§3.3.3 The "Frequent Extension" Lemma

We now prove another theorem on the existence of imbedding extensions, showing that under certain conditions the statement "very frequently" well-founded.

Let $\alpha > \omega_1$ be regular. Let $\delta > \alpha$, $\bar{\delta} = \alpha$. Let $\mathcal{P} = \mathcal{A}^\lambda$ be a sufficiently large cardinal, let $A^\lambda$ be a normal set, $\alpha^\lambda = \alpha$.

Let $\mathcal{P}_\delta = \mathcal{J}_\delta^\lambda$, where $\mathcal{P}_\delta = \mathcal{P}$ and $\lambda^\delta < \delta^\lambda < \alpha$ for $\delta < \alpha$. Let commuting maps $\mathcal{Q}^\delta_\delta$ $(\delta < \delta < \alpha)$ be given such that:

(a) $\mathcal{Q}^\delta_\delta : \mathcal{P}_\delta \rightarrow \mathcal{P}_\delta$

(b) $\alpha^\delta = \text{crit}(\mathcal{Q}^\delta_\delta)$ $(\delta < \delta)$

(c) $\mathcal{P}_\lambda = \bigcup \mathcal{Q}^\delta_\delta$ for limit $\lambda$ $\lambda < \alpha$

Let $\Delta \subseteq \lambda$ be a stationary set of points of cofinality $> \omega_1$. For $\delta \in \Delta$ let $\lambda^\delta < \lambda^\delta < \delta^\delta$ and

Let $\lambda^\delta = \sup \{ \delta^\delta : \delta < \lambda^\delta \}$, $\delta^\delta = \delta^\delta$.
Set \( Q_3 = J A_3 \), \( Q^* = J A^* \),

Then \( \overline{Q_3} : Q_3 \rightarrow Q^* \) cofinally.

Suppose moreover that:

(d) \( A_3 \) has a largest cardinal \( \beta_3 \),
and that for each \( 3 \in S \) we have chosen a more \( M_3 \) which "end extends" \( Q_3 \) - i.e.,

(e) \( Q_3 = J E \) where \( M_3 = \langle J_8, F \rangle \)

(f) \( \nu_3 = \text{On} \cap M_3 \) or \( \nu_3 \) is regular in \( M_3 \).

We also assume:

(g) \( M_3 \) is bound to \( Q_3 \) (i.e.,

if \( \omega \rho^{n+1} < \nu_3 \leq \omega \rho^n \), then \( M_3 \in M^3_3 \)

The closure of some \( \nu_3 \in \Sigma \) under good \( \Sigma_4 \) functions.

For each \( 3 \in S \) we form \( D_3 \) in the attempt to produce a canonical extension of \( \overline{Q_3} : Q_3 \rightarrow Q^* \) with \( M_3 \).
Formal There is a cub $C \subseteq V_\delta$

$\textnormal{D}_3$ in well founded for $\exists \in C_3$.

The proof is rather lengthy and will require some sublemmas.

Suppose not. Let $X$ be minimal for counterexamples. Let $S' \subseteq S$ be stationary s.t. $\text{D}_3$ is not well founded for $\exists \in S'$. Let $\nu_3$ be chosen minimally for the existence of $M_3$ with this property and let $M_3$ be $C$-minimal (in the sense of §2.4) for this property at $\nu_3$.

Sublemma 1.1 Let $\exists \subseteq S$. Let $Z \subseteq M_3$ be countable. There is a sequence $q_i \in \text{D}_3 = \text{D}_3 \cup M_3$

Letting $X = \cap_{i = 0}^{\infty} q_i$.

(a) $Z \subseteq X$

(b) $\nu_3 \subseteq X$, $\nu_3 \subseteq X$ if $\nu_3 \subseteq M_3$

(c) There is an $\pi: N \rightarrow \langle X, \in, \text{En}_X, \text{Fin}_X \rangle$

such that $N$ is a move and

$\pi: N \rightarrow \text{M}_3\ (\text{where } M_3 = \langle \text{D}_3, \in \rangle$
Proof of Sublemma 1.1

First select \( \langle a_i, f_i \rangle \) (\( i \leq \omega \)) which witness the failure of well-foundedness in \( D_3 \) - i.e.

\[ \langle a_{i+1}, f_{i+1} \rangle \in \langle a_i, f_i \rangle \text{ in } D_3 \]

Pick \( p_i \in M_3 \) s.t. either \( f_i = p_i \in M_3 \) or \( f_i \) is a good \( \Sigma_1 \) \( (M_3) \) function in the parameter \( p_i \) (by a functionally absolute definition), where \( \beta_3 \leq \text{ord} \ M_3 \).

Assume (w.l.o.g.) that

\[ d_3 \beta_3 \beta p, \quad p_i \in \mathbb{Z} \quad (i \leq \omega) \quad \text{and} \]

\[ \nu_3 \in \mathbb{Z} \quad \text{if} \quad \nu_3 \in M_3. \]
Assume (w.l.o.g.) $\frac{1}{3}, \frac{1}{3}, P \in M_3, P \in Z (i < 0)$

We proceed by case as follows:

**Case 1** $\frac{1}{3} \leq p_n^m$ for all $n$.

Let $\langle F_i \mid i < \omega \rangle$ enumerate the fans which, for some $n$, are $\Sigma^m_1 (M_3)$ to $H^n$ (lightface), with a many repetitions of each fan. Let $\langle \xi_i \rangle$ enumerate $Z$.

Define $g_i$ by induction on $i$ as follows:

$g_i = \{ \langle \xi_i, 0 \rangle \}$

$g_{3i}$ is defined by

$g_{3i+1} (\langle s, s' \rangle) = F_i (g_{3i} (s), g_{3i} (s'))$

$g_{3i+2} = \text{id} | \tau_i$

where $\tau_i = \text{the least } \tau > \sup_{n < i} \tau_n$,
\[ \text{Note that } \text{dom}(g_i) \text{ is a bounded subset of } \nu_i \text{ for } i \leq \omega. \]

\[ \text{Let } X = \bigcap_i \text{dom}(g_i). \text{ Then} \]

\[ (1) \quad X \cap \nu_3 = \bigcap_i \text{dom}(g_i) = \bigcap_i 
\]

\[ \text{Set } \bar{X} = X \cap \nu_3. \]

\[ (2) \quad \bar{F}_X \cdot X^n \subset X \text{ for all } \text{q-val } \Sigma^k \]

\[ \text{functions.} \]

\[ \text{Set: } \bar{X} = \langle X, E_{\nu_3} \cap X, E_{\nu_3} \cap X \rangle. \]

\[ \text{Then:} \]

\[ (3) \quad \bar{X} \leq \Sigma^k \]

\[ \text{where } X^n \in \Sigma^k \]

\[ \text{taken as ranging over } X \cap H_{M_3} \cap X. \]

\[ \text{Let } \pi : N \rightarrow \bar{X} \text{ where } N = \langle X, E_{\nu_3}, E_{\nu_3} \rangle \]

\[ \text{is transitive. Then } \bar{E}_{\nu_3} = N \text{ and} \]

\[ (4) \quad \pi^{-1} \nabla = \text{id} \text{ and } \pi(\bar{V}) = \nu_3 \text{ if } V \in M_3 \]

\[ \text{Set: } H_n = \pi^{-1}(X \cap H_{M_3}). \]

\[ \text{Then } H_n = \bigcup \bar{E}_{\nu_3} \text{ where} \]

\[ \omega \rho^m = \text{on } \cap H_n. \text{ Clearly:} \]
\[
\pi : N \rightarrow \Sigma^* M_3, \text{ if } x^n \text{ is taken as ranging over } H_m \text{ in } N,
\]

Thus to prove that \( \pi \) is really \( \Sigma^* \)-preserving we must show:

Claim \( \rho^m = \rho^n \quad (1 \leq m < \omega) \)

\( \leq ) \text{ Let } A \text{ be } \Sigma^{(m-1)}_1 (N), \)
\[ y^m = A \land x^n \text{ in } \Sigma^{(m)}_0 (N) \text{ for } \sigma. \]
Since \( \sigma : N \rightarrow \Sigma^* M_3 \) under the above interpretation, we conclude that \( x \land A \in H_m \text{ for } x \in H_n \)

\( \geq ) \text{ Let } p = \rho_{M_3}^m, \overline{p} = \pi^{-1} (p) \)

Let \( A \) lie \( \Sigma^{(m-1)}_1 (N) \) in \( \overline{p} \) by the same definition as in \( p \) over \( M_3 \).

Then \( \overline{A} \in H_m \). Claim \( \overline{A} \notin N. \)

Suppose not. Let \( \overline{A} = x \). Then
\[ \overline{A} = x \land H_m \text{ is a } \Pi^m_1 (N) \text{ statement in } x, \overline{p}, \text{ hence the same statement holds in } M_3 \text{ in } \overline{\pi}(x), \overline{p}, \]
Hence \( A = \overline{\pi}(x), \overline{H}_m \in M_3 \).

Contra! \( \quad \text{QED (Claim)} \)
Thus \( \pi : N \to \mathbb{E}^* \) \( \Xi_3 \) and hence \( N \) is a monoid.

Set: \( \overline{Q} = \int_{\mathbb{E}^*} A \), \( \overline{\nu} = \sup \overline{\nu}_* \), \( \overline{Q} = \int_{\mathbb{E}^*} \).

\( \overline{\nu}_* = \overline{\nu} / \overline{Q} \). Then

(5) \( \overline{\nu} : \overline{Q} \to \overline{\nu}_* \) is finitely and

\( \overline{\nu} \) is regular in \( N \), where

\( \overline{Q} = \int_{\mathbb{E}^*} \).

Hence we can form \( \overline{ID} = \overline{ID}_\overline{\nu}, \overline{Q}, N \).

Let \( f_c \) be defined in \( \Xi_3 \) from \( \overline{P}_c = \pi^{-1}(p_c) \) on \( f_c \) was defined in \( \Xi_3 \) from \( p_c \). Then

(6) \( \pi(f_c) = f_c \).

(losing of commutation).

(7) \( \langle u, v \rangle \mid f_c(u) = \overline{f_c}(u) \rangle \}

But then \( \overline{ID} \) is not well founded.

By the minimal choice of \( \nu_3 \), it follows that:

(8) \( \overline{\nu} = \nu_3 \).

The remaining verification are

trivial \( \quad \Omega \quad \epsilon \quad \Xi_3 \) (Case 1)
Case 2 \( \omega \rho_{M^3}^q < \nu^3 \)

Let \( \forall y \exists \varphi_i (y, x, z) \) \((i < \omega)\) enumerate the \( \Sigma_0 \) formulas \( \omega \) in two free \( \omega \)-languages.

Define \( y_0, \tau_i \in \nu^3, i \leq \tau_0 \) by induction on \( i \):

\[
\begin{align*}
\gamma_{3i} &= \{ \langle z, 10 \rangle \} \quad \text{the}\ <M^3_{\gamma_3} - \text{least}\ \gamma_i + 1
\end{align*}
\]

\( \gamma_i \in SE_{M^3_{\gamma_3}} \) s.t.

\[
\varphi_i (y_1, y_2, (\tau_i, 0), (\tau_i, 1))
\]

where \( \gamma_i = \text{least } \gamma > \tau_i \ \text{p} \ \gamma_h \)

\( \text{with } z \in SE \) and \( E \cap SE \in SE \)

\( \tau_h \in SE \) \((h < i)\), where

\( M^3_{\gamma} = \langle \gamma, E, F \rangle \)

\( g_h = \{ \text{id} \} \cap \gamma \) \( \text{p} \) \( \tau_i \)

\( \text{with } \text{dom}(g_h) \subset \gamma \) and \( \text{rangen}(g_h) \subset \gamma \text{ for } h < i \).
Set $\omega^e = \omega^e \cdot \omega$. Then

$$M = \langle J^E, F \cap J^E \rangle$$

is a $J$-model $\bar{t}$, in fact, a premodel. Hence $M = M_3 \mid \gamma$ by the initial segment property. Hence $M$ is a model, but $f_i = p_i \in \bar{M}$ for $i < \omega$. Hence $D_{\omega_3} \bar{M}$ in not well-founded.

Hence $M = M_3$ and $\gamma = \omega$ by the $< \omega$-minimality of $M_3$.

Hence, letting $X = \bigcup \text{rng}(g_i)$, (1) $X \leq M_3$ cofinally.

Set $\bar{u} : N \to \bar{M}_3$, $\bar{u}(n) = \bigcup \text{rng}(g_i) = \bigcup \text{dom}(g_i)$. Then

$$\bar{t}(\bar{u}) = \bar{t}(\bar{u}) = \bigcup \text{rng}(g_i) = \bigcup \text{dom}(g_i).$$

$\bar{u}(\omega^e) = \bigcup \text{rng}(g_i) = \bigcup \text{dom}(g_i)$. 

Since $H^m = \bigcup_{\omega^e}^{\omega^e} \in J^E \leq N_1$,

we clearly have

(2) $\bar{t}(\bar{u}) = \bigcup \text{rng}(g_i) = \bigcup \text{dom}(g_i)$.

(3) $\bar{t} : N \to \bar{M}_3$, interpreting $x^m$ as ranging over $H^m$. 


Just as in the previous case we then get:

\[ \mathcal{F}^n_{M_x^3 \rightarrow N} \]

Hence we have \( N \rightarrow \mathcal{F}^n_{M_x^3} \) and \( N \) is in a monico. But, letting \( \bar{f}_c \) have \( \bar{f}_c(u) \in f_c(u) \), we have \( \{ \langle u, v \rangle \mid \bar{f}_c(u) \in f_c(u) \} \) and hence \( \bar{f}_c \) is not well founded.

Where \( \bar{f}_c \) is based on

\[ \bar{f} : \bigcup_{V}^{E} \rightarrow \bigcup_{V}^{E} \]

\( \bar{f} = \bar{f}_c \mid \bigcup_{V}^{E} \). By the minimal choice of \( V_x \), we conclude:

\[ (5) \mid V_x = \bar{f} \]

\( \text{QED (Case 2)} \)
Case 3 The above cases fail. Then \( \omega \rho^{m+\nu_3} < \nu_2 \leq \omega \rho^{m} \) for all \( m > 0 \).

Let \( \tau \in \mathcal{M}_{\mathcal{L}} \), with \( \mathcal{M}_{\mathcal{L}} \) a bound to \( \nu_2 \).

We combine the methods of the previous cases. Let \( \langle F_i \mid i < \omega \rangle \) enumerate the set of \( \Sigma^1_1 \mathcal{L} \) formulas of two variables for \( h = m \), with \( \omega \) many repetitions of each.

Let \( \langle V_y \varphi_i (y, x, z) \mid i < \omega \rangle \) enumerate the \( \Sigma^1_{\omega_1} \) formulas, again with \( \omega \) repetitions, but the \( \Sigma \mathcal{L} \). Shoenfield function \( h \mathcal{L} \) be uniformly defined by:

\[
y = h(i, x) \iff \forall z \forall i' (i, x, y, z) \quad (1)
\]

where \( \forall \in \Sigma_{\omega_1} \). We define \( \varphi_i, 
\tau_i < \nu_2, \varphi_i < \omega \rho^{m} \) by induction on \( i \) as follows:

\[
\varphi_i = \{ \langle z_i, \nu > \}
\]

\[
\varphi_{i+1} (x, z_i) = F_{\varphi_i} (x, \varphi_i (z), \varphi_i (z_i))
\]

\[
\varphi_{i+2} (x, z_i) = \text{the } \mathcal{L} \text{-least } y < \mathcal{E}_{\zeta_i} \mathcal{L} \text{ with } \varphi_i (x, y, z_i)
\]

\[
\text{Let } \mathcal{M}_{\mathcal{L}} \mathcal{L} = \{ y, \varphi_i (x, z_i), \varphi_i (z_i) \}
\]

where \( i \).
\[ z_0 = \text{the least } y > \sup_{h \leq c} \gamma \text{ s.t.} \]

(a) \( \text{for } h < c \) \( z_h \triangleq \nabla \)

(b) \( \forall j, V_p < \beta, \forall z \in J_C^E \quad \exists \, \gamma \in \mathcal{U}(j, z, 2, 8) \)

(c) \( \forall_j \forall p < \beta, \forall z \in J^E \quad \exists \, \gamma \in \mathcal{U}(j, z, 2, 8) \)

for all \( h < c \)

This is possible by our condition on \( z \).

\( h > \id_{\tilde{c}} \), where \( \tilde{c} = \text{the least } \)

\( \forall h \in \mathbb{N} \quad \text{dom}(y_h) \subset \mathcal{T} \text{ and } \gamma_h \)

\( \forall \exists \, \gamma \subset \text{range}(y_h) \subset J_C^E \text{ for } h < c \).
Set \( \psi_h^b = \text{Con} n + h, \quad (h \leq n) \)

(4) \( \psi_h^b = \psi_{\frac{h}{N}} \quad (h \leq n) \)

**Proof**,

For \( h < n \) exactly as in Case 1:

\( \psi^m \leq \psi^m_{\frac{N}{n}} \), since, letting \( \overline{A} = \)

\[ \prod_{i \in M_3} (A^{n, n} \cap J^E_i) \]

we have

\[ \langle H_n, \overline{A} \rangle \in \text{co-final} \text{ and} \]

\[ \prod_{i \in M_3} \langle H_n, \overline{A} \rangle \rightarrow \Xi_0, \quad M_3 \]

it follows easily that \( \overline{A} = A^{n, n} \cap H_n \), where \( \pi(x) = x \).

Finally, we note that

\[ \overline{\nu} \leq \psi_{\frac{N}{n}} (\beta_3^2) \]

where \( \overline{\nu} = \beta_3^N \). Hence

\[ \psi^m_{\frac{N}{n}} \leq \psi^m_n \]

\( \Box \)

The proof of (4) also shows:

(5) \( \psi^m_{\frac{N}{n}} < \overline{\nu} \leq \psi^m_{\frac{N}{n}} \) and

\( \overline{N} \in \overline{\nu} \)-closed (in the sense of § 3.3.2).
Then \( \langle H_n, \overline{A} \rangle \) is amenable as \( \pi \rightarrow \Pi H_n \rightarrow \langle H_n, \overline{A} \rangle \rightarrow \Xi_0 \rightarrow M_n \).

Furthermore, we find that \( \overline{A} = A^m \overline{N} \wedge \overline{H}_n \), where \( \overline{\alpha}(\overline{\alpha}) = r \). This proves \( \rho_n^m \leq \rho_n^\overline{A} \). But by construction, \( \overline{N} \ni h \langle H_n, \overline{A} \rangle \) is cofinal in \( \overline{N} \).

Hence \( \overline{A} \neg \in \overline{N} \), since \( \rho_n^m \geq \rho_n^\overline{A} \geq \overline{N} \), and \( \overline{N} = \beta^+ \overline{N} \). Hence \( \rho_n^m = \rho_n^\overline{A} \).

Q.E.D. (4).

The proof of (4) showed:

(5) \( \rho_n^m < \overline{N} \leq \rho_n^\overline{A} \) and

\( \overline{N} \in \overline{V} \) - clear (in the sense of §3.3.2).

But if \( \pi(\overline{g}_i) = \overline{g}_i \) (i < \( \omega \)), then by our construction, for each \( \overline{g}_i \), there \( \exists i \in h_\overline{N} \) \( (\overline{\beta^+}) \) such that \( \overline{g}_i = \overline{g}_i \) or \( \overline{g}_i \in \text{ a good } \Xi_1^{(2)}(N) \) function in \( \overline{N}, \overline{\alpha} \). Hence:

(6) \( \overline{N} \) is bounded to \( \overline{V} \) as witnessed by \( \overline{\alpha} \).
Now let \( \overline{G} = \bigcup_{n \geq 1} E_n \), \( G = \bigcup_{n \geq 1} E_n^* \), where \( E_n^* \subseteq E_n \). Then
\[
\pi_{1} \overline{G} : \overline{G} \to \overline{G} \text{ cofinally}.
\]
By the interpolation lemma there is \( \pi_{0} \overline{G} : \overline{G} \to \overline{G} \) cofinally extending \( \pi_{1} \overline{G} \) and \( \overline{G} \). Since \( \overline{G} \) is bounded to \( w_{m}^N \) we witnessed by \( \bar{\delta} \) (by (61)), we have:

(7) \( \pi_{0} \overline{G} : \overline{G} \to \overline{G} \) cofinally.

Moreover there is a unique \( \pi_{1} \overline{M} \) such:

(6) \( \pi_{1} \overline{M} : \overline{M} \to \overline{M} \) \( \overline{M} \in \mathcal{E}_0 \), \( \bar{\delta} \pi_{1} \overline{M} = \bar{\delta} \).

By (8), \( \pi : \overline{M} \to \overline{M} \) \( \overline{M} \in \mathcal{E}_1 \) hence by \( \S 2.2 \) Lemma 2 i

(9) \( \overline{M} \) is iterable above \( w_{m-1}^N \).

Straightforward use of the initial segment property gives:

(10) \( E_{\overline{M}} = E_{\overline{M}}^* \).

Hence \( \overline{N}, \overline{M} \) are iterable.

Let \( R \) be a common iterate of
of $N, M_3$ and a simple iterate of one.

If $R$ is a non-simple iterate of $M_3$, then $N$ is a measure and $N \preceq \times M_3$.

But by our construction

(11) $\sigma^t \omega_3 \longrightarrow \omega^t$ has no canonical extension unit $N$, contradicting the $\times$-minimality of $M_3$. If $R$ is a non-simple iterate of $N$, then, letting

$\sigma^t = \mu^{-1}(\sigma) = \mu^{-1}(\omega) - t$, we have:

$A_m^{\omega_n} = A_m^{\omega_n} \cap J_{E} \notin N$.

This is a contradiction, since

$\omega_n \in R^m \setminus N$ by § 3.3.2 Lemma 4.

Hence $R$ is a simple iterate of both $M_3$ and $N$, and in an iterate of $N$ above $\gamma = \omega_0^{\omega_n}$, Hence $\gamma = \omega_0^{\omega_n}$.

Hence, since $\pi = \pi_1 \pi_0$:

(12) $\overline{\pi} : \overline{N} \rightarrow \frac{\omega}{m} \times M_3$.

Finally, at follows from just as in Case 2 that

(13) $\overline{\pi} : \overline{N} \rightarrow \frac{\omega}{m} \times M_3$ and

$H^h_{\overline{N}} = H^h_{\overline{M}_3}$ for $h \not\preceq m$. 
Thus $\bar{N}$ is a universe. Obviously $\exists Y \subseteq X$ cannot be extended with $\bar{N}$. Hence, as before, 
$\tilde{V} = \nu_3$ by minimality.

QED (Sublemma 1.1)

Note that, by this analysis, $cf(\nu_3) = \omega$ for $z \in S$. Recall that we had defined $\bar{Q}_3 = \check{\bar{A}}_3$ where $\bar{P}_3 = \check{\bar{A}}_3$ and $\bar{A}_3 < \nu_3 \leq \bar{A}_3$; and $\bar{V}_3 = \check{\bar{V}}_3$. $\bar{Q}_3 \rightarrow \bar{Q}_3^{\omega_0}$ cofinal, where $\check{\bar{V}}_3 : \bar{P}_3 \rightarrow \bar{P}$, $\bar{P} = \bar{P}$.

We now get:

Sublemma 1.2. There is a cut $C \subset d$ not. for all $z \in C \cap S$, we have:
$\bar{Q}_3 = \bar{P}_3$ and $\bar{Q}_3^{\omega_0} = \bar{P}$ (hence $\bar{V}_3 = \check{\bar{V}}_3$ is cofinal and $\bar{V}_3(\beta_3) = \beta_3$).

$\text{M}$. We first show that $\nu_3 = \nu_3$ for $z \in C \cap S$, $C \subset d$ a cut not.

Suppose not. Then $\nu_3 < \nu_3$.
for $3 \in S'' \subset S'$, where $S''$ is stationary.

Then for $3 \in S''$ we have $\nu_3 = \frac{\omega}{\omega_3}(\nu)$, $\bar{3} < \omega_3$.

It follows easily that there is a stationary $S''''$ + fixed $\bar{3}$, $\nu$ a.t.

$\nu_3 = \frac{\omega}{\omega_3} (\nu)$ for $3 \in S''''$. Set:

$\nu^* = \frac{\omega}{\omega_3} (\nu_3) = \frac{\omega}{\omega_3} (\nu)$ for $3 \in S''''$. Then we can get a counterexample to Lemma 3 with $\nu^*$ in place of $\delta > \nu^*$, contradicting the minimality of $\delta$.

Thus $\nu_3 = \delta_3$, for $3 \in Cn S'$. But $\nu_3$ is co-cofinal, it follows that

$\frac{\omega}{\omega_3}$ is cofinal into $\omega_3$ for sufficiently large $3 < \omega$.

QED (Sublemma 1.2)

From now on assume w.l.o.g. that $Q \equiv P$, $Q^* \equiv P$ for $3 \in S'$.

We also note $Q = \nu^*$.
We also note that we may assume w.l.o.g. that

\[(*) \quad E^{M_3}_{\kappa^3} = \emptyset \quad \text{for} \quad \kappa^3 \in S'.\]

To see this, note that otherwise we may replace $M_3$ by $M'$ where $\kappa^3 \mapsto^* M'$. After $E^{M_3}_{\kappa^3}$, $\langle f_m | m < \omega \rangle$ witnesses the non well-foundedness of $D_{\kappa^3}, M_3$. Then $\text{Kof}_m \in M' (m < \omega)$ by standard methods (since $\text{dom}(f_m) \leq \kappa^3$ in $M_3$) and $\langle \text{Kof}_m | m < \omega \rangle$ witnesses the non well-foundedness of $D_{\kappa^3}, M_3'$.

From now on assume $(*)$. 
For $\mathfrak{s} \in S$, let $\langle y^3 \mid i < \omega \rangle$ be the sequence of Sublemma 1.1. (In particular, we suppose it to be defined as in the proof of Sublemma 1.1.) We recall that the def. of $\langle y^3 \rangle$ made use of a sequence $\langle f^3 \rangle$, $\langle a^3 \rangle$, which witnesses the mon- well-foundedness of $D^3_\mathfrak{s}$ i.e., $\langle a^3_i, f^3_i \rangle \in \langle a^3_i, f^3_i \rangle (i < \omega)$.

Let $\mathfrak{s}^* = \text{ the least } \mathfrak{s}^* > \mathfrak{s} \text{ s.t. } a^3 \in \text{ rng } (\mathfrak{s}^* \cdot d)$ for all $i < \omega$.

Then if $\mathfrak{s}, \mathfrak{s} \in S', \mathfrak{s}^* \leq \mathfrak{s}$, then $\mathfrak{s}^* : \mathfrak{s} \rightarrow \mathfrak{q}_\mathfrak{s}$ has no canonical extension w.t. $M^g_{\mathfrak{s}}$. Define a cut $C \subseteq d$ by:

$C = \text{ the set of limit pte } \mathfrak{s}$ of $\mathfrak{s}'$ s.t. $\mathfrak{s}^* < \mathfrak{s}$ for $\mathfrak{s} \in S'$.
Fix $x \in C \Lambda \subseteq S'$. We shall define a countable $W \subseteq Q_\Lambda$ and observe that $W \subseteq \text{rg } \left( \frac{8}{\Lambda} \right)$ for a $\exists \in S \forall \lambda$, since $\left( x \right) > \omega$. We then use this to derive a contradiction.

We shall use the machinery developed in the proof of Sublemma 3.1 and shall also consider the same three cases.

**Case 1** \[ \nu_\Lambda \leq \omega^m \] for all $m < \omega$.

For each $x \in X$ find a $q = q(x, i_1, \ldots, i_m)$ and each $i_1, \ldots, i_m < \omega$, let:

\[ T_{x, i} = \text{the set of } <x_1, \ldots, x_m> \text{ s.t. } \]

\[ x_h \in \text{dom}(g_h), \quad q(x_h) \in H_{\nu_\Lambda} \quad (h = 1, \ldots, m) \]

and $M_\Lambda \supseteq Q \left[ a_1, \ldots, a_m \right]$, where $<a_h>$ are the canonical maps developed in the proof of Sublemma 3.1.
Let $W$ be the set of all $t_{\phi,i}$ and suppose that $W \leq \text{rng} \ (\sigma_3, \chi)$ for a fixed $\xi \in S \setminus \chi$.

Set: $c_i = \text{dom}(g^i_c)$

$E_{i,j} = \{<x, y> \mid g^i_c(x) \in g^j_c(y)\}$

$I_{i,j} = \{<x, y> \mid x = y\}$

$E_i = \{x \mid g^i_c(x) \in E\}$, $F_i = \{x \mid g^i_c(x) \in F\}$

where $M_\chi = \langle E, F \rangle$. Then

$d_{c_i} \in E_{i,j}, I_{i,j}, E_i, F_i \in \text{rng} \ (\sigma_3, \chi)$.

Set: $\tilde{D} = \{<i, x> \mid x \in d_{c_i}\}$ and define $\tilde{e}, \tilde{I}, \tilde{E}, \tilde{F}$ on $\tilde{D}$ by:

$<i, x> \tilde{e} <i, y>$ if $<x, y> \in E_{i,j}$

Sim. for $\tilde{I}, \tilde{E}, \tilde{F}$. Set:

$\tilde{\Omega} = \langle \tilde{D}, \tilde{e}, \tilde{I}, \tilde{E}, \tilde{F} \rangle$.

Then $k: \tilde{\Omega} \rightarrow M_\chi$ where $k(<i, x>) = g^i_c(x)$.

Moreover, if we interpret $\sigma^n_m$ ranging over $\tilde{H}_m = \{<i, x> \mid g^i_c(x) \in H^\chi_m\}$ in $\tilde{\Omega}$, we have $k: \tilde{\Omega} \rightarrow \Sigma^* M_\chi$.
Now let $\overline{S}_3$ \((\overline{d}_i, \overline{e}_i, \overline{E}_i, \overline{F}_i) = \overline{d}_i, \overline{e}_i, \overline{E}_i, \overline{F}_i\) and define:

$$\overline{ID} = (\overline{d}, \overline{e}, \overline{I}, \overline{E}, \overline{F})$$

from $\overline{d}, \overline{e}, \overline{E}, \overline{F}$ as $\overline{ID}$ was defined from $d, e, E, F$. Define a pseudo satisfaction relation on $\overline{ID}$ by:

$$\models \phi([^i_1, x_1], \ldots, [^i_m, x_m]) \iff$$

$$\models x_1, \ldots, x_m \in \overline{t}_{\phi, i}$$

where \(\overline{S}_3 \lambda (\overline{t}_{\phi, i}) = \overline{t}_{\phi, i} \)\.

Using $\overline{S}_3 \lambda : Q_3 \rightarrow \overline{E}$ finally, it is easily seen that for $\overline{m}$,

$$\models (\overline{V} \overline{d} \phi)[\overline{z}] \iff$$

$$\iff \overline{V} x \in \overline{H}_j, \models \phi[^x, \overline{z}],$$

where $\overline{H}_j = \overline{S}_3 \lambda \overline{H}_j$. At follow
easily that $\sigma_3^\lambda : \overline{D} \rightarrow \overline{E} \\
$ letting $\nu^m$ range over $\overline{H}_m$, $\overline{H}_m$ resp. in $\overline{D}$, $\hat{D}$. [Note: We could in fact show that $\sigma_3^\lambda$ in $\Sigma^m_2$ preserving in this sense for $m < \omega$]. But then $\overline{D}$ is well formed and satisfies extensionality. Hence there is $\overline{h} : \overline{D} \rightarrow \overline{N}$, where $\overline{N}$ is transitive. Set: $\pi = k \sigma_3^\lambda \overline{h}^{-1}$.

Then $\pi : \overline{N} \rightarrow \overline{M}_3$, letting $\nu^m$ range over $\overline{H}_m = \overline{h}^{-1} \overline{H}_m$ in $\overline{N}$. Since $g_i = \text{id} \cup t_i \in \text{rng}(\sigma_3^\lambda)$, and $\nu_3 = g_i(0) \in \text{rng}(\pi)$ for some $i$, we have: $\nu_3 + 1 \in \overline{N}$ and $\pi(\nu_3) = \nu_3$, $\pi(\nu_3) = \sigma_3^\lambda \nu_3$. Hence $\overline{\theta}_3 = \overline{J}_3$, where $\overline{N} = \langle \overline{J}_3, \overline{E} \rangle$ and $\overline{\pi}(\overline{\sigma}_3) = \overline{\theta}_3$. Now let $H_m = \overline{J}_3^{\text{mth}}$.

Exactly as in the proof of
Sublemma 1.1 Case 1: we get \( \rho^m = \rho^m_N \) (\( m < \omega \)). Hence

\[ \pi \cdot \overline{N} \rightarrow_{\overline{M}} \sum \lambda \]  
and \( \overline{N} \) is a monic. Form \( D_{\overline{M}} \), \( \overline{N} \) in the usual way. Clearly there are \( f_m \in \overline{N} \) not \( \pi \cdot \overline{M} \) and \( \overline{N} \) hence \( \langle a_{m+1}, a_m \rangle \in \overline{M} \) for \( m < \omega \). Hence \( D_{\overline{M}}, \overline{N} \) is not well founded and \( \overline{M} \) has no extension with \( \overline{N} \).

Now coiterate \( \overline{N}, \overline{M} \) to \( \overline{N}', \overline{M}' \). The first point moved is \( \overline{M} \).

Hence, by a trivial argument, \( D_{\overline{M}}, \overline{N}' \) and \( D_{\overline{M}}, \overline{M}' \) are well founded. But then \( \overline{N}' \) is not a proper initial segment of \( \overline{M}' \) and \( \overline{M}' \) is a simple iterate.
of $M_3$, since otherwise $\bar{N} \leq^* M_3$, contradicting the $\leq^3$-minimality of $M_3$. But there is an iterate $N'$ of $M_\lambda$ and a map $\pi : \bar{N} \to N'$ s.t. $\pi' \pi^{-1} = \pi M_3 N' \pi$. Hence $\pi' M_3 M' : M_3 \to N'$ for some initial segment $M''$ of $N'$. Hence $\forall \lambda M_3$ is well founded, contradicting the def. of $C$. Contr. QED (Case 1).

Case 2 $\omega \psi_1 < \chi_0$.

$W = \{ \text{set of } t_{\psi_i} \text{ s.t. } \psi = \psi_0 \psi_1 \ldots \psi_n \}$ in a $\Sigma_0$-fmla. Define $\tilde{D}$, $k$ as before. Then $d_i : E_i \vdash I_i \vdash E_i$ if $E_i \in \text{ rng } (\bar{\varphi}_{t_{\chi}})$ + we can define $\tilde{D}$ as before. As before we get:

$$\forall \lambda M_3 \to \tilde{D} \in \Sigma_0.$$

But $\forall \lambda$ we
cofinal wrt. $\mathbb{E}$, since there are arb. large $\eta < \kappa$ s.t. $\gamma \in \mathbb{E}$

(M$\chi$ = $\langle \mathcal{U}^{E}_{\beta}, F \rangle$) for appropriate $\beta$, i.

Hence $\overline{\mathbb{E}}_{\beta} \rightarrow \mathbb{D}_{\beta}$. As before,

there is $\bar{b} : \overline{\mathbb{E}}_{\beta} \rightarrow \mathbb{N}$, $\mathbb{N}$ trans,

t hence $\overline{\pi} : \mathbb{N} \rightarrow M_{\chi}$ cofinally,

where $\overline{\pi} = \bar{b} \overline{\mathbb{E}}_{\beta}^{-1}$. As before,

$\mathbb{Q} \subseteq \mathbb{N}$, $\overline{\pi}(\mathbb{Q}) = \overline{\mathbb{E}}_{\beta}$, and

$\overline{\pi}(\gamma) = \gamma$. As before, $\mathbb{D}_{\overline{\mathbb{E}}_{\beta}}, \mathbb{N}$ in

not well founded. We can

then form the ($\ast, 0$) - coiteration

of $\mathbb{M}_{\beta}, \mathbb{N}$, terminating in $\mathbb{M}', \mathbb{N}'$.

Exactly as before, $\mathbb{M}'$ is a segment

of $\mathbb{N}'$ + a simple iterate of $\mathbb{M}_{\beta}$,

since otherwise $\mathbb{N}' <_{\ast} \mathbb{M}'$. But this

yields a contradiction exactly

as before. QED (Case 2)
Case 3 \( \omega \rho^m + 1 < \nu_\lambda \leq \omega \rho^m \) \( (m > 0) \)

Again let \( W \) be the set consisting of \( \varphi, i \) fin \( \Sigma_1^{(m)} \) formulas \( \varphi = \chi(\vec{y}) \) and \( i < \omega \). Define \( \overline{\Delta}, \overline{D} \) as before. Then \( \varphi_3 \lambda \downarrow \overline{D} : \overline{D} \to \Sigma_1^{(m)} \overline{D} \)

interpreting \( \nu^h \) by the appropriate \( \overline{\nu}_h, \overline{H}_h \) in \( \overline{D}, \overline{D} \) resp. Again we get \( \overline{\pi} : \overline{D} \rightarrow \overline{N} \), \( \overline{N} \) transitive, and define \( \overline{\pi} = \varphi_3 \lambda H^{-1} \). Then \( \pi : \overline{N} \rightarrow \Sigma_1^{(m)} M_\lambda \) finally,

with \( \varphi^h \), \( h \) ranging over \( H_h = \pi^{-1} \varphi^h \) in \( \overline{N} \). Again we have \( \varphi \leq \overline{N}, \overline{\pi} \varphi = \varphi_3 \lambda \).

Letting \( H_h = \beta_\overline{E} \)

\( (\overline{N} = \{ \beta_\overline{E} \}, \overline{F}) \), we get \( \rho^m = \rho^m_N \) by the methods of Sublemma 3.1 Case 3. We also get \( \overline{N} \in \nu_{\overline{E}} \) - closed and bounded to \( \nu_{\overline{E}} \).
Hence \( \omega \beta_3 \leq \beta_3 < \nu_3 \) and \( \tilde{N} \) is iterable above \( \beta_3 \) by §2.2 Lemma 2. We coiterate \( \tilde{N}, M_3 \) and derive a contradiction as before.

QED (Lemma 1)

We now improve Lemma 1 to Lemma 2. Let \( \langle Q_3 \rangle, \langle M_3 \rangle \) etc.

be as in Lemma 1. There is a cube \( C \subset A \) s.t. for all \( 3 \in C \) we have:

(a) The canonical extension

\[ \overline{Q}_3 : M_3 \to M_3^* \text{ of } \overline{Q}_3 \text{ s.t. } M_3 \]

exists.

(b) \( M_3^* \) is iterable above \( \nu_3^* \).

The proof is virtually the same, once we have developed a small amount of additional machinery. Hence we first develop this...
machinery and then briefly sketch
the modifications to be made.

Suppose we are given a premonit
M and an iteration \( \langle M_i, i < \Theta \rangle \) wit
indices \( \langle \nu_i, \lambda_i \rangle \) and maps \( \pi_i \).
Now let \( \sigma : \Sigma \rightarrow M \). There is
a canonical way of trying to "mirror" the iteration of
M by an iteration \( \langle \overline{M}_j \rangle \) of
\( \overline{M} \) with indices \( \langle \overline{\nu}_i, \overline{\lambda}_i \rangle \) and
maps \( \overline{\pi}_i \), simultaneously
producing embeddings
\( \overline{\sigma} : \overline{M}_i \rightarrow \Sigma \times M \) set \( \overline{\sigma} \overline{\pi}_i = \overline{\pi}_i \overline{\sigma}_i \).

The definitions are as follows:

\( \overline{\sigma}_0 = \sigma. \) Now let \( \overline{M}_i, \overline{\sigma}_i \) be
defined. We attempt to
define \( \overline{\nu}_i, \overline{\lambda}_i \) by
\[ \overline{V}_i = \begin{cases} 0 & \text{if } \overline{E}_{\overline{K}_i} = \emptyset \\ \sigma_i^{-1}(\overline{K}_i) + \overline{M}_i & \text{if } \text{not} \end{cases} \]

\[ \overline{w}_i = \begin{cases} \text{On} \cap \overline{M}_i & \text{if } \overline{w}_i = \text{On} \cap \overline{M}_i \\ \sigma_i^{-1}(\overline{w}_i) & \text{if } \text{not} \end{cases} \]

This defines \( \overline{M}_{i+1}, \overline{\pi}_{i,i+1} \) and we define \( \overline{\sigma}_{i+1} \) by:

\[ \overline{\sigma}_{i+1} = \overline{\sigma}_i \cap \overline{M}_{i+1} \text{ if } \overline{E}_{\overline{K}_i} = \emptyset, \]

otherwise:

\[ \overline{\sigma}_{i+1}(\overline{\pi}_{i,i+1}(f(\overline{x}_i))) = \overline{\pi}_{i,i+1}(f(\overline{x}_i)) \]

for \( f \in \Gamma(\overline{\pi}_{i,i+1} \overline{d}_i) \).

For limit \( \lambda \) define \( \overline{\sigma}_\lambda \) by:

\[ \overline{\sigma}_\lambda \overline{\pi}_\lambda = \overline{\pi}_\lambda \overline{\sigma}_\lambda \]

At this point, it is clear that this definition can break down only at a successor stage. We now
formulate a condition which is sufficient to prevent breakdown.

Def Let $\mathcal{M}, \langle M_i | i < \theta \rangle$ etc. be as above. We assign to each $x \in M_i$ an element $u_i(x)$ of $M_i$ such that

$$\pi_i(x) = x \mapsto u_i(x) = u_h(x).$$

We define $u_i$ by induction on $i$ as follows:

$$u_0(x) = x, \quad \text{for } x \in M,$$

$$u_{i+1}(x) = u_i(x) \quad \text{if } \pi_{i,i+1}(x) = x.$$

If there is no such $x$, let $x = \pi_{i,i+1}(f(M_i))$, where $f \in \Gamma(M_i, M_i)$. Pick $p \in t$, either $f = p \in M_i$ or else $f$ is a good $\Sigma^m_1(M_i, 1)$ function in $p$ (by a functionally abs. def.) for $n$ s.t. $\kappa_i < \omega^{m+1}_n \cdot M_i(\kappa_i)$. 

\[\]
Set: \( u_{i+1}(x) = u_i(p) \).

For limit \( \lambda < \Theta \) set \( u_\lambda(x) = u_\lambda(\bar{x}) \) for \( i < \lambda \) set \( u_\lambda(x) = \bar{x} \).

This completes the definition.

*Sublemma 2.0* Let \( M_i, \langle M_i \rangle, \bar{M}_i, \sigma \) etc. be as above. Assume:

(a) \( u_i(x_i) \in \text{rung}(\sigma_i) \) for \( E_{M_i} \neq \emptyset \)

(b) \( u_i(\omega x_i) \in \text{rung}(\sigma) \) for \( \omega x_i \in M_i \).

Then \( \bar{M}_i, \sigma_i \) are defined for \( i < \Theta \). Moreover:

\( u_i(x) \in \text{rung}(\sigma) \rightarrow x \in \text{rung}(\sigma_i) \) for \( x \in M_i, i < \Theta \).

**Proof.** Induction on \( i \).

The details are left to the reader.
Note that the construction of the "mirror" iteration \( \langle \overline{\mu}_i \rangle \) and the maps \( \overline{\sigma}_i \) goes thinner under somewhat weaker assumptions than \( \overline{\sigma} : \overline{M} \to M \). We can replace this by:

\((\star)\) \( \overline{\sigma} : \overline{M} \to M \) and \( \overline{\nu} \in \overline{M} \) in

(a) \( \overline{\sigma}(\overline{\nu}) = \nu \)

(b) \( \overline{\nu} \leq \omega \rho^m \) \( \bigwedge \nu \leq \omega \rho^m \) \( (m < \omega) \)

(c) \( \overline{\sigma} : \overline{M} \to M \) whenever \( \nu \leq \omega \rho^m \)

\( \overline{M} \), \( \overline{\sigma}_i \) are then defined as before, with \( \overline{\sigma}_i : \overline{M}_i \to M_i \) whenever \( \nu \leq \omega \rho^m \) \( (\text{This is just as in} \ S2.2 \text{Lemma 2}) \). Sublemma 2.10 continues to hold.

Now suppose Lemma 2 to be false and let \( \langle Q_3 \overline{13} \overline{5} \rangle, \langle M_3 \overline{13} \overline{5} \rangle \), etc. be a counterexample.

Let \( 8 \) be chosen minimally, as before.
By Lemma 1 we may suppose £.d.o.g. that \( \sigma_3 : \omega_3 \to \omega_3 \) has a canonical extension \( \sigma_3^* : M_3 \to M_3^* \) with \( M_3^* \) for \( \mathfrak{S} \in S \). Let \( S \subseteq S' \) be stationary s.t. \( M_3^* \) is not iterable above \( \nu_3^* \) for \( \mathfrak{S} \in S' \).

We let \( \nu_3 \) be chosen minimally for \( M_3 \) with this property and \( M_3 \) be \( <_\ast \) minimal for this property at \( \nu_3 \). As before:

**Sublemma 2.1** Let \( \mathfrak{S} \in S' \). Let \( \mathcal{Z} \subseteq M_3 \) be countable. There is a sequence \( q \in \mathcal{P}_{\nu_3} = \Pi_{\nu_3} \mathcal{M}_3 \) (\( \nu_3 < \omega_3 \)) s.t.

\[(a) \quad \mathcal{Z} \subseteq X \]
\[(b) \quad \nu_3 + 1 \subseteq X \]
\[(c) \quad \text{There is } \bar{\pi} : N \overset{\sim}{\to} (X, \in, E_\mathcal{N}X, E_{\mathcal{N}}X, \text{t.}, N \text{ in a mouse and } \bar{\pi} : N \to M_3 \text{ (where } M_3 = \langle \mathcal{D}_\mathcal{N}, \mathcal{I} \rangle \rangle).\]

The proof is a straightforward modification of that of Sublemma 1.1. An place of
The sequence \( \langle a_i, f_i \rangle \) which formed a counterexample to well-foundedness of \( \mathcal{F}_2 \); we use a sequence \( n_i = \tilde{\mathcal{F}}_2 (f_i) (a_i) \) which witnesses - in a manner still to be specified - the noniterability of \( \mathcal{M}^* \).

\( \langle a_i | i < \omega \rangle \) is the enumeration of a countable set \( R < \mathcal{M}^* \) which is defined as follows:

Let \( \langle \tilde{M}_i | i < \Theta \rangle \) be the iteration of \( \mathcal{M}_2^* \) above \( \mathcal{F}_2^* \) which cannot be continued. Let \( \langle \tilde{\nu}_i, \tilde{\xi}_i \rangle \) be the indices of this iteration + \( \tilde{\mu}_i \) the maps.

\[ \text{Case 1 } \Theta = \gamma + 1 \]

Then there is \( \omega \tilde{\nu} \leq \omega \alpha_i \tilde{\mu} \) and \( \tilde{\nu} = \tilde{\nu} + \tilde{\mu} \) with \( \tilde{E} = E_{\tilde{\mu}} \neq \emptyset \) and \( \tilde{\mu} \) is not \( \approx \)-extendable by \( \tilde{E} \).
In particular there are \( q_i \in \Gamma (\tilde{\nu}, \tilde{M}) \)

\( \text{a.t. } \{ q_i \in q_i (\tilde{\nu}) \} \in E \) \( \text{for } i < \omega \).

Pick \( p_i \in \tilde{M} \) a.t.

either \( p_i = q_i \in \tilde{M} \) or else in

a ground \( \Sigma_1^1 (\tilde{M}) \) map \( \tilde{\nu} \) in \( \tilde{p}_i \) \( \text{by a func. abs. def.} \) \( \forall \tilde{S} \subseteq \tilde{\nu} \)

\( \tilde{\alpha} = \tilde{\gamma}_0 \), \( \tilde{\nu} = \tilde{\nu}_0 \), \( \tilde{\nu} = \tilde{\nu}_0 \).

Let \( R = \) the not containing:

(a) \( u_i (\tilde{\alpha}_i) \) for \( i < \omega \) a.t. \( E_{\tilde{p}_i}^{\tilde{\nu}_i} \neq \emptyset \)

(b) \( u_i (\tilde{\omega}_i) \) for \( i < \omega \) a.t. \( \tilde{\omega}_i \in \tilde{M}_i \)

(c) \( u_{\tilde{p}_i} (p_i) \) \( \text{for } i < \omega \).

Case 2: \( \text{fin. (B)} \).

Then there is a monotone sequence \( \langle \tilde{\nu}_m \mid m < \omega \rangle \) and \( \tilde{\nu}_m \in \tilde{M}_m \).

\( \text{a.t. } \tilde{\nu}_{m+1} \in E_{\tilde{p}_m}^{\tilde{\nu}_m} (\tilde{\nu}_m) \) \( \text{for } m < \omega \).

Let \( R = \) the not containing:

(a) \( u_i (\tilde{\alpha}_i) \) for \( i < \Theta \) a.t. \( E_{\tilde{p}_i}^{\tilde{\nu}_i} \neq \emptyset \)

(b) \( u_i (\tilde{\omega}_i) \) \( \Rightarrow \) a.t. \( \tilde{\omega}_i \in \tilde{M}_i \)

(c) \( u_{\tilde{p}_m} (\tilde{\nu}_m) \) \( \text{for } m < \omega \).
This gives us $\tau_i = \sigma^* (f_i) (\alpha_i) \langle i < \omega \rangle$.

We again pick $p \in M_3$ s.t. within
$p_i = f_i \in M_3 \wedge f_i \in \text{a good } \Xi_{\alpha_i}^{\omega_i}(M_3)$
in $p_i$ (by a fancier abuse of def.) where $i \leq \omega P^{\omega_i+1}$. We make the previous assumptions on $Z$ (in particular $d_3 \uparrow \omega_3 \uparrow P_3 \uparrow P \in \text{Z (i < \omega)}$) and construct the four $g_i$ (i < \omega) exactly as before, using the same three cases.

We then repeat the proof of Lemma 1.1, replacing non-well-foundedness by non-intercalability in a rather mechanical way. We exemplify this by sketching the changes to be made in Case 1.

We form $X = \bigcup_{i < \omega} (g_i)$

and \( \bar{X} = \langle X, \in, E\cap X, F\cap X \rangle \)
\( (M = \langle \psi, F \rangle \) as before and
form \( \pi : N \xrightarrow{\cong} X \) as before.

Exactly as before we prove that
\( \pi : N \rightarrow X \) + hence that \( N \) is
a monic. We again set:

\[
\bar{\nu} = X \cap \nu_3 = \bigcap_{i \in \text{dom}(g_i)} \psi_i \nu_i .
\]

As before:

1. \( \pi^{\dagger} \bar{\nu} = 1d, \quad \pi (\bar{\nu}) = \nu. \)

We wish to show: \( \bar{\nu} = \nu. \)

Set:

\[
\bar{\alpha} = \int_{\bar{\nu}} A_3, \quad \bar{\nu}^* = \nu - \sigma_3^{-1} \bar{\nu},
\]

\[
\bar{\alpha}^* = \int_{\bar{\nu}^*} A, \quad \bar{\sigma}_3 = \bar{\sigma}_3 \upharpoonright \bar{\alpha}, \quad \text{then:}
\]

2. \( \bar{\tau} : \bar{\alpha} \rightarrow_{\Sigma_3} \bar{\alpha}^* \) cofinally and
\( \bar{\tau} \) is regular in \( N, \quad \bar{\alpha} = \int_{\bar{\nu}} E N. \)

But \( \sigma_3^* \pi : N \rightarrow_{\Sigma_3} M_3^* \) and

\[
\sigma_3^* \pi \bar{\alpha} = \bar{\sigma}_3 , \quad \text{Hence by the interpolation lemma } \bar{\sigma}_3 \text{ has a}
\]
canonical extension \( \bar{\sigma}^* : N \rightarrow \mathbf{N}^* \)

\( \mathbf{N}. \) Moreover there is
\[ \pi^* : N^* \to \sum_i M_i^* \text{ s.t. } \pi^* \circ^* = \circ_3^* \pi. \]

\[
\begin{array}{c}
N^* \xrightarrow{\pi^*} M_i^* \\
\circ^*_i \uparrow \quad \circ_3^* \uparrow \\
N \xrightarrow{} M_i^* \\
\end{array}
\]

But by our construction:

\[ \tilde{\alpha}_i^* = \circ^* (f_i^*) (a_i^*) \in N^* \quad (i < \omega) \]

where \( \pi^* (\tilde{\alpha}_i^*) = \tilde{\alpha}_i^* \). Hence there is an iteration \( \langle \tilde{N}_i \mid i < \theta \rangle \) of \( N^* + \sum_i \tilde{N}_i \to \sum _i M_i \) as given by Sublemma 1.0. But, using Sublemma 1.0, our construction ensures that \( \langle \tilde{N}_i \rangle \) cannot be continued. Hence \( N^* \) is not iterable above \( \tilde{V}^* \).

Hence \( \tilde{V} = V_3 \) by the minimality of \( V_3 \). QED (Case 1)

The other cases are similar but require

QED (Sublemma 2.1)
We again have: \((v_3) = \omega\) for \(3 \in S'\).

Arguing exactly as before:

Sublemma 2.2 There is a cut \(C \subset \delta\)
\(\text{s.t. for all } 3 \in C \cap S', \text{ we have}\)
\(Q^*_3 = P_3 \quad \text{and} \quad Q^*_3 = P_3 \quad \text{(hence} \quad \overline{\delta}_3 = \overline{\delta}_3 \quad \text{is cofinal)}).

From now on assume w.l.o.g.,
\(Q^*_3 = P_3 \quad \text{and} \quad Q^*_3 = Q = \omega \cdot P \quad \text{for } 3 \in S'.\)

Without loss of generality we also assume:
\[(**): E_{v_3}^{M_3} = \emptyset \quad \text{for } 3 \in S'.\]

To see this, note that we may replace \(M_3\) by \(M'\) where
\(k: M_3 \rightarrow M', \quad \text{let } \sigma^*: M' \rightarrow M'^*\)
be the canonical extension of \(\overline{v}_3\)
\(\text{w.r.t. } M'^*\). Define \(k^*: M^*_3 \rightarrow M'^*\)
by:
\(k^*(\overline{v}_3(f)(a)) = \sigma^*(k \circ f)(a),\)
Then \(k^* \overline{v}_3 = \sigma^* k\). At is straightforward.
forward (e.g. by the remark after §2.3(a) or 2.2) to prove that \( k^* : M_3^* \rightarrow E_3^* \).

But then the non-iterability of \( M_3^* \) above \( \gamma_3^* \) implies the non-iterability of \( M_1^* \) above \( k^*(\gamma_3^*) \).

For \( \bar{s} \in S' \) let \( < f_i^3 \mid i < \omega > \), \( < a_i^3 \mid i < \omega > \) be the sequences witnessing the non-iterability of \( M_3^* \). Let \( \bar{s}^* = \text{the least } \bar{s}^* > \bar{s} \text{ not } \)

\( a_i^3 \in \text{rng}\left( \overline{f_i^3}^{\bar{s}^*} \right) \) \( \text{ for all } i < \omega. \)

If \( \bar{s}, \bar{s}' \in S' \), \( \bar{s}' \leq \bar{s} \) \( \text{ then the canonical extension of } \overline{f_3^3} : \mathcal{A}_3 \rightarrow \mathcal{A}_3 \text{ at } M_3^* \) is not iterable above \( \gamma_3^* \), as can be seen using Sublemma 2.2.

Define a cub \( C \subseteq \mathbb{D} \) by:
$C$ = the set of limit points of $S'$
not $S^* < x$ for $z \in S' \setminus \lambda$.

Fix $x \in C \cap S'$. We define a countable $WCA \lambda$ exactly as in
the proof of Lemma 1 and
pick $z \in S' \setminus \lambda$ not. $WCA \lambda (z, \lambda)$
We derive a contradiction
exactly as before.

QED (Lemma 2)

This proof shows more than we claimed. Recalling our justification of the assumption
(**), we have:

Corollary 3. There is a cub $C \subseteq$
not $z \in C \cap S'$ we have (a), (b) and:
(c) $M^k_3$ is iterable above $\beta = \beta_+$.
Henceforth we refer to the conjunction of Lemmas 1, 2, 3 as the "frequent extension lemma".