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More on Iterability

In our earlier note [ANFS] (Addendum to "A New Fine Structure...") we showed that under appropriate assumptions various notions of iterability for 1-small premice coincide. Let $\Theta$ be a strongly inaccessible cardinal. We made the further assumptions:

A1 Either no $\tau < \Theta$ is Woodin in an inner model or else $V_\Theta$ is closed under $\#$.
A2 Let $M \subseteq V_\Theta$ be a 1-small premouse and $J$ a normal iteration of $M$ of length $\Theta$. Then $J$ has a cofinal branch.
A3 $\Theta$ is Mahlo.

From these assumptions we showed that normal iterability implies full iterability in $V_\Theta$. Moreover, weak iterability implies weak iterability. (A mouse is said to be weakly (normally) iterable iff whenever $\sigma: M \rightarrow \mathcal{E}^M$, $M$ and $\mathcal{M}$ is countable, then $\mathcal{M}$ is countably (normally) iterable.) We now show that these results follow from A1, A2 alone without A3. Our main
Lemma does not involve A1 or A2 but needs a certain weakening of A2:

(+) Every countably normally iterable 1-ema premouse is \( \omega_1 + 1 \) iterable.

(\text{Note} (+) will hold if \( \omega_1 \) is not Woodin in an inner model or if A# exists for all \( A \subseteq \mathcal{V} \).
Similarly \( A_2 \) holds if \( \Theta \) is not Woodin in an inner model or A# exists for \( A \subseteq \mathcal{V} \).)

The main lemma states that if \( \Theta \) is inaccessible, (+) holds, and Q6 VQ \( \Theta + 1 \) - normally iterable, then Q in weakly iterable. (This is proven for 1-small Q. The proof will also work for any premouse whose normal iterations always have unique cofinal branches.) In proving this we follow Steel's basic technique of constructing a "background array" of weakly iterable premice \( N_i, M_i \) (\( i \leq \xi < \Theta \)) and showing that Q embeds into some \( N_i \).

In this case, however, the \( N_i \) are obtained directly by normal iteration of Q. We construct a sequence of normal iterations \( T_i \) of Q s.t.,
\[ l_h(Y_i) = i+1 \]. The \( Y_i \) have a tree structure under the relation \( (i < j \iff Y_i = Y_j \setminus \{i+1\}) \). Letting \( Q_i \) be the ultimate model in \( Y_i \), we define \( N_i \) to be a "segment" of \( Q_i \) (i.e. either \( N_i = Q_i \setminus \beta \) or \( N_i = \langle \beta, E_{Q_i}^\beta \rangle \), where \( E_{Q_i}^\beta \neq \emptyset \)). The process terminates only when \( N_\beta = Q_\beta \) and \( Q_\beta \) is a simple iterate of \( Q \). We first show that the process does terminate below \( \Theta \). This gives us an iteration map from \( Q \) to \( N_\beta \). We then use Steel's argument to show that each \( N_i \) is weakly iterable. (However, in §1 we do not carry out the full iterability proof, but only sketch a special case which illustrates several of the main ideas. The full proof is given in §3, which is based straightforwardly on Steel's proof in [S] §9.)
An [ANFS] we considered a weaker notion of iterability ("MS-iterability") which seems to be implicit in the work of Mitchell and Steel. The difference is that MS-iterations do not allow unrestricted linear "cascading" of normal iterations. It followed that MS-iterability coincides with full iterability under the assumptions A1-A3 (similarly for "weakly MS-iterable" and "weakly iterable").

We now, of course, have the same result under the weakened assumption.

The Mitchell-Steel iterations differ from ours; however, in another essential respect: Because their extenders are indexed differently, the pattern in which they are applied in the course of a normal iteration differs from ours. At is possible, however, using our premise, to define an alternative notion of "normal iteration" in which the
Extenders are applied in the Mitchell-Steel fashion. At these iterations
\[ \mathcal{J} = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \gamma_i \rangle, \langle \tau_{i} \rangle, T \rangle, \]
we define \( \lambda_i = \sigma(\nu_i) \) to be the "natural length" of \( E_{\nu_i}^{M_i} \) in Steel's sense, and use \( \lambda_i \)
in place of \( \lambda_i = \rho h (E_{\nu_i}^{M_i}) \). At place of \( \nu_i \) we then have \( \nu_i^+ = \) the successor of \( \nu_i \) in the ultra power by \( E_{\nu_i}^{M_i} \).
We call such \( \mathcal{J} \) a normal \( \lambda \)-iteration. The notion of good \( \lambda \)-iteration is then defined as before.

It is possible to do \( \lambda \)-coiterations. (In fact the whole of [MS] could easily be recast in terms of \( \lambda \)-iterations of our premice.) In §2 we develop the basic theory of \( \lambda \)-iterations and then show, under the assumptions A1 and A2, that for 1-small premice in \( V_\Theta \) the notions "iterable", "normally \( \lambda \)-iterable" and "\( \lambda \)-iterable" coincide.
References

[MS] Mitchell, Steel Fine Structure and Iteration Theory (Lecture Notes in Logic)

[S] Steel The Core Model Aterability Problem (Lecture Notes in Logic)

[NFS] Jensen A New Fine Structure Theory for Higher Core Models (Handwritten Notes)

[ANFS] Jensen Addendum to NFS (Handwritten Notes)