§ 3.3.2 Upward Extensions of Embeddings

As a preliminary to proving the weak covering lemma, we develop some fine structural lemmas.

Suppose we are given acceptable $\bar{M} = \langle J_{\beta}^A, B \rangle$ and $\bar{Q} = J_{\gamma}^A$, where $\gamma = \omega \bar{\beta}$ or $\bar{\gamma} \in M$ is regular in $\bar{M}$.

Suppose furthermore that $\sigma : \bar{Q} \to \bar{Q}$ cofinally. Under what circumstances can we "lift" $\sigma$ to a "nice" embedding $\bar{\sigma} : \bar{M} \to M$. An particular, when do we find $\bar{\sigma} > \sigma$ and $M = \langle J_{\beta}^A, B \rangle$ such that $\bar{Q} = J_{\gamma}^A$ and $\bar{\sigma} : \bar{M} \to M$ for all $n$ such that $\bar{\gamma} \leq \omega_{\bar{\rho}^n}$?
We try to construct such $\Sigma, M$ by imitating the $\Sigma$-ultraproduct construction.

$\text{Def } \Gamma = \prod_{\bar{Q}, M} \bar{\nu}, M$

The set of maps $f$ s.t. dom$(f)$ is bounded in $\bar{Q}$ (i.e. $\nu \in \bar{Q}$ dom$(f) \subseteq u$ and either $f \in \Sigma_{1}^{m}(M)$ for

an $m$ s.t. $\bar{v} \leq \omega_{\bar{M}}^{m+1}$ or $f \in H_{\bar{M}}^{m}$ where $\bar{v} \leq \omega_{\bar{M}}^{m}$.

(Note: If $f \in \Sigma_{1}^{m+1}(M)$ and dom$(f)$, say $(f) \text{ are bounded in } H_{\bar{M}}^{m}$, then $f \in H_{\bar{M}}^{m}$.)

(Note: dom$(f) \in \bar{Q}$ for $f \in \Gamma$)

$\text{Def } D = \{(a, f) | f \in \Gamma \land a \in \delta$(dom$(f)$)

We define a pseudo $\preceq$ - relation $\preceq$ and a pseudo-identity $I$ on $D$ as follows:
\[ \text{Def } \langle a, f \rangle \in \langle b, g \rangle \text{ if } \] 
\[ \langle a, b \rangle \in \sigma(\{\langle u, v \rangle \mid f(u) \in g(v)\}) \] 

Similarly:

\[ \text{Def } \bar{A}(\langle a, f \rangle) \text{ if } a \in \sigma(\{u \mid f(u) \in A\}) \] 

\[ \bar{B} \]

(This obviously generalizes to the case of many predicates or predicates with many argument places).

\[ \text{Def } D = D_{\sigma, \tau, \bar{r}} = \left< D, \sigma, \tau, \bar{r}, \bar{m} \right> = \left< D, \sigma, \tau, \bar{m} \right> = \left< D, \sigma, \tau, \bar{m} \right> = \] 

Exactly as in §1.3 Lemma 1.1 we get for theorem for \( \Sigma_0 \) formulae:

\[ \text{Lemma 1.1 Let } \langle a_1, f_1 \rangle, \ldots, \langle a_m, f_m \rangle \in D \text{ and let } \phi \text{ be a } \Sigma_0 \text{ formula.} \]

\[ \text{Then } \phi[\langle a_1, f_1 \rangle] \iff \] 
\[ \iff \bar{a} \in \sigma(\{\bar{u} \mid \bar{m} = \phi[\langle f(u) \rangle]\}) \]
In place of §1.3 Lemma 1.2 we use:

**Lemma 1.2** Let $R(y^m, x^{i_1}, \ldots, x^{i_r})$ be $\Sigma_1^{(m)}(\bar{M})$ where $j \leq m$. Let $m \geq m$ and let $f_1, \ldots, f_r \in \Gamma$ be good $\Sigma_1^{(m)}(\bar{M})$ maps, where $f_i$ is to $H^1$. Then there is a good $\Sigma_1^{(m)}(\bar{M})$ map $g \in \Gamma$ with $\text{dom}(g) = \text{dom}(f_1) \times \ldots \times \text{dom}(f_r)$ and

$$
\forall y^m \in R(y^m, f(\bar{u})) \leftrightarrow R(g(\bar{u}), f(\bar{u})),
$$

for $\bar{u} \in \text{dom}(f_1) \times \ldots \times \text{dom}(f_r)$.

The proof is the same as in §1.3 Lemma 1.2.
Thus I is an equality relation for D and I satisfies \( x \sim x \) -
tensionality. Until further notice assume:

(*) ID is well-founded,

Then there is a structure preserving \([\ ] : D \rightarrow M\), where M is transitive and:

\[
[x] = [y] \iff x I y \\
[x] \in [y] \iff x \in y .
\]

Def \( \tilde{\sigma} : M \rightarrow M \) is defined by:

\( \tilde{\sigma}(x) = \langle \langle 0, \{x, 0\} \rangle \rangle \)

Our hope is that \( \tilde{\sigma} \) is the derived extension of \( \sigma \). We shall now be able to prove this; however, without a further assumption on \( M \). For the moment we content ourselves with proving:

\( \tilde{\sigma} : M \rightarrow \bigcup M \) for \( \tilde{\sigma} \leq \omega \rho^M \).
Corresponding to §1.3 Lemma 1.3:

**Lemma 1.3** \( \bar{A} \bar{B} = 0 \)

Set \( \bar{A} = \bigcup \bar{A} \) as the set of all \( [a, f] \) of \( f \in \bar{A} \). Then \( \bar{A} \) is transitive and

\[ [a, f] \in [b, g] \iff \sigma(f)(a) \in \sigma(g)(b) \]

Define a map \( \bar{A} \to \mathbb{N} \) by

\[ [a, f] \to \sigma(f)(a) \] This is clearly seen to be onto, hence the identity.

Hence \( [a, f] = \sigma(f)(a) \) and

\[ \sigma(x) = [\langle 0, \{x, 0\} \rangle] = \sigma(x) \quad \text{QED} \]

We are now ready to prove:

**Lemma 2** Let \( \bar{M} \) be \( \bar{N} \)-linear.

Then \( \bar{f} : \bar{M} \to \mathbb{N} \) for \( \bar{N} \leq \omega \bar{P}_M^n \).

For the most part our proof will closely follow that of §1.3 Lemma.

Just as there we define:

\[ \phi_P = \begin{cases} \{ \langle f, \sigma(f)(x) \rangle \mid f \in P \} & \text{if } \omega \bar{P}_M^n \geq \bar{N} \\ \{ \langle f, x \rangle \mid f \in P \} & \text{if } \omega \bar{P}_M^{n+1} < \bar{N} \leq \omega \bar{P}_M^n \end{cases} \]
\[ \text{Def: } H_n = \{ \langle a, f \rangle \mid f \in \Gamma_n^m \land \langle a, f \rangle \in D \} \]

iff \( \Gamma_n^m \) is defined.

As in §1.3 Lemma 2:

(1) \( H_n \) is transitive.

For \( \bar{v} \leq \omega \Gamma_M^n \) we interpret \( \Sigma_n^{(m)} \) formulae over \( M \) by letting \( x^n \) range over \( H_n \).

Just as in §1.3 we get:

(2) Let \( \langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle \in D \) and let \( \varphi \) be either a \( \Sigma_1^{(m)} \) formula, where \( \bar{v} \leq \omega \Gamma_M^{m+1} \), or a \( \Sigma_0^{(m)} \) formula, where \( \bar{v} \leq \omega \Gamma_M^n \). Then:

\[ M \models \varphi \left[ \langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle \right] \iff \bar{a} \in \sigma (\{ \bar{u} \mid M \models \varphi [\bar{f} \bar{u}] \}) \]

Since \( \bar{a} \models H_M^n \) is cofinal into \( H_n \), for \( \omega \Gamma_M^{m+1} < \bar{v} \leq \omega \Gamma_M^n \) we conclude:

(3) \( \bar{a} : M \rightarrow \Sigma_1^{(m)} M \) for \( \omega \Gamma_M^n \geq \bar{v} \).

As in §1.3 we also get:

(4) \( \bar{a} : M \rightarrow \Sigma_0^{(m)} M \) for \( \omega \Gamma_M^n \geq \bar{v} \), and:

...
(5) \[ M \rightarrow \frac{\Sigma^{(m)}_2 \bar{M}}{\omega \bar{p}^{n+1}_M} \] for \( \omega \bar{p}^{n+1}_M \geq \bar{v} \).

An particular, since \( \varphi : \bar{M} \rightarrow M \), we have:

(6) \( M \) is acceptable.

Using (5), it then follows easily that:

(7) \( H_m = \bigcup B^p_m \) where \( M = B^B \)

and \( \omega \bar{p}^m = 0 \) on \( H_m \).

Thus it remains only to prove:

Claim \[ p^m = \frac{p^m}{M} \text{ if } \bar{v} \leq \omega \bar{p}^{m+1}_M \]

and \[ p^m \leq \frac{p^m}{M} \text{ if } \bar{v} \leq \omega \bar{p}^m_M \]

The proof is by induction on \( m \). \( m = 0 \) is trivial, so assume \( m > 0 \).

To prove \( \leq \) we let \( A \subset W^p_m \) be \( \Sigma^{(m-1)}_1 (\bar{M}) \) and imitate \S 1.3 Lemma 2.8 to show: \( \langle H_m, A \rangle \) is amenable.

To prove \( \geq \) (with \( \bar{v} \leq \omega \bar{p}^{m+1}_M \)) we imitate the proof of \S 1.3 Lemma 2.13, let \( A \subset W^p_m \) be \( \Sigma^{(m-1)}_1 (\bar{M}) \) in \( p \), let \( A \) be \( \Sigma^{(m-1)}_1 (\bar{M}) \) in \( \bar{M} \) by the same definition.
Assume \( A \in M \). Claim \( A \wedge \omega^\alpha \in M \).

Suppose not. Let \( A \wedge \omega^\alpha = \{ \langle a, f \rangle \} \).

The statement \( A \wedge \omega^\alpha = \alpha \) is expressed by:
\[
\Lambda z^n (z^n \in A \rightarrow z^n \in \alpha)
\]
which in \( \Pi_1^m \) is \( \exists (p) = \alpha \).

\[ A \wedge \omega^\alpha = \alpha \rightarrow M \models C(q_0 [\langle a, f \rangle], \exists (p)) \]
where \( q_0 \) in \( \Pi_1^m \). Hence
\[
a \in C(q_0 [\exists (p) = f(a)], \exists (p)) = \tau(\{ u | A = f(u) \})
\]
by Lemma. Hence \( \forall u \bar{A} = f(u) \in M \),
Contr.
AED (Lemma 2).

As a corollary of the proof:

\[ \text{Lemma 2.1} \quad \forall : \bar{M} \rightarrow \bar{M} \text{ for } \bar{v} \leq \omega \bar{\omega}^m \]

\[ A \upharpoonright \omega \bar{\omega}^m \geq \bar{v} \quad (p^m = \inf \{ p^m | m < \omega \}) \]

This lemma gives us all the information we want. Otherwise
\[ \omega \bar{\omega}^m + 1 < \bar{v} \leq \omega \bar{\omega}^m \] and we must impose an additional condition.
Before we can show \( \Sigma_1^m \) preservation
Unless specified otherwise, we shall in the following regard $M^m$ as an abbreviation for $M^m <\nu_0,\ldots,\nu_{m-1} > = M^m <\nu,\ldots,\nu,\nu_{m-1} > (\nu_0 = \nu, \nu_{m-1} = 0)$.

**Def**: $\bar{M}$ in $\bar{v}$ is clear if

(a) There is a largest cardinal $\bar{\beta} \in \bar{v}$ (i.e. $\bar{v} = \bar{\beta} + \bar{v}$),

(b) $\exists \bar{\alpha} \in \bar{M} \exists \nu_0 < \bar{v} \leq \omega \bar{\alpha}$. Then there is $\bar{\alpha} \in \bar{M}$ s.t. $\bar{h}_{\bar{M}\bar{\alpha}} (\bar{\beta} \cup \{\bar{\nu}_0\})$ is cofinal in $\bar{v}$.

[Note By (a), $\bar{v} = \bar{\beta} + \bar{M}$ and $\omega \bar{\alpha} < \bar{v}$ is equivalent to $\omega \bar{\alpha} \leq \bar{\beta}$. Moreover $\bar{h}_{\bar{M}\bar{\alpha}} (\bar{\beta} \cup \{\bar{\nu}_0\}) \cap \bar{v}$ is cofinal in $\bar{v}$ if $\exists \bar{v} \in h_{\bar{M}\bar{\alpha}} (\bar{\beta} \cup \{\bar{\nu}_0\})$]

[Note: If (a) holds and $\bar{M}$ is a model, then (b) holds].

Let us also observe that $[<a, f>] = \bar{e}(f)(a)$ in $\bar{M}$, where $\bar{e} = (\bar{f})$ is defined in the obvious way (cf. §1.3 (a.2, 2)).
Lemma 3. Let $\bar{M}$ be $\bar{\nu}$-closed, where \[ \omega \nu^m < \bar{\nu} \leq \omega \nu^m . \]
Then

(a) $\tilde{\sigma} : \bar{M} \rightarrow \Sigma (\bar{\nu}) \bar{M}$ cofinally

(b) $\tilde{\sigma}(\bar{\nu}) = \nu$ \hspace{1cm} (Taking $\tilde{\sigma}(\omega \nu^m) = \omega \nu^m$)

(c) $\omega \nu^{m+1} < \nu \leq \omega \nu^m$

(d) $M \in \nu$-closed.

Proof.

We first prove (b). Let $\tilde{\sigma} = \tilde{\sigma}(f)(a) < \tilde{\sigma}(\bar{\nu})$ where $\langle a, f \rangle \in D$. We can assume w.l.o.g. that $\tilde{\sigma}(f) < \bar{\nu}$. But it follows readily that $\tilde{\sigma}(f) \in \bar{M}$ is not cofinal in $\bar{\nu}$. Hence, letting $\tilde{\sigma}(f) \in \gamma < \bar{\nu}$, we have $\tilde{\sigma}(f)(a) < \tilde{\sigma}(\gamma) < \nu$. Using (1),

It is then obvious that $\beta^+ M = \nu$, where $\tilde{\sigma}(\bar{\nu}) = \beta$. $\nu \leq \omega \nu^m$ lies,

Lemma 2. To see that $\omega \nu^{m+1} \leq \beta < \nu$,

we observe that $h_{\omega \nu^m} (\beta \cup \omega \nu^m) \leq \tilde{\sigma}'' h_{\omega \nu^m} (\beta \cup \omega \nu^m) \leq \tilde{\sigma}'' h_{\omega \nu^m} (\beta \cup \omega \nu^m)$, where
\( \bar{\sigma} \) witnesses the clarity of \( \bar{M} \), and \( \bar{\sigma} \in \Sigma^{(m)}_0 \) preserving. Hence \( \forall \bar{\sigma} \exists h^m \bar{\rho}_{\bar{m}}^p (\beta \cup \Sigma^p \bar{m}^3) \) is confinal in \( \forall \), preserving (c) and (d). To prove (a), observe that if \( H_m \in \bar{a} \) in the proof of Lemma 2 and \( A \) is defined by:

\[ \bar{\sigma}^{f_7} (H_m, \bar{a}) \rightarrow (H_m, A) \text{ confinal in } \Sigma_0, \]

Then \( A = A^m \cap H_m \) and

\[ h^m (\beta \cup \Sigma^p \bar{m}^3) \supseteq \bar{\sigma}^{f_7} h^m \bar{\rho}_{\bar{m}}^p (\beta \cup \Sigma^p \bar{m}^3) \]

\( h^m (\beta \cup \Sigma^p \bar{m}^3) \) is confinal in \( V \). Hence \( A \in \bar{M} \).

Hence \( \rho^m \subseteq \rho^m \) (\( \rho^m \) are in the proof of Lemma 2). But \( \bar{a} \) was proven in Lemma 2. QED (Lemma 3).

We now give a condition which can be used to prove \( \Sigma^{(m)}_7 \) preservation when \( \bar{\sigma} \)-clarity fails.
Def. \( \bar{M} \) is bound to \( \bar{v} \) (as witnessed by \( \bar{p} \)) iff either \( \bar{w}^{\bar{m}+1} \geq \bar{v} \) for all \( \bar{m} \) or

There is \( \bar{m} \) s.t. \( \bar{w}^{\bar{m}+1} < \bar{v} \leq \bar{w}^{\bar{m}} \)

and \( \bar{M} = \) the closure of \( \bar{v} \cup \bar{p}_{\bar{m}+3} \)

under good \( E^{(m)} \) form.

\[ \text{Note: The last condition is equivalent to} \]
\[ H_{\bar{m}}^n \subset h_{\bar{m}}h_{\bar{m}}^n (\bar{v} \cup \bar{p}_{\bar{m}+3}) \] and
\[ H_{\bar{m}}^n \subset h_{\bar{m}}h_{\bar{m}}^n (\bar{w}^{\bar{m}+1} \cup \bar{p}_{\bar{m}+3}) \] for \( \bar{m} < \bar{n} \).

\[ \text{Lemma 4: Let } \bar{M} \text{ be bound to } \bar{v} \text{, where} \]
\[ \bar{w}^{\bar{m}+1} < \bar{v} \leq \bar{w}^{\bar{m}} \text{. Then} \]

(a) \( \bar{v} : \bar{M} \rightarrow E^{(m)} \bar{M} \) s.t. fin.

(b) \( \bar{w}^{\bar{m}+1} \leq \bar{v} \)

(c) \( \bar{M} \) is bound to \( \bar{v} \) (in fact, \( \bar{M} \) in the closure of \( \bar{v} \cup \bar{p}_{\bar{m}+3} \) under good \( E^{(m)} \) form).

(\text{Note: We cannot infer } \bar{w}^{\bar{m}+1} < \bar{v} \text{.})
proof of Lemma 4.

We first show (a). Let $\rho^m$, $H^m$ be as in the proof of Lemma 1. At first, let us show:

Claim: $\rho^m = \omega^m$.

At $m = 0$ this is trivial. Let $m > 0$.

Define $A$ by:

$$\omega^m | H^m_M : M^{m+1} \to E_0$$

Then $<H^m, A> \leq H^m | \rho^m$, where $\rho = \omega^m | \rho^m$. By the downward extension of imbedding lemma.

There are $\hat{H}^m, \hat{\rho} \in E_0$ such that $\hat{H}^m | \rho^m = <H^m, A>$ and $\hat{\rho} \in F^{n}_{\infty}$. Moreover, there is $\pi: \hat{H}^m \to E_0$ such that $\pi | H^m = id$

and $\pi(\hat{\rho}) = \rho$. Since $\rho \in F^{n}_{\infty}$, there is $\hat{\omega} : \hat{M} \to E_0$ such that $\omega^m | H^m_M = \tilde{\pi} | H^m_M$ and $\omega^m(\hat{\rho}) = \tilde{\hat{\rho}}$.

But then $\omega(\hat{f}(a)) = \omega((f)(a))$ for $(a, f) \in D$. Hence $\pi \omega = \tilde{\omega}$ and $\pi$ is onto. Hence $\pi = id$, $\omega = \tilde{\omega}$ and $\hat{M} = M$. Q.E.D (a)
But then each \( x \in \mathcal{L}_M^\mu \) has the form \( \bar{\sigma}(f)(a) \), where \( f \in \mathcal{L}_M^\mu \) and \( a \in \bar{x}(\text{dom}(f)) \). Thus, if \( f = h(i, \langle \beta, \bar{r}^\mu \rangle) \) for some \( \beta < \nu' \), then
\[
\bar{\sigma}(f)(a) = h \left( j, \langle \beta, a \rangle, \bar{r}^\mu \right)
\]
for some \( j \). Hence \( \bar{M}^{\mu \nu'} \subseteq h(\nu \cup \{\beta\}) \) and (b), (c) hold.

QED (Lemma 4)

Finally, we note that if there is a \( \sigma^* : \overline{M} \rightarrow M^* \) which is \( \Sigma_0^{(m)} \) preserving for \( \overline{w} \geq \nu \) and let
\[
\sigma^* (\overline{a}) = \overline{\sigma} \quad \text{Then the well-foundedness of ID is ensured, since}
\]
there is \( k : \text{ID} \rightarrow M^* \) defined by
\[
k(\langle a, f \rangle) = \sigma^*(f)(a),
\]

Pursuing this line of reasoning, we easily get the interpolation Lemma.
Lemma 5 Let $\sigma^*: \overline{M} \rightarrow M^*$ for $\omega_{\overline{M}} \geq \overline{\nu}$ n.t. $\sigma^* \Gamma \sigma = \sigma$. Then

(a) The canonical completion
$$\overline{\sigma}: \overline{M} \rightarrow M$$
of $\sigma: \overline{A} \rightarrow \overline{A}$ exists.

(b) There is a unique $\overline{\pi}: \overline{M} \rightarrow \overline{\Sigma}_{\overline{\omega}}^m$

n.t. $\overline{\pi} \overline{\sigma} = \overline{\sigma}^*$ and $\overline{\pi}: \overline{M} \rightarrow \overline{M}^*$

for $\omega_{\overline{M}} \geq \overline{\nu}$, $\overline{\pi}$ is defined

by: $\overline{\pi} (\overline{\sigma} (f)(a)) = \overline{\sigma}^* (f)(a)$.

(c) If $\omega_{\overline{M}} < \overline{\nu} \leq \omega_{\overline{M}}$

$\overline{\sigma}: \overline{M} \rightarrow \overline{\Sigma}_{\overline{\omega}}^m M$ is finally, then

$\overline{\pi}: \overline{M} \rightarrow \overline{\Sigma}_{\overline{\omega}}^m M^*$. 