§ 2.4 Some Consequences

We recall that by § 2.1 Lemma 3, no $M$ can be both a simple and non-simple iterate of a mouse $\overline{M}$.

Def Let $M, N$ be mice.

$M \sim^* N \iff M, N$ have a common non-simple iterate.

Lemma 1 $\sim^*$ is an equivalence relation on mice.

Proof.

$M \sim^* M$, $M \sim^* N \rightarrow N \sim^* M$ are trivial. We prove transitivity.

Let $M \sim^* N \sim^* Q$. Let $M'$ be a common simple iterate of $M, N$ and $Q'$ a common simple iterate of $N, Q$.

By coiteration there is a common iterate $P$ of $M', Q'$ which is a simple iterate of one of them. Suppose e.g. that $P$ is not a simple iterate of $M'$.

Then it is a non-simple iterate of $N$. But it is a simple iterate of $Q'$, hence of $N$. Contradiction!

QED (Lemma 1.1)
Def. $M <_* N$ if there is a common iterate $Q$ which is a simple iterate of $M$ but not of $N$.

Lemma 2 <* is a linear ordering of mice modulo the congruence relation $\sim_*$.

The proof stretch over a few sublemmas:

Lemma 2.1 $M <_* N \lor M \sim_* N \lor N <_* M$

pf. By coiteration.

Lemma 2.2 $M \sim_* N \rightarrow M <_* N$

pf. Suppose not. Let $Q$ be a common simple iterate of $M, N$. Let $M'$ be a common iterate of $M, N$ which is simple of $M$ but not of $N$. Let $P$ be a common iterate of $Q, M'$ which is simple of one of them. If $P$ is a simple iterate of $Q$, then it is a simple iterate of $N$. 
which is impossible since \( P \) is an iterate of \( M' \). But then \( P \) is a simple iterate of \( M' \), hence of \( M \), and \( P \) is a non simple iterate of \( Q \), hence of \( M \).

\textbf{Contr.} \hspace{1cm} \textbf{QED (Lemma 2.2)}

\underline{Lemma 2.3} \hspace{1cm} M \not< N \rightarrow N \not< \*M

\textbf{Proof.} Suppose not.

Let \( M' \) be an iterate of \( M \), \( N \) which is simple of \( M \) but not of \( N \). Let \( N' \) bear the same relation to the pair \( N, M \). Let \( Q \) be an iterate of \( N', M' \) which is simple of one of them. \( Q \) is not simple of \( M' \), since it is an iterate of \( N' \). Hence \( Q \) is not simple of \( M' \). Similarly, \( Q \) is not simple of \( N' \).

\textbf{Contr.} \hspace{1cm} \textbf{QED (Lemma 2.3)}

\underline{Lemma 2.4} \hspace{1cm} \*M \not< N \*Q \rightarrow Q \not< \*M

\textbf{Proof.} Suppose not.

Let \( M' \) be an iterate of \( M, N \) which is simple of \( M \) but not \( N \). Let \( N' \) bear the same relation to \( N, Q \) and \( Q' \) to \( Q, M \). By contraction there is \( P \) which is a common
iterate of $M', N', Q'$ and a simple iterate of one of them. This is easily seen to be false. Contd.

QED (Lemma 2.4)

**Lemma 2.5** $M < * N < * Q \rightarrow M < * Q$.

Proof. Suppose not.

Then $M < * N < * Q < * M$ by Lemma 2.1, 2.5. Let $M'$ be an iterate of $M, N$ which is simple of $M +$ not of $N$.

Let $N'$ bear the same relation to $N, Q$. Then $M' < * M < * N$ and $M', Q$ have a common simple iterate $Q'$. Let $P$ be common iterate of $N', Q'$.

An easy argument again shows that $P$ is not a simple iterate of $N'$ or $Q'$.

Contradiction! QED (Lemma 2.5)

This proves Lemma 2

**Note** The relation $< *$ is well-founded, as can be seen by the nonexistence of degenerate iterations.