§1.4 **Extendability**

**Def.** \( N \) is *extendable* by \( E \) iff

\[
\text{there exists } \pi : N \rightarrow^* N'.
\]

In this section we develop criteria of extendability.

**Def.** \( \sigma : \langle N, E \rangle \rightarrow^* \langle N', E' \rangle \) iff

the following hold:

(a) \( \pi : N \rightarrow^* N' \);

(b) \( E \) is a weak extender on \( N \) such \( \tilde{E} = \{ \langle a, x \rangle \mid x \in E_a \} \) is closed over \( N \)

in a parameter \( p \);

(c) \( E' \) is a weak extender on \( N' \) and

\( \tilde{E}' \) is closed over \( N \) in \( \sigma(p) \) by the

same definition.

**Lemma.** Let \( \sigma : \langle N, E \rangle \rightarrow^* \langle N, E \rangle \) and

\( \pi : N \rightarrow^* N' \). Then there is \( \bar{\pi} : \tilde{N} \rightarrow^* \tilde{N}' \).

Moreover, there is a unique \( \sigma' : \tilde{N} \rightarrow N' \)

such that \( \bar{\pi} \sigma = \pi \bar{\pi} \sigma' \) and \( \sigma' \pi \bar{\pi} = \sigma \pi \bar{\pi} \).

where \( \tilde{E} \) is at \( \bar{\pi}, \tilde{N} \). \( \sigma' \) is defined

by: \( \sigma'(\bar{\pi}(f)(a)) = \pi \sigma(f)(\sigma(a)) \) for

\( f \in \Pi(\tilde{E}, \tilde{N}) \), \( a \in [\tilde{N}] <\omega \).
proof of Lemma 1.

Let $\mathcal{D} = \langle \bar{D}, \bar{E}, \bar{I}, \bar{I} \rangle$ be defined from $\bar{N}, \bar{E}$ in the usual way. Then

\[ \langle \alpha, \beta > \bar{I} \langle \beta, \gamma > iH \quad \exists u \mid f^{\alpha}(u) = g^{\beta}(u) \rangle \in \bar{E}_c \]

\[ iH \quad \exists u \mid (f)(u) = (g)(u) \]

\[ \in E_{\sigma(c)} \]

\[ iH \quad \pi \sigma(f)(\sigma(a)) = \pi \sigma(g)(\sigma(b)) \]

Similarly:

\[ \langle \alpha, \beta > \bar{E} \langle \beta, \gamma > iH \quad \pi \sigma(f)(\sigma(a)) \in \pi \sigma(g)(\sigma(b)) \]

Thus $\bar{E}$ is well formed and $\pi : \bar{N} \rightarrow^{E} \bar{N}'$ exists. Clearly there is a structure preserving map $\sigma' : \bar{N}' \rightarrow N'$ defined by:

\[ \sigma'((\pi(f)(a)) = \pi \sigma(f)(\sigma(a)) \]

Claim:\n
$\sigma' : \bar{N}' \rightarrow^{E} N'$.

proof:

1. Let $\bar{e} \in \omega_{\bar{N}}^{m+1}$. Then $\sigma' : \bar{N}' \rightarrow^{E} N'$.

1. Let $\varphi \in \Sigma^{m}$.\n
$\bar{N}' = \varphi \{ \pi(f)(a) \} \leftarrow \{ u \mid \bar{N} \varphi \{ f^{\alpha}(u) \} \} \in \bar{E}_c$\n
$\leftarrow \{ u \mid \bar{N} \varphi \{ \sigma(f)(u) \} \} \in E_{\sigma(a)}$\n
$\leftarrow \bar{N}' = \varphi \{ \pi \sigma(f)(\sigma(a)) \}$. QED(1)
(2) Let \( \omega^{m+1}_N \leq \bar{n} < \omega^m_N \). Then

\[
\sigma' : \bar{N}' \rightarrow \Sigma^m_\bar{n} \nabla \nabla \nabla
\]

Proof.
Let \( \bar{A} \) be \( \Sigma^m_\bar{n} (\bar{N}') \) and let \( A \) be \( \Sigma^m_1 (N') \) by the same definition.
Let \( x \in \bar{N}' \). We claim:

Claim: \( \bar{A}(x) \leftrightarrow A(\sigma'(x)) \).

Let \( x = [\langle a, f \rangle] = \overline{\pi f}(a) \) when \( a \in [\bar{\nu}]^m \), \( f \in \Gamma(\bar{\nu}, \bar{N}') \). Then

\[
\sigma'(x) = [\langle \sigma(a), \sigma(f) \rangle] = \overline{\pi f}(a),
\]

Choose \( p \in \bar{N} \) s.t. \( f = p \) or \( f \) is a \( \nu \)-fold \( \Sigma^{m-1}_1 (\bar{N}) \) function in \( p \) by the same absolute definition. Then \( \sigma(f) \) bears the same relation to \( \sigma(p) \) in \( \bar{N} \).

The proof of (10) in §1.3 Lemma 2 shows that there is \( \bar{A}^* \) which is \( \Sigma^m_\bar{n} (\bar{N}) \) in \( p \) (uniformly in the def. of \( \bar{A}^* \)) s.t. \( \bar{A}(a) \leftrightarrow \bar{A}([\langle a, f \rangle]) \) for all \( a \in [\bar{\nu}]^m \). Similarly, if \( A^* \) is \( \Sigma^m_1 (N) \) in \( \sigma(p) \) then...
The same definition, we have:
\[ \overline{A} \left( \overline{\pi \left( f \right) \left( a \right)} \right) \leftrightarrow \overline{A}^* \left( a \right) \]
\[ \leftrightarrow \overline{A}^* \left( \sigma \left( a \right) \right) \]
\[ \leftrightarrow \overline{A} \left( \overline{\pi \left( \overline{\sigma \left( f \right)} \right) \left( \sigma \left( a \right) \right)} \right) \]

QED (2)

But \( H_{N' \overline{K}} = H_{N' \overline{K}} \) and \( \overline{\pi} \cap H_{N' \overline{K}} = \text{id} \).

We can carry the above argument a step further and show that if \( \overline{A} \left( \overline{y}, \overline{x} \right) \in \Sigma_{1}^{(m)} \left( N' \right) \) and \( x = [\sigma \left( a \right)] \) and \( p \) are as above, there is \( \overline{A}^* \) which is \( \Sigma_{1}^{(m)} \left( N \right) \) in \( p \) (unif.

in the def. of \( \overline{A} \left( f \right) \)) such

\[ \overline{A} \left( \overline{y}, \overline{\pi \left( f \right) \left( a \right)} \right) \leftrightarrow \overline{A} \left( \overline{y}, a \right) \]

for all \( \overline{y} \in H_{N' \overline{K}} \). Since \( \overline{\sigma} \cap H_{N' \overline{K}} = \overline{\sigma} \cap H_{N' \overline{K}} \) for \( a \) and \( \overline{\sigma} \) is \( \Sigma^* \) preserving, we may conclude:

(3) \( \overline{\sigma} : N' \rightarrow \Sigma_{1}^{(m)} \left( N' \right) \) for \( a \)

This proves that \( \overline{\sigma}' \) is \( \Sigma^* \) preserving. The uniqueness of \( \overline{\sigma}' \) is straightforward.

QED (Lemma 1)
Note An lemma we can ob-
viously replace the assumption

$\pi : N \to N'$ by:

$\pi : N \to x \in \mathcal{E}$

$\pi : N \to x \in \mathcal{E}$ s.t. $\text{crit}(\pi) = n$

and $a \in \pi(x)$ iff $x \in E_a$,

whenever $a \in \mathcal{E}$ and $\mathcal{E} \subseteq \mathcal{E}$.

Define let $E$ be a weak extender
at $n$, $\nu$ on $N$. $E$ is $\omega$-complete
iff for every sequence $\langle a_i, X_i \rangle$
$i.e. X_i \subseteq E_{a_i}$ ($i < \omega$). There is
an order preserving map

$\delta : \bigcup_{\omega} a_i \to \nu$ s.t. $\delta(a_i) \subseteq X_i$

for $i < \omega$. 
Lemma 2.1 Let $\sigma : \langle \bar{\mathbb{N}}, \bar{E} \rangle \rightarrow^* \langle \bar{\mathbb{N}}, \bar{E} \rangle$
where $\bar{\mathbb{N}}$ is countable and $E$ is $\omega$-complete. Then $\bar{\pi} : \bar{\mathbb{N}} \rightarrow^{\ast} \bar{\mathbb{N}}$ exists. Moreover there is a $\bar{\sigma} : \bar{\mathbb{N}} \rightarrow^{\ast} \bar{\mathbb{N}}$ such that $\bar{\sigma} = \bar{\pi}$.

Proof.
Let $E$ be at $u, v$ and $\bar{E}$ at $\bar{u}, \bar{v}$. By $\omega$-complete there is a $\delta : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}}$
which in order preserving: for$
\delta (a) < \delta (b) \text{ whenever } x < E a$.
From $\bar{\mathbb{N}} = \langle \bar{\mathbb{N}}, \bar{E}, \bar{I}, \bar{B} \rangle$ an usual,
At is easily seen that:
$\langle a, f \rangle \bar{I} \langle b, g \rangle \iff \bar{\delta} (f) (\bar{\delta} (a)) = \bar{\delta} (g) (\bar{\delta} (b))$
Then $\bar{\mathbb{N}}$ is well founded: hence $\bar{\pi} : \bar{\mathbb{N}} \rightarrow^{\ast} \bar{\mathbb{N}}$ exists. Moreover there is a structure preserving
$\bar{\sigma} : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}}$ defined by:
$\bar{\sigma} (\bar{\pi} (f) (a)) = \pi \delta (f) (\delta (a))$.
A virtual repetition of the proof of Lemma 1 then shows that $\bar{\sigma}$ is $\bar{E} \ast$ preserving.

QED (Lemma 2.1)
Corollary 2.2 Let \( E \) be a weak extender on \( N \) not \( E \) in \( \mathfrak{I} \) over \( N \). If \( E \) is co-complete, then \( N \) is *-extendable by \( E \).

**Proof.** Suppose not. Let \( N \notin \mathfrak{H}_\theta \) for a regular \( \theta \) and let \( N \in X < \mathfrak{H}_\theta \) where \( X \) is countable. Let \( F: X \xrightarrow{\sim} X \) where \( X \) is transitive. Set: \( \bar{N} = F^{-1}(N) \), \( \bar{E} = F^{-1}(E) \). Then \( \bar{N} \) is not extendable by \( \bar{E} \), by absoluteness. But \( \sigma : \langle \bar{N}, \bar{E} \rangle \to \langle N, E \rangle \) where \( \sigma = F \upharpoonright \bar{N} \). Hence \( \bar{N} \) is extendable. Contradiction! \( \Box \)
Finally we note a variant of Lemma 1 which will be used in the sequel. We first alter the def. of $\rightarrow^*$ as follows:

\[ \text{Def} \quad \sigma : \langle N, E \rangle \rightarrow^{(m)} \langle N', E' \rangle \quad \text{iff} \]

\[(a) \quad \sigma : N \rightarrow^{(m)} N' \quad \text{and} \]

\[(b), (c') \quad \text{as in the def. of } \rightarrow^*. \]

The proof of Lemma 1 then trivially gives:

\[ \text{Lemma 1'} \quad \text{Let } \sigma : \langle \bar{N}, \bar{E} \rangle \rightarrow^{(m)} \langle N, E \rangle \]

where $\bar{E}$ is an extender on $\bar{\kappa} \geq \omega_1^m$.

\[ \text{Let } \bar{\pi} : \bar{N} \rightarrow^{(m)} \bar{N}' \quad \text{Then there is} \]

\[ \bar{\pi} : \bar{N} \rightarrow^{(m)} \bar{N}' \quad \text{Moreover there is a} \]

\[ \text{unique } \sigma' : \bar{N}' \rightarrow^{(m)} \langle N', E' \rangle \quad \text{such that} \]

\[ \sigma' \bar{\pi} = \bar{\pi} \sigma \quad \text{and} \quad \sigma' \bar{\pi} = \bar{\pi} \sigma \bar{\nu} \quad \text{where } E \vdash \bar{\kappa}, \bar{\nu}. \quad \sigma' \quad \text{is defined by} \]

\[ \sigma' (\bar{\pi} (f) (a)) = \bar{\pi} \sigma (f) (\sigma' (a)) \quad \text{for } f \in \Gamma (\bar{\kappa}, \bar{N}) \quad \text{a} \in [\bar{\nu}] < \omega. \]