§ 3.3 Mitchell Covering Lemma

§ 3.3.1 The Theory of $\mathcal{O}^n$

**Def.** An $n$-premonotone (rpm) in a $\mathcal{J}$-model $M = \langle J_\beta, E_\beta, E_{\beta+1} \rangle$ is:

(a) $\langle J_\beta, E_\beta \rangle$ is a p.m.; $E_\beta \neq \emptyset$.

(b) $E_{\beta+1}$ is a normal measure on $\mathcal{R}_n$ in $M$ (where $n = \text{crit}(E_\beta)$).

(c) $\forall \pi : M \rightarrow M'$, Then $E_{\beta+1}^{M'} \upharpoonright \beta = \langle J_\beta^{M'}, E_\beta \rangle$.

A simple iteration $\langle M_i, i < \theta \rangle$ of an rpm with indices $\gamma_i \leq \beta_i + 1$ ($i+1 < \theta$), where $M_i = \langle J_{\beta_i}, E_{\gamma_i}, E_{\beta_i+1} \rangle$ is defined in the obvious way.

Note that a proper truncation of an rpm is a p.m. Hence the notion of an arbitrary iteration with indices $\langle \gamma_i, i \rangle$ has an obvious definition. The notion of standard and normal iter.
Definition. An $r$-mouse is an $r$-pm which is iterable in the sense that any iteration may be continued.

Call $\mathcal{N}$ a \textit{generalized mouse} ($\text{gm}$) iff $\mathcal{N}$ is a mouse or an $r$-mouse. For $\text{gm}$'s the following are easily verifiable:

(a) \S2.1 Lemma 1.1 - 1.3 go through as before.

(b) \S2.1 Lemma 2 (The monoinexistence of a degenerate iteration) goes then since any proper truncation of an $r$-mouse is a mouse.

(c) \S2.1 Lemma 3 (The "Doob-Lawson Lemma") goes then as before.

(d) The definition of \text{coiteration} $\langle N_i^h \mid i < \theta \rangle$ ($h = 0, 1$) of $\vec{N}$, $\bar{N}$ with indices $\nu$, goes then as before, using the convention:

$$E_{\beta + \gamma}^N = \emptyset \text{ if } \beta \leq \mu < \nu < \beta + \gamma.$$
(e) §2.2 Lemmas 1.1 - 1.4 go through as before.

Note that no $s$-mouse can be a proper segment of any $g$-mouse. Hence the coreflection of two $s$-mice either truncates on both sides or on neither.

(f) §2.2 Lemmas 2 - 5 go through as before.

(g) By §2.3 any $s$-mouse is HP (hereditarily pure). But then the argument of §2.3 goes through for $s$-mice.

In particular, one side of any coreflection must be simple. By the above remark, two $s$-mice must coreflect simply to a common $s$-mouse. Now let $M$ be an $s$-mouse and set:

$$M = h_M(\phi), \quad M \text{ transitive.}$$

Then $\omega = \omega^M_0 \uparrow \phi \in R^1_M$. By §2.1 Lemma 2 it follows that $M$ is an $s$-mouse. But
since any two $\kappa$-mice centerate to a common $\kappa$-mouse, it follows easily that $\mathcal{M}$ is the common core of all $\kappa$-mice.

Def $0^\kappa$ = the core of all $\kappa$-mice

By our remarks, $\omega_0^\kappa = \omega$ ($m < \omega$)
and $P_0 = \emptyset.$

We let $\mathcal{C}$ be the statement that $0^\kappa$ does not exist.

This section is devoted to an important consequence of $\mathcal{C}$.

We prove Mitchell's "weak covering lemma" which says that for certain universal $\kappa$ the successors of sufficiently large singular cardinals are absolute. This holds in particular for the canonical $\omega$-complete $\kappa$-ultrafilter. In order to state the theorem, we define:
Def Let $\lambda > \omega$ be regular. A weakly $\omega$-compact $\text{W}$ is called $\omega$-full iff whenever $\kappa$ is a cardinal in $\text{W}$ such that $(\kappa)^{\langle \lambda \rangle} = \kappa$ and $\nu = \lambda^+$, and if $F$ is $\text{W}$-complete set, then $\langle \nu, \text{W}, F \rangle$ is a premeasure. Then $\text{F} = \text{E}_{\nu}$.

The proof which showed the canonical $\omega$-complete weakly $\omega$-compact to be universal also shows every $\omega$-full weakly $\omega$-compact to be universal.

Theorem Assume $\lambda^{\omega^2} = \lambda^+$. Let $\text{W}$ be an $\omega$-full weakly $\omega$-compact, $\beta > \omega$ be a singular cardinal. Then $\beta^+ = \beta^{++}$.