By §4 Lemma 3.

**Note** An earlier version of "$L$-forcing" theorem proof in §4 Lemma 3, 1 contained an error. Having set $\tilde{\beta} = \sup \tilde{\omega}_x, \beta^p_x \ (p 11)$ and $\tilde{M} = \bigcup_{\beta^p_x}^{\beta^p}$, we claimed that $\tilde{\pi}_{\beta^p_x, \omega_1}^i : M^p_x \prec \tilde{M}$. This does not hold if $\beta$ is not regular.
However, we didn't use it in the proof. For purposes of the proof it suffices to note that if for such $a \in \mathbb{P}$ we let $\bar{a} = a \land \bar{\mu}$, then $\pi_{\bar{a}}^G : \langle M^a, \bar{a} \rangle \rightarrow \langle \bar{\mu}, \bar{a} \rangle$ is a cofinal $\Sigma_0$-preserving map, where $\langle a, \bar{a} \rangle \in \mathbb{F}$. Finally, we note that — mutatis mutandis — the proof of §3.6 Lemma 3 in "Subproper and subcomplete forcing" shows:

Lemma 4: $\mathbb{F}$ is subcomplete.

We have thus proven:

Thm 1 Let $2^\omega = \omega_1$, $2^\beta = \beta'$, where $\beta > \omega_1$. There is a subcomplete forcing $\mathbb{F}$, with $G \in \mathbb{F}$ generic, then in $V[G]$ we have:

(a) Let $\delta \in (\omega_1, \beta]$ be regular. Then there is a countable $X \leq H_\delta$ s.t. $H_\delta = \text{the smallest } Y \subseteq H_\delta \text{ with } \omega_1 \setminus X \subseteq Y$.

(b) If $\text{cf}(\beta) = \omega_1$, Then $\beta^+ = \omega_2$ in $V[G]$. Note (a) is equivalent to: There is a countable $H$ and a $\sigma : \bar{H} \prec H$ which in $\omega_2^H$ is cofinal.
Note. If \( \text{cf}(\beta) \neq \omega_1 \), it follows easily that (a1) holds for \( \gamma = \beta \). Thus \( \text{cf}(\beta) = \omega_1 \) in \( V[G] \) and it follows by §4 Lemma 4.1 that \( 2^\beta \) has cardinality \( \omega_1 \) in \( V[G] \). But then \( 2^\beta \) has cofinality \( \omega_1 \) in \( V[G] \), since otherwise \( (2^\beta)^+ \) would be collapsed in \( V[G] \) by §4 Lemma 4.1. Thus it is impossible, since \( \bar{\beta} < (2^\beta)^+ \).

For the case that \( \beta \) is strongly inaccessible, we can get another result:

Thm 2. Let \( 2^\omega = \omega_1 \). Let \( \beta \) be strongly inaccessible. There is a subcomplete forcing \( \bar{\beta} \) s.t. whenever \( G \in \beta \) for some complete \( \bar{\beta} \) generating, then in \( V[G] \) we have:

(a) Let \( \delta \in (\omega_1, \beta) \) be regular. There is a countable \( X \subseteq H_\delta \) s.t. \( H_\delta = \text{The smallest } Y \subseteq H_\delta \text{ with } \omega_1 \cup X \subseteq Y \).
(b) \( \text{cf}(\beta) = \omega_1 \) and \( \beta^+ = \omega_2 \) in \( V[G] \).
The proof is a repetition of the proof of ..., except that in place of \(L\) we use the language \(L'\) obtained by replacing the axiom \(\beta_0 = \beta\) by \(\beta_i < \beta\) for \(i < \omega_1\). We must first prove:

**Lemma 5** \(L'\) is consistent.

**Proof.**

Let \(M\) be a solid model of the previous language \(L\). Since \(L\) is strongly inaccessible, there is \(X \prec N = \langle H_{\beta+1}, M, <, \in \rangle\) s.t. \(V_{\beta} \subseteq X\) and \(\bar{X} = \beta\) for \(\alpha < \beta\)

s.t. \(\beta' = \bar{z} \bar{c'}\) and \(c_t(\beta'') = \omega_1\). Let \(\sigma: N' \rightarrow X\). Then \(\sigma: N' \prec N\). Let \(\sigma(L'') = L'\). At this point to show:

**Claim** \(L''\) is consistent.

Note that \(N' \models V_{\beta} \preceq \text{core}(\text{Ord})\); hence

\[M' = \sigma^{-1}(M) = L_{\beta'}^{A'} \in M'_{\beta'} \preceq M\]. Pick \(i_0 < \omega_1\)

s.t. \(M' \subseteq \text{rng}(\beta_{i_0}^{\omega_1})\). Set:

\[M_{i_0}' = (\pi_{i_0}^{\omega_1})^{-1}(M') (i_0 < \omega_1)\).
\[ \pi'_{i, j} = \pi_{i_0 + i, j_0 + j} \big| M'_i \ (i' \leq i \leq \omega_1), \]

Using the fact that of (β') = \omega_1 in \( V'_{\beta} \) (hence in \( V' \)), we easily get:

(1) \( \bar{\beta}'_i < \beta'_i \) for \( i < \omega_1 \), where \( \bar{\beta}'_i = \cup \bar{\beta}'_i \).

Note that \( M'_i = V'_{\bar{\beta}'_i} \) where \( M = M'_{\omega_1} \).

Since \( \pi'_{i_0 + i, \omega_1} : M'_{\omega_1} \rightarrow M \) is \( \pi'_i \) - cofinal, where \( \pi'_i = \omega'_{\pi'_i} = \omega_{\pi'_i + i} \), it follows that

\[ \pi'_{i, \omega_1} : M'_{\omega_1} \rightarrow \bar{M}'_i \quad \text{in} \quad \bar{\tau} - \text{cofinal}, \]

where \( \bar{M}'_i = L^A_{\bar{\beta}'_i} \). (Hence \( M'_i \) = \( V'_{\bar{\beta}'_i} \).)

Thus:

(2) \( \pi'_{i, \omega_1} : M'_i \rightarrow \bar{M}'_i \) is the lift up of \( M'_i \) by \( \pi'_{i, \omega_1} : M'_{\omega_1} \rightarrow \bar{M}'_i \).

Let \( \Omega = \langle A, \mathcal{E}_{\mathcal{O}}, M'_{\omega_1}, \pi'_{\omega_1} \rangle \) follow that \( \Omega' = \langle A, \mathcal{E}_{\mathcal{O}}, M'_i, \pi' \rangle \) models \( \mathcal{L}' \), where \( M' = \langle M'_i \ (i' \leq i \leq \omega_1) \rangle \), \( \pi' = \langle \pi'_{i', i} \ (i' \leq i \leq \omega_1) \rangle \). QED (Lemma 5)
We then prove the analogues of Lemmas 2, 3, and 4 for \( IP' = IP_{\mathcal{L}} \) exactly as before. (In particular, the analogue of Lemma 3 says that \( \beta^+ \) remains a cardinal in \( V[G] \).) An \( V[G] \) every cardinal \( \beta < \beta^+ \) is collapsed to \( \omega_1 \) and \( \beta = \omega_1 \). Hence \( cf(\beta) = \beta = \omega_1 \) in \( V[G] \). Hence \( \beta^+ = \omega_2 \).

\[ \text{QED (Thm 2)} \]

\text{Note} Let \( IP \) be the forcing of Thm 1. Let \( G \) be \( IP \)-generic. Let \( U \) be any countable set in \( V[G] \) which is \( \omega \)-closed in \( H^{V[G]} \), then \( V[U] \) in fact, accomplishes all that \( G \) was intended to accomplish - i.e. in \( V[U] \) we have:

Fix Skolem functions for \( H^{V[G]} \) and let \( X_d \) be the Skolem closure of a \( U \) for \( d \leq \omega_1 \). Set: \( \bar{E}_d : \mathcal{Q}_d \leftrightarrow X_d \).

Then for every \( \bar{z} \in \{ \omega_1, \beta \} \) there are \( d \leq \omega_1 \), \( \bar{H} \in H_{\omega_1} \) s.t. \( \mathcal{Q}_d = H^{\bar{H}} \) and \( \bar{E}_d \) lift to \( \bar{z} : \bar{H} \lessdot H \). Hence \( cf(\bar{z}) = \omega \) in \( V[U] \), since \( \bar{z} \in V[U] \),
We also note that if \( u \)' is also countable and cofinal in \( H_{\omega_2} \), then \( V[u] = V[u'] \).

To see (3) let \( \bar{\epsilon} \) be big enough that \( u' \in \text{rng}(\bar{\epsilon}) \). Then \( u' = \Pi_{\bar{\epsilon}} \bar{u} \) for a \( \bar{u} \in H_{\omega_1} \).

However, we have now defined \( \Pi \) in such a way that \( V[\bar{u}] \neq V[u] \) for any such \( u \). An otherwize, the canonical complete Boolean algebra \( \hat{B} = BA(\Pi) \) over \( \Pi \) is not identical to the sub-algebra \( \hat{B}' \) generated by \( \Pi \). We could rectify this in the case that \( \text{cf}(\beta) \neq \omega_1 \) by adding to \( \hat{B} \) the axiom:

\[
\text{rng}(\Pi_{\bar{\epsilon}+1}, \omega_1) = \text{the smallest } Y \subseteq M \text{ act. } \text{card. } Y \text{ such that } \text{rng}(\Pi_{\bar{\epsilon}}, \omega_1) \subseteq Y.
\]

(Thur is exactly what we did in \( \S 3 \).) The case \( \text{cf}(\beta) = \omega_1 \) is somewhat more complex. Suffice it at this place to state without proof that \( \hat{B} \) is, indeed,
Isomorphic to the complete BA over condition $\tilde{P} \equiv \tilde{P} \equiv$ which are reversible. (That $\tilde{H}$ is subcomplete follows straightforwardly by the fact that $\tilde{H} \subseteq \tilde{H}$ and $\text{card}(\tilde{H}) = \text{card}(\tilde{H})$, where $\tilde{H}$ is subcomplete.)

The same remarks apply mutatis mutandis to the forcing $\tilde{P}'$ of Thm 2. (In this case the fact that $\beta = \omega_2$ in $\mathcal{V}[u]$ is established as follows:

Let $\mathcal{Q} \in H_{\omega_1}$ be the set of $\langle \alpha, \tilde{H}, \eta \rangle$

s.t. $\alpha < \omega_1$, $\tilde{H} \models ZFC$, $\eta \in \tilde{H}$,

$Q_\alpha = \tilde{H}^{\tilde{H}}_{\omega_2} = \tilde{H}$ 

and $\tilde{H}$ lifts up to

$\tilde{P}' : \tilde{H} \preceq \tilde{H}$ for a regular $\gamma < \beta$. Set $\tilde{F}(\langle \alpha, \tilde{H}, \eta \rangle) = \tilde{P}'(\eta)$. Then $F \in \mathcal{V}[u]$ maps $\mathcal{Q}$ onto $\beta$. Since $\text{cf}(\beta) = \omega_1$ in $\mathcal{V}[u]$ of $\mathcal{V}[u]$, it follows that $\text{cf}(\beta) = \omega_1$ in $\mathcal{V}[u]$. )