§1.2 Abstract Projects

Let \( N = \langle J^N, \bigcup_i, B_i \rangle \) be acceptable.

For \( m \in \omega \) we define the \( m \)-th project

\[ \rho^m = \rho^m \text{ in such a way that} \]

\[ \rho^0 = \beta \text{ and } \rho^{m+1} \leq \rho^m \text{ and each} \]

\[ \rho^m \text{ is a cardinal in } N. \text{ For each} \]

\[ \rho \in \bigtimes_{i \leq m} H^N_{\omega_i \rho_i} \text{ we simultaneously}\]

define the \( m \)-th \underbar{reduct} \( N^{m+1}_{\rho} \text{ int} \)

\[ L_N^{m+1} = H^N_{\omega_i \rho_i}. \text{ For } m \geq 1 \text{ we have}\]

\[ N^{m+1} = \langle J^N, A^{m+1} \rangle \text{ and, we call}\]

\[ A^{m+1} \text{ the } m \text{-th standard code}. \]

\[ \text{Def } \rho^0 = \beta, \text{ i.e. } N^{0, \beta} = N; \]

\[ \rho^{m+1} = \min \left\{ \rho_{N^{m+1}} \mid \rho \in \bigtimes_{i \leq m} H^N_{\omega_i \rho_i} \right\}; \]

\[ N^{m+1} \rho = (N^{m+1} \rho^m)^{m+1} \rho^{m+1}. \]

\[ \text{Def } H_n^N = H^N_{\omega_n \rho_n}; \text{ i.e. } H_n^N = \bigtimes_{i \leq n} H^N_{\omega_i \rho_i}. \]

We define \( P^m \text{ and } \rho^m \text{ if } m \leq \omega \)

We define \( P^m \text{ and } \rho^m \text{ if } m \leq \omega \)

follow:
Def \( P^0 = \{ \emptyset \} \)

\[ P^{m+1} = \{ p \in P^{m+1} \mid p \in P^m \wedge \]
\[ \wedge \exists p^{m+1} \in P^{m+1} \wedge p^{m+1}(n) \in P^m \} \]

\( P^m \) then has the same def. with \( R \) in place of \( P \). Hence:

\[ p \in P^m \iff \exists \bar{p} \in \mathbb{R}^{n \times \bar{p}} \text{ for } \bar{i} \leq m. \]

Trivially: \( R^n \subseteq P^m \neq \emptyset \)

Def \( N \) is \( m \)-round iff \( R^m = P^m \)

\( N \) is \( \overline{m} \)-round iff \( N \) is \( m \)-round for \( m < \alpha \)

The downward ext. of imbedding lemma for iterated projects in the conjunction of the following three lemmata, which follow trivially from the corresponding lemmata for one projectum:

Lemma 1.1 Let \( \pi : \mathbb{R}^{n \bar{p}} \to \mathbb{N}^{m \bar{p}} \) \( (\bar{n} < \omega) \)

where \( \bar{p} \in \mathbb{R}^n \). There is a unique

\( \bar{\pi} \supseteq \pi \) s.t. \( \text{dom}(\bar{\pi}) = \mathbb{N}, \bar{\pi}(\bar{p}) = p \) and,

setting \( \bar{\pi}_i = \bar{\pi} \cap N^{\bar{i}} \),

\[ \bar{\pi}_i : \mathbb{N}^{i \bar{p}_i} \to \mathbb{E}_i, \mathbb{N}^{i \bar{p}_i} \text{ for } \bar{i} \leq m. \]

Moreover, for \( \bar{i} \leq m \) the map \( \bar{\pi}_i \) is \( \Sigma \) preserving.
Lemma 1.2 Let \( \pi : \tilde{N}^m \tilde{p} \to N^m \tilde{p} \) where \( n, l < \omega, \tilde{p} \in \tilde{R}_{\tilde{N}}, \tilde{p} \in R_{\tilde{N}} \). Let \( \tilde{\pi} \) be as above. Then
\[
\tilde{\pi}_i : \tilde{N}^i \tilde{p} \tilde{p}_i \to N^i \tilde{p} \tilde{p}_i \quad (i \leq m),
\]
\[
\sum m - i + l
\]

Lemma 1.3 Let \( \pi : M \to \tilde{N}^m \tilde{p} \) for \( m < \omega, \tilde{p} \in \tilde{R}_{\tilde{N}} \). There are unique \( \tilde{N}, \tilde{p} \) s.t. \( \tilde{p} \in R_{\tilde{N}} \) and \( M = \tilde{N}^m \tilde{p} \).

These lemmas naturally suggest a notion of "\( \Sigma^m_0 \) preserving imbedding" – i.e. imbeddings \( \pi : \tilde{N} \to \tilde{N} \) s.t. \( \pi \mid \tilde{N}^m \tilde{p} : \tilde{N}^m \tilde{p} \to N^m \tilde{p} \) whenever \( \tilde{p} \in \tilde{R}_{\tilde{N}} \) and \( \pi (\tilde{p}) = \tilde{p} \).

At turn out that these are the imbeddings which are elementary with respect to satisfaction of \( \Sigma^m_0 \) formulae, a notion which we now develop.
Let $L^*$ be a language with predicate $e_i = i, \leq, \geq$ and variable $v^c$ of type $i < \omega$

We interpret $L^*$ in $N$ in the obvious way, letting $v^c$ range over $\mathbb{N}$, $i < \omega$

(Thus larger indices mean smaller domains.)

By induction on $m < \omega$ define set $\Sigma_k^{(m)}$ of formulae ($k < \omega$):

$\Sigma_0^{(m)} = \text{the smallest set } \Sigma \text{ of formulae }$

such that:

(a) if $x^c \in y^i$, $x^c = y^i$, $B_k^c x^i$, $B_k' x^i$ are in $\Sigma$

(b) $\Sigma_1^{(m)} \subseteq \Sigma$ for $m < \omega$

(c) $\Sigma$ is closed under $\wedge, \vee, \rightarrow, \leftrightarrow$

(d) $\forall \phi \in \Sigma$, $\varphi \Delta x^c y^i \psi$, $\forall x^m \in y^c \psi$ for $i \geq m$.

For $k > 0$ we then set:

$\Sigma_k^{(m)} = \text{the set of formulae }$

$\forall x_1^m \wedge x_2^m \ldots \wedge x_k^m \psi$ ($\phi \in \Sigma_0^{(m)}$

$\Pi_k^{(m)} = \text{the set of formulae }$

$\forall x_1^m \vee x_2^m \ldots \vee x_k^m \psi$ ($\phi \in \Sigma_0^{(m)}$

We also set: $\Pi_0^{(m)} = \Sigma_0^{(m)}$

Finally, set: $\Sigma^* = \Sigma^{(\omega)} = \bigcup_m \Sigma_0^{(m)}$. 
When dealing with the $\mathcal{L}^*_{\text{ar}}$ model theory of $N$ we informally use $x^i$ as a variable for elements of $\text{Th}_N$. Using this notation we define:

**Def:** $R(\bar{x}^i) \in \Sigma^{(m')}_{\bar{k}'}(N)$ relation of type $\langle i^1, \ldots, i^p \rangle$

(in the parameters $\bar{q}^{d1}, \ldots, \bar{q}^{dm}$).

Hence $R$ is defined by a $\Sigma^{(m')}_{\bar{k}'}$ formula $\varphi(\bar{x}^i)$ (in the parameters $\bar{q}^{d}$).

$R \in \Sigma^{(m')}_{k'}(N) \iff R \in \Sigma^{(m')}_{k}(N)$ in some parameters.

Similarly for $\Pi^{(m')}_{k}(N)$, $\prod^{(m')}_{k}(N)$.

Also: $\Sigma^*(N) = \Sigma^{(m')}_{k}(N) = \bigcup_{m} \Sigma^{(m')}_{k}(N)$.

(Similarly for $\Pi^*(N)$).

We say that $f$ is a $\Sigma^{(m')}_{k}$ function to $N$ of argument type $\langle i^1, \ldots, i^p \rangle$ if $f(\bar{x}^i) \in \Sigma^{(m')}_{k}$ relation of type $\langle i^1, \ldots, i^p \rangle$.

[Note: A relation in this sense is not characterized by its graph. It could be identified with a pair consisting of its graph and its type.]
Lemma 2.1 Let \( n, l \leq \omega \). If \( R(x^k, x^l) \in \Sigma^m_l \) and \( k \geq h \), then \( R(x^k, x^l) \in \Sigma^m_l \).

Proof: trivial

We call \( R(x^k, x^l) \) a specialization of \( R(x^k, x^l) \).

Lemma 2.2 Let \( n, l \leq \omega \). If \( R(x^k, x^l) \in \Sigma^m_l \) and \( k \geq h \geq m \), then \( R \) is a specialization of a relation \( R(x^k, x^l) \) which is \( \Sigma^m_l \).

Proof: trivial.

Thus every \( \Sigma^m_l \) relation is obtainable by specialization from one whose arguments are of type \( \leq m \). It is also apparent that the \( \Sigma^m_l \) relations are closed under permutation of arguments and insertion of dummy arguments.

Another obvious consequence of our definitions is.
Lemma 3.1 Let \( m, l < \omega \). \( R(\overrightarrow{x}^{m+1}, \ldots, \overrightarrow{x}^0) \) in \( \Sigma_\ell^{m+1}(N) \) if the relation

\[ R_x^\omega = \{ \langle \overrightarrow{x}^{m+1} \rangle \mid R(\overrightarrow{x}^{m+1}, \overrightarrow{x}) \} \]

is uniformly \( \Sigma_\ell (\langle H^{m+1}, Q_x^\omega \rangle) \), where each \( Q_x^\omega \) has the form

\[ Q_x^\omega = \{ \langle \overrightarrow{x}^{m+1} \rangle \mid Q_x(\overrightarrow{x}^{m+1}, \overrightarrow{x}) \} \]

and \( Q_x(\overrightarrow{x}^{m+1}, \overrightarrow{x}) \) is \( \Sigma_\ell^{m+1}(N) \).

Note The reduction in Lemma 3.1 is uniform for all \( N \), with the \( \Sigma_\ell^{m+1} \) definitions of the \( Q_x \) depending only on the \( \Sigma_\ell^{m+1} \) def. of \( R \).

By a straightforward induction on \( m \), using Lemma 3.1:

Lemma 3.2 Let \( m < \omega \), \( 1 \leq l < \omega \).

\( R(\overrightarrow{x}^m, \ldots, \overrightarrow{x}^0) \) in \( \Sigma_\ell^m(N) \) if

\[ R_x^\omega = \{ \langle \overrightarrow{x}^m \rangle \mid R(\overrightarrow{x}^m, \overrightarrow{x}) \} \]

is uniformly \( \Sigma_\ell (N^{m+1}, p(\overrightarrow{x})) \), where

\[ p(\overrightarrow{x}) = \langle \langle \overrightarrow{x}^{m-1} \rangle, \ldots, \langle \overrightarrow{x}^0 \rangle \rangle. \]

Similarly:

Lemma 3.3 Let \( m < \omega \). If \( R(\overrightarrow{x}^m, \ldots, \overrightarrow{x}^0) \) in \( \Sigma_0^m(N) \), then \( R_x^\omega \) is uniformly

reducible in \( N^{m+1}, p(\overrightarrow{x}) \).
Def \( \pi : \tilde{N} \rightarrow \Sigma^{(m)} \tilde{N} \) \( i \vdash \pi : \tilde{N} \rightarrow \tilde{N} \)

and whenever \( \varphi(v_1, \ldots, v^m) \) is a \( \Sigma^l \) formula with \( |v_i| \leq m \) and \( \pi \in \tilde{H}^d \) for \( i = 1, \ldots, m \), Then \( \tilde{N}(x_i) \in \tilde{H}^d \) and
\[
\tilde{N} \models \varphi[\tilde{x}] \iff \tilde{N} \models \varphi[\pi(x)].
\]
(Hence \( \pi^{-1}(H^i) \subseteq H^i \) for \( i \leq m \).) For \( l > 0 \) or \( i < m \) it follows that \( \pi^{-1}(H^i) \subseteq H^i \).
\[
\text{Lemma 3.2, 3.3 Then give us!}
\]

**Lemma 4.1** Let \( m, l < \omega \). Then
\[
\pi : \tilde{N} \rightarrow \Sigma^{(m)} \tilde{N} \ i \vdash \pi : \tilde{N} \rightarrow \tilde{N} \text{ and }
\]
whenever \( \bar{p} \in \Gamma^m \tilde{N} \), \( \bar{p} = \pi(\bar{p}) \), Then \( \bar{p} \in \Gamma^m \tilde{N} \) and \( \pi^{-1}(H^i) \subseteq \tilde{N}^m \pi^{-1}(H^i) \rightarrow \tilde{N}^m \).
(Note For \( l = 0 \) the proof uses the fact that for \( \Gamma \)-modules, \( \pi : \tilde{M} \rightarrow \Sigma^l M \) implies \( \pi : \tilde{M} \rightarrow \text{mod } M \).

An order to reformulate the extension of imbeddings, Lemma in terms of \( \Sigma^l \) imbeddings we prove:
Lemma 4.2 Let \( \varphi : \vec{N} \rightarrow N \) s.t.

\[
\pi \varphi \vec{N} \rightarrow \vec{N}^\ell, \vec{p} \rightarrow \vec{N}^\ell, \vec{p} \text{ for } i < n
\]

and \( \pi \varphi \vec{N}^n \rightarrow \vec{N}^n, \vec{p} \rightarrow \vec{N}^n, \vec{p} \), when \( \vec{p} \in \vec{N}^n \) and \( p = \pi(\vec{p}), \ell \leq \omega \). Then

\[
\pi : \vec{N} \rightarrow \vec{N}^\ell, \vec{p} \rightarrow \vec{N}^\ell, \vec{p}
\]

Proof:

Let \( \vec{q} \in \vec{N}^n \) and \( q = \pi(\vec{q}) \). By induction on \( i \leq n \)

we show:

\( \vec{A}^{\ell, \vec{q} \vec{p} \vec{e}} \) is real in \( N^\ell, \vec{p} \vec{e} \) in a parameter

\( x \) and \( \vec{A}^{\ell, \vec{q} \vec{p} \vec{e}} \) is real in \( N^\ell, \vec{p} \vec{e} \) in \( \vec{N}^n \) by the same. - end of proof.

The conclusion follows by Lemma 5.1.

QED (Lemma 5.2).

This enables us to reformulate Lemma 4.1 as

Lemma 4.1' Let \( \pi : \vec{N}^n \rightarrow \vec{N}^n, \vec{p} \rightarrow \vec{N}^n, \vec{p} \) s.t. \( \pi(\vec{p}) = p \) and \( \pi : \vec{N} \rightarrow \vec{N}^n, \vec{p} \rightarrow \vec{N}^n, \vec{p} \).
Lemma 4.3. Let \( \pi : \mathbb{N} \rightarrow \mathbb{Z^m} \). Then
\[
\pi^{-1}(P^m_N) \subset P^m_N.
\]
\(\text{Pf.}\)
Suppose not. Let \( p \in P^m_N \), \( \bar{p} = \pi^{-1}(p) \notin P^m_N \)
Then \( m > 0 \). Let \( A \) be \( \Sigma^m_1(N) \) in \( p \)
\( \text{i.e.} \ A \cap \omega p \notin N \ (p = p^m) \). Let
\( \bar{A} \) have the same definition in \( \bar{p} \)
over \( \bar{N} \). Then \( \bar{a} = \bar{A} \cap \omega \bar{p} \notin \bar{N} \ (\bar{p}^m = \bar{p}^m) \). Let \( a = \pi(\bar{a}) \). Then
\( (\star) \land \bar{z}^m (\bar{z}^m \in \bar{N}) \leftrightarrow \bar{z}^m \in \bar{a} \)
holds in \( \bar{N} \). \( (\star) \in \Pi^m_1(\bar{N}) \) in \( \bar{a}, \bar{p} \).
Hence the same \( \Pi^m_1 \) statement
holds of \( a, p \) in \( N \). Hence
\( A \cap \omega p = a \cap \omega p \notin N \). Contrad!
\( \square \) (Lemma 4.3)

Note that if \( p \in P^m_N \), then \( \phi = \langle \rho_1, \rho_2, \ldots, \rho_N \rangle \in P^m_N \), since \( A^m \) is
uniformly and in \( A^m \phi \).
Since
\( \text{there is } m \text{ s.t. } p = p^m \)
\( = p \min \{ p^m \mid m < \omega \} \), we con-
clude that \( P^* \neq \emptyset \) !
Def: \( P^*_N \) = the set of \( p \in \mathbb{N} \) s.t. \( (p, 1, 0, -1, 0) \in P^*_N \) for all \( n \.

(Similarly for \( P^*_N \)). Then

Corollary 4.4 Let \( \pi : \bar{\Sigma} * \mathbb{N} \to \bar{\Sigma} * \mathbb{N} \). Then

\[ \pi^{-1} P^*_N \subseteq P^*_N \]

Note \( P^*_N \) is the set of \( p \in \mathbb{N} \) s.t. for each \( m \) there is \( A \) which is \( \Sigma^m_1 \) in \( p \) and \( A \cap \omega^m \cap N \).

\( \Sigma^m_1 \) relations are not, in general, closed under substitution of \( \Sigma^m_1 \) functions.

There are, however, important cases where such substitution is possible. One such is:

Lemma 5.1 Let \( p \leq m \leq \omega \), \( 1 \leq l \leq \omega \). Let \( R(x^0, \ldots, x^0) \) be \( \Sigma^m_1(N) \). Let \( F^0, \ldots, F^0 \) be s.t. each \( F^i(x^0, \ldots, x^0) \) is a \( \Sigma^1_1 \) map to \( H^i_N \). Then \( R(F(x)) \in \Sigma^m_1(N) \).

[Note The \( F^i \) may be partial on \( \times H^i_N \). As usual, \( R(F^i(x)) \) is taken to hold if \( x \) is in \( \Sigma^1_1 \), the \( F^i(x) \) exist, and \( R \) holds of \( F^i(x) \).]
proof of Lemma 5.1. By induction.

The case $n = 0$ is trivial. Let $n = m + 1$. Let $R^m = \{ < x^m > | R(\bar{x}^m, \bar{x}) \}$ be uniformly $\Sigma^e (\bar{H}^m, \bar{Q}^m)$. Define

$$Q_i, \bar{z} = \{ < u^m > | Q_i(u^m, \bar{z}) \}$$

and $Q_i \in \Sigma^{(m+1)}(N)$. Let $Q_i(\bar{u}^m, \bar{z})$ be a specialization of $Q_i(\bar{u}^m, \bar{z})$, where this is $\Sigma^{(m+1)}(N)$. By the induction hypothesis, $Q_i(\bar{u}^m, \bar{F}'(\bar{z}')) \in \Sigma^{(m)}(N)$, where $\bar{F}' = \bar{F}^m, \bar{F}'$.

Hence, by specialization, so is $Q_i(\bar{u}^m, \bar{z}) \leftrightarrow Q_i(\bar{u}^m, \bar{F}'(\bar{z}'))$. Let $R'_z$ be $\Sigma^e (\bar{H}^m, \bar{Q}^m)$ by the same definition as $R^m$ over $\langle \bar{H}^m, \bar{Q}^m \rangle$. Clearly

$$R^m \rightarrow R'_z (\bar{z}') \leftrightarrow R(z', \bar{F}'(\bar{z}'))$$

where $\bar{F}'(\bar{z}')$ exist. Hence,

$$R(z', \bar{F}'(\bar{z}') \leftrightarrow (\bar{F}'(\bar{z}') \leftrightarrow R'_{z'} (\bar{z}')) \in \Sigma^{(m)}$$

But $R(\bar{F}'(\bar{z}') \leftrightarrow \forall \bar{x}^m (\bigwedge_i x_i = \bar{F}'(\bar{z}')) \wedge R(\bar{x}^m, \bar{F}'(\bar{z}'))$

QED (Lemma 5.1)
(Note) It is apparent from the proof that
the defining $\Sigma_1^{(m)}$ formula of $P(E(z))$
depends uniformly for all $N$ only on the
defining formulae of $R, E$.}

One consequence of this is that $\Sigma_1^{(m)}$
relations are essentially characterized
by their graphs: Let $R(x^{i_1}, \ldots, x^{i_m})$
and $R(x^{j_1}, \ldots, x^{j_m})$ have the same
graph $i$, where $i_1, \ldots, i_m, j_1, \ldots, j_m \leq m$. Then
one $i \in \Sigma_1^{(m)}$ iff the other $i$. To see this,
note that we can convert one to the
other by composition with the identity
function $y_i = x^{i_1}$. It follows in particular
that a relation $i \in \Sigma_1^{(m)}$ iff the relation
with the same graph and arguments of
type $0 \in \Sigma_1^{(m)}$. 
In order to extend this result we define

**Def The good $\Sigma^{(m)}_\lambda(N)$ fama comprise**

the smallest class s.t.

(a) Each partial $\Sigma^{(m)}_\lambda(N)$ map

$F(x_1^{i_1}, \ldots, x_p^{i_p})$ to $H^\lambda_\nu$ is good $(i_1, \ldots, i_p \leq m)$

(b) if $F(x_1^{i_1}, \ldots, x_p^{i_p})$ is good and

$G_c(z)$ is a good map to $H^\lambda_\nu$ $(c = 1, \ldots, p)$

(the arguments of $G_c(z)$ all being of

type $\leq m$), then $F(G_c(z))$ is good.

**Lemma 5.2** Let $R(x_1^{i_1}, \ldots, x_p^{i_p})$ be

$\Sigma^{(m)}_\lambda(N)$ $(m < \omega, n \leq \lambda < \omega, i_1, \ldots, i_p \leq m)$.

Let $F_c(z)$ be a good $\Sigma^{(m)}_\lambda(N)$ map

to $H^\lambda_\nu$ $(c = 1, \ldots, p)$. Then $R(F_c(z))$

is $\Sigma^{(m)}_\lambda(N)$.

**Proof** $R(F_c(z))$ results from iterated

applications of Lemma 5.1 (and

permutations of arguments). Q.E.D. (5.2)

The remarks at the end of Lemma 5.1

obviously apply to 5.2 as well.

(Nota Good functions are closed under

compositions.)
Lemma 5.3 Let \( R(y^n, x^1, \ldots, x^m) \) be \( \Sigma^m_1 \) \((j \leq n)\). There is a \( \Sigma^m_1 \) function \( F \) to \( \mathbb{N}^m \) s.t.

(a) \( \text{dom}(F) = \{ y^n | R(y^n, \bar{x}) \} \)

(b) \( \forall y^n \ R(y^n, \bar{x}) \iff R(F(x^1), \bar{x}) \).

Proof:

We prove it for \( R(y^n, \bar{x}^m, \ldots, \bar{x}^0) \).

Set \( p(x^j) = \langle \langle x^0 \rangle, \ldots, \langle x^{j-1} \rangle \rangle \). Then the claim holds for appropriate:

\[
F(\bar{x}^j) = \bigwedge_{N^m, p(x^j)} (0, \langle \bar{x}^m \rangle)
\]

\( \Box \) E D (5, 3)

\[
\Sigma^m_\omega = \bigcup_{m<\omega} \Sigma^m_1
\]

\[
Q^m_1 = \text{the set of formulae } Qx^m \varphi = \forall z^m \forall x^m (z^m \in x^m \land \varphi), \text{ where } \varphi \in \Sigma^m_1.
\]

Let \( \pi : \bar{N} \to N \) then has the obvious meaning.

We set : \( Q^x = \bigcup_n Q^{m_1}_1 \).
Lemma 5.4 There is a good \( \Sigma_{\ell}^{(m)}(N) \) for } 
\[ F \text{ s.t. if } p \in \mathbb{R}_{N}^{m+1}, \text{ then each } x \in X \] 
has the form \( F(u_1, p) \), where \( u \in H_{N}^{m+1} \). 
\[ \text{prov. And, on } m, \]
\[ m = 0 : F(u_1, p) \approx h_{N}((u), <(u_1), p>) \]
\[ m = m+1 : \text{Let } x = G(v_1, p, m) \text{ where } v \in H_{N}^{m} \]
and \( G \) is a good \( \Sigma_{\ell}^{(m)}(N) \) map. Then \( u = H(u_1, p) \approx h_{N}^{m+1}, p_{m}((u), <(u_1), p>) \)
where \( u \in H_{N}^{m+1} \). Hence \( x = G(H(u_1, p), p, m) \).
\[ \text{QED (Lemma 5.4)} \]

Note This gives a new proof of Lemma 4.

Cor 5.5 Let \( p \in \mathbb{R}_{N}^{m} \). Then \( \Sigma_{\ell}^{(m)}(N) \supset \Sigma_{\ell}^{(m)}(N) \) for \( \ell \geq 1 \).

Thus if \( N \) is round we have:
\[ \Sigma_{\omega}^{(m)}(N) = \Sigma_{\omega}(N), \]
\[ (\Sigma_{\omega}^{(m)} \subset \Sigma_{\omega} \text{ always holds, but the converse may not).} \]
Note: An subsequent sections we
sometimes write:

\[ \pi : M \rightarrow \Sigma_0 \]

\[ \Sigma_0 \]

for mean:

\[ \pi : M \rightarrow \Sigma_0 \]

\[ \Sigma_0 \]

This clearly implies:

\[ \pi | H^m_M : M^m \rightarrow N^m \]

\[ \Sigma_0 \]

for all \( p \)

hence:

\[ \pi : \Sigma_1 \]
Functional absoluteness

Every good $\Sigma_1^{(m)}$ function $f(x^{i_1}, \ldots, x^{i_m}) (i_1, \ldots, i_m \leq m)$ has a $\Sigma_1^{(m)}$ definition which is functionally absolute in the sense that it defines a $\Sigma_1^{(m)}$ function over every acceptable structure of the appropriate structural type.

To see this, note that a $\Sigma_1^{(m)}$ function $y^m = F(x^m, \ldots, x^0)$ has a functionally absolute definition of the form:

$$y^m = h_{N^m \cdot \mathcal{P}(x^m)} (c, x^m)$$

where $\mathcal{P}(x^m) = \langle x^{n+1}, \ldots, x^0 \rangle \ (\mathcal{P}(x^m) = \emptyset$ if $m = 0)$. But every good function is obtained by composition and argument permutation from such functions.