Def. \( U \subseteq \mathcal{P}(\kappa) \) is suitable for \( \kappa \) if

(a) \( \kappa \) is p.r. closed,

(b) Let \( A_1, \ldots, A_n \in U \), \( \kappa_1, \ldots, \kappa_m \in \kappa \), let \( B \in \text{On}^{<\kappa} \) be p.r. in predicates \( A_1, \ldots, A_n \) and the parameters \( \kappa_1, \ldots, \kappa_m \in \kappa \).

Then \( B \cap \kappa \in U \).

It follows that if \( U \) is suitable, then:

(i) \( \kappa \in U \)

(ii) \( A, B \in U \implies A \land B, A \cup B, A \setminus B \in U \)

(iii) \( \kappa \leq \kappa \implies \kappa, \exists \forall \exists \in U \).

If we set \( U^m = \{ A^m \mid A \in U \} \), where \( A^m = \{ \langle \xi_1, \ldots, \xi_m \rangle \mid \langle \xi_1, \ldots, \xi_m \rangle \in A \} \),

then it follows from (b) that if \( A_1, \ldots, A_n \in U \) and \( B \in \text{On}^{<\kappa} \) is p.r. in \( A_1, \ldots, A_n \) and the parameters from \( \kappa \),

then \( B \cap \kappa^m \in U^m \).
Define \( U \subseteq \mathcal{P}(\kappa) \) be suitable for \( \kappa \).
Let \( \kappa < \lambda \), where \( \lambda \) is p.r. closed.

\[
F : U \rightarrow \mathcal{P}(\lambda) \text{ is an extender on } U \\
\text{with length } \lambda \text{ iff whenever } B_{A_1, \ldots, A_n} \subseteq \text{On in uniformly p.r. in } A_1, \ldots, A_n \in U \text{ and parameters from } \tau.
\]

Then \( F(\kappa \cap B_{A_1}) = \kappa \cap B_{F(A_1)} \).

We get:

(i) \( F(\kappa) = \kappa \), \( F(\{\in\}) = \in \) for \( \kappa \in \lambda \)

(ii) \( F(\emptyset) = \emptyset \)

(iii) \( F(A \cap B) = F(A) \cap F(B) \), \( F(A \cup B) = F(A) \cup F(B) \)

(\( F(A \setminus B) = F(A) \setminus F(B) \))

(iv) \( A \subseteq B \Leftrightarrow F(A) \subseteq F(B) \)

(v) \( \kappa \cap F(A) = A \), since

\( \nu \in A \Leftrightarrow \{\in\} \subseteq A \Leftrightarrow \{\in\} \subseteq F(A) \Leftrightarrow \in \notin F \)

We can extend \( F \) to \( F'' \) on \( U'' \) by

\[ F''(A'') = F(A)'' \]
We write $F(A^1)$ for $F^n(A^1)$ when $A \in U^n$. Then:

(i) $F(\kappa A^1 B_A^{\alpha}) = \kappa B_{F(A^1)}$ if $B_A^{\alpha} \subset \alpha^n$ in $\kappa$ with $A \in A_1 \ldots A_n \in \Gamma$ and parameters from $\alpha$.

(ii) $F(\nu_1 \times \ldots \times \nu_n) = \nu_1 \times \ldots \times \nu_n$ and $F(\langle \nu_1, \ldots, \nu_n \rangle^3) = \langle \nu_1, \ldots, \nu_n \rangle^3$ for $\nu_1, \ldots, \nu_n < \kappa$. 
(Note that $F^m = \mathcal{E}(\lambda_1, \ldots, \lambda_m)^3 = \mathcal{E}$ for $\lambda_1, \ldots, \lambda_m < \lambda$.)

**Def.** Let $F$ be an extender on $U$ of length $\lambda$. $\mathcal{E} = \langle F_\alpha | \alpha < \lambda \rangle$ is the associated hypermeasure where $F_\alpha = \{x | x \in F(x)\}$.

**Def.** $F = \langle F_\alpha | \alpha < \lambda \rangle$ is a hypermeasure iff it is associated with an extender $\mathcal{E}$ of length $\lambda$ and $F$ is an extender, where $F(x) = \{d | x \in F_d\}$.

**Def.** $\mathcal{E} = \langle 1M_1, \mathcal{E}, A_1, \ldots, A_m \rangle$ is a suitable hypermeasure $\mathcal{E}$, $M$ is transitive, and closed and $\mathcal{E} A_1 \times \cdots \times A_m \subseteq M$ whenever $A_1, \ldots, A_m \subseteq \mathcal{E}(\kappa)$.

**Def.** $M$ is suitable for $\kappa$ iff $M$ is suitable, $\kappa \in M$ is p.r., closed and $\mathcal{E} A_1 \times \cdots \times A_m \subseteq M$ whenever $A_1, \ldots, A_m \subseteq \mathcal{E}(\kappa) \cap M$. (Hence $U = \mathcal{E}(\kappa) \cap M$ is suitable.)
Def Let \( M \) be suitable.
\[ \pi : M \to N \quad \text{iff} \quad \]
(a) \( N \) is transitive
(b) \( \pi : M \to N \) is finally
(c) \( \kappa = \text{crit}(\pi) \), where \( M \) is suitable for \( \kappa \)
(d) \( F = \{ \chi \cap \pi(X) \mid X \in \mathcal{P}(\kappa) \cap M \} \), where \( \kappa < \chi \leq \pi(\kappa) \), \( \chi \) is p.n. closed, and
(e) \( N = \text{the } \Sigma_0 \text{ closure of } \text{any } (\pi(\kappa)) \cup \chi \).
(Equivalently to (e):
\[ N = \{ \pi(f)(x) \mid \exists \lambda, f : \kappa \to M, f \in M \} \]

Lemma 1 \( \Delta : \pi : M \to N, \kappa = \text{crit}(\pi) \),
Then \( F \) is an extender of length \( \chi = F(\kappa) \) on \( \mathcal{P}(\kappa) \cap M \) and \( N, \pi \) are uniquely determined by \( F \).

1. \( F \mid \kappa = \text{id} \), \( F(\kappa) = \lambda \)
2. \( F(\kappa \cap B_{A_1, \ldots, A_m}) = \chi \cap B_{F(A_1), \ldots, F(A_m)} \)
   for \( B \) and \( \chi \) p.n. in \( A_1, \ldots, A_m \in \mathcal{P}(\kappa) \cap M \),
   \( \chi \in \mathcal{P}(\kappa) \cap B_{\pi}(A) \)

and $\lambda \mapsto B_{\pi(A)} = \lambda \mapsto B_{\lambda \in \pi(A)}$ so

$= \lambda \mapsto B_{\pi(A)}$, since $\lambda \in$ p.s. closed. QED (2)

Hence $F$ is an extender. For $\Sigma_0$ formulas $\phi$ we have for Thm 1:

(4) $\{ \phi(F(\xi^2) \in \kappa^m | M \models \phi\} \in \mathcal{F}$

$= \{ \phi(F(\xi^2) \in \kappa^m | N \models \phi(\pi(f_1(\xi)), \ldots, \pi(f_m(\xi))) \}

= \{ \phi(\pi(f_1(\xi)), \ldots, \pi(f_m(\xi))) \}

$ \text{for } f_i : \kappa \to M, f \in M (i = 1, \ldots, m).

Since $N = \{ \pi(f(\xi)) | f : \kappa \to M, f \in M, \xi \in \kappa \}$

it follows that $N, \pi$ are uniquely determined by $F$.

QED (Lemma 1)

Def $\pi : M \to N$ weakly iff

(b) - (c) hold as above and

(a') $N = \langle \mathcal{I} \mathcal{N}, \mathcal{E}^N, \mathcal{A}^N \rangle$ in $\Lambda$, $w$-core $\langle \mathcal{N} \rangle$ is transitive and $\mathcal{X} \subseteq w$-core $\langle \mathcal{N} \rangle$. 
[This involves a slight abuse of notation.] Clearly the same proof yields:

**Corollary 2** If \( \pi : M \rightarrow N \) weakly, \( \kappa = \text{crit}(\pi) \), then \( F \) is an extender of length \( \lambda = F(\kappa) \) on \( \#(\kappa) \setminus M \) and \( N, \pi \) are uniquely determined by \( F \) (up to isomorphism).

Finally, using an "ultrapower" construction:

**Lemma 3** Let \( M \) be suitable for \( \kappa \), \( U = \#(\kappa) \setminus M \) and \( F \) an extender on \( U \) of length \( \lambda \). There are \( \pi, N \) s.t. \( \pi : M \rightarrow N \) weakly.

\[ \pi \]

Define a term model \( ID = \Pi(M, F) \)

by:

\( ID = \langle D, \equiv, \hat{\varepsilon}, \hat{A} \rangle \) where:

\[ D = \{ \langle d, f \rangle | f \in M, f: \kappa \rightarrow M, \alpha < \chi \} \]
\[ \langle d, f \rangle \equiv \langle \beta, g \rangle \quad \Rightarrow \quad \langle d, \beta \rangle \in F(\{3, 5\} \mid f(3) = g) \]

\[ \tilde{\varepsilon} \quad \Rightarrow \quad \langle \tilde{\varepsilon}, \varepsilon \rangle \]

\[ \tilde{A} \langle d, f \rangle \quad \Rightarrow \quad d \in F(\{3\} \mid f(3) \in A) \]

By induction on \( \Sigma \). For the \( \varphi \) we get

For Thm 1:

(1) \( \mathcal{D} \vdash \varphi(\langle d_1, f_n \rangle, \ldots, \langle d_m, f_m \rangle) \quad \Rightarrow \quad \langle \tilde{d}, \tilde{f} \rangle \in F(\{3\} \mid \mathcal{M} \vdash \varphi(f_1(3), \ldots, f_m(3))) \)

(2) \( \langle d, f \rangle \ovv{\varepsilon} \langle \beta, \text{id} \rangle \quad \rightarrow \quad \forall \gamma < \beta \quad \langle d, f \rangle \equiv \langle \gamma, \text{id} \rangle \)

\[ \mu f. \quad \exists f. \text{I} = f \cdot \text{id} = \{3, 5\} \mid f(3) = 3; \quad \langle d, \beta \rangle \in F(\{3, 5\} \mid f(3) < 5) \]

\[ \langle d, \beta \rangle \in F(\{3, 5\} \mid f(3) < 5) \quad \Rightarrow \quad \exists f. \text{I} = f \cdot \text{id} = \{3, 5\} \mid f(3) < 5 \]

\[ \exists f. \text{I} = f \cdot \text{id} = \{3, 5\} \mid f(3) < 5 \quad \Rightarrow \quad \exists f. \text{I} = f \cdot \text{id} = \{3, 5\} \mid f(3) < 5 \quad \Rightarrow \quad \exists f. \text{I} = f \cdot \text{id} = \{3, 5\} \mid f(3) < 5 \]

Hence there is \( \gamma < \beta \) s.t.

\[ \langle d, \gamma \rangle \in F(\text{I}) \quad \Rightarrow \quad \langle d, f \rangle \equiv \langle \text{id}, \gamma \rangle \]

\[ \Box \quad \text{ED (2)} \]
By (1), \( \cong \) is a congruence relation for ID. Let \( p : ID \rightarrow ID/\cong \) be the natural projection. Let \( \sigma : (ID/\cong) \rightarrow N \), where \( wfc(N) \) is transitive. Set:

\[
[t] = \sigma p(t) \quad \text{for } t \in ID
\]

\[
\pi(x) = [\langle 0, \text{cut}_x \rangle] \quad \text{for } x \in M.
\]

(3) \( \pi : M \rightarrow \Sigma_0 N \) cofinally.

\[\Sigma_0 \text{ preservation follows by } \Sigma_0 \text{ Thm.}\]

Cofinality follows by:

\[
[\mu f] \subset \pi (\text{using } (f)) \text{ in } N.
\]

QED (3)

(4) \( [\alpha, \text{id}] = \alpha \) for \( \alpha < \lambda \)

\[\Sigma_0 \]

Set: \( \tilde{\lambda} = \langle \alpha, \text{id} \rangle \). Then \( \tilde{\lambda} \in wfc(ID/\cong) \). By (2) and hence:

\[
[\tilde{\lambda}] \in wfc(N), \text{ where } wfc(N) \text{ is transitive. Hence}
\]

\[
[\tilde{\lambda}] = \{[t] | ID = t \in \tilde{\lambda} \} \text{ for } \alpha < \lambda.
\]

By induction on \( \alpha \), using (2), we get:

\[
[\tilde{\lambda}] = \alpha. \quad \text{QED (4)}
\]
\( (5') \quad [\alpha, f] = \pi(f)(d) \)

\[ \frac{\text{mut}}{D \vdash I = \top \text{ if } f \in \{\langle 3, 5 \rangle | f(5) = f'(5)\}}. \]

\[ [\alpha, f] = \pi(f)(d): \quad \langle \rightarrow \rangle. \]

\[ \langle \rightarrow \rangle \quad \text{ID} \vdash \{\langle \alpha, f \rangle = \langle 0, \text{cm} f \rangle (\langle \alpha, \text{id} \rangle)\} \]

\[ \langle \rightarrow \rangle \quad \langle 0, \alpha, d \rangle \in F(\{\langle 3, 5, \gamma \rangle | f(5) = f'(5)\}) \]

\[ \langle \rightarrow \rangle \quad \text{id} \in \{\langle 3, 5, \gamma \rangle | \langle 5, \gamma \rangle \in F(I)\} \]

\[ \langle \rightarrow \rangle \quad \langle d, d \rangle \in F(I) \quad \langle \rightarrow \rangle \quad \langle d, f \rangle \equiv \langle d, f \rangle. \]

\[ \langle \rightarrow \rangle \quad [\alpha, f] = [\alpha, f]. \quad \text{QED (5')} \]

Hence,

\[ (6) \quad \mathcal{N} = \{\pi(f)(\alpha) | \alpha \in \lambda, f : \kappa \rightarrow M, f \in M^3\} \]

\[ (7) \quad \pi \kappa = \text{id} \]

\[ \frac{\text{mut}}{L \vdash \langle 0, \nu \rangle \in F(\{\langle 3, 5 \rangle | \text{cm}^\nu(\nu) = \text{id}(5)\})} \]

for \( \nu < \kappa \), \( \kappa \)-inco \( F(A) \cap \kappa = A \).

Hence, \( \pi(\nu) = \langle 0, \text{cm}^\nu \rangle = [\nu, \text{id}] = \nu \).

\[ \text{QED (7')} \]

\[ (8) \quad \alpha < \pi(\kappa) \quad \text{in} \quad \mathcal{N} \quad \text{for} \quad \alpha \in \lambda \]

\[ \frac{\text{mut}}{L \vdash \langle d, \text{id} \rangle < \langle 0, \text{cm}^\kappa \rangle, \quad \kappa \text{-inco} \}

\[ \langle 0, 0 \rangle \in F(\{\langle 3, 2 \rangle | \text{id}(3) < \kappa\}) = F(\kappa) = \lambda \]

\[ \text{QED (8')} \]
Then:

(4) \( \kappa = \text{crit}(\pi) \)

It remains only to show:

(10) \( F(A) = \lambda \cap \pi(A) \) for \( A \in \mathcal{P}(\kappa) \cap M \)

Let \( \lambda \). Then \( \lambda \in \pi(A) \iff \iff \left[ \text{crit}(A) \right] \in \left[ 0, \text{crit}(A) \right] \iff \iff \left< \kappa, 0 \right> \in F(\{ \left< \kappa, 5 \right> \left| \kappa \in A \} \) \iff \lambda \in F(A) \)

QED (Lemma 3)

Lemma 4 Let \( \pi : M \rightarrow N \) where \( \pi = \text{crit}(\pi) \) is the largest cardinal in \( N \).

Assume \( F(\kappa) = \pi(\kappa) \). Then \( \langle N, F \rangle \) is amenable.

Let \( x \in N \). Claim \( x \cap F \in N \).

Let \( x \subseteq \pi(X) \). Then \( x \cap F = x \cap (\pi(X) \cap F) \) and it suffices to show:

Claim \( \pi(X) \cap F \in N \).

But \( F \) is a function and \( \left< a, F(a) \right> \in \pi(X) \rightarrow F(a) \in \bigcup \pi(X) = \pi(U^\pi(X)) \). Hence \( F \cap \pi(X) \subseteq \bigcup \pi(U^\pi(X)) \) and it suffices
- 12 -

to show! Claim \( F \mid x \in N \) for \( x \in M \).

Assume w.l.o.g. \( F \mid x \neq \emptyset \) (i.e., \( \#(x) \land x \neq \emptyset \)). Let \( f \in M \) s.t.
\( f: x \in \mathcal{U} \rightarrow \#(x) \land x \). Then
\( \mathcal{U}(f): \mathcal{U}(x) \in \mathcal{U}(x) \rightarrow \#(x \mid x \in N \).

Moreover, \( f = \{ \langle \mathcal{U}(f)(x) \rangle \mid x \in N \} \in N \).

But \( F(f(x)) = \mathcal{U}(f(x)) = \mathcal{U}(f(x)) \)
for \( x \in N \). Hence :
\( F \mid x = \{ \langle f(x), \mathcal{U}(f(x)) \rangle \mid x \in N \} \in N \).

QED (Lemma 4)

---

**Def** Let \( F \) be an extender on \( U \) of length \( \lambda \), \( \kappa = \text{crit}(F) \). \( F \) is **weakly amenable** iff whenever \( X \subseteq U \),
\( \delta < \lambda \), then \( \{ \exists \xi \mid X'' \exists \xi \subseteq F_{\delta} \} \subseteq U \).

**Def** Let \( \mathcal{U}: M \rightarrow N \) weakly. \( F \downarrow \Sigma_1 \)-**amenable** wrt. \( M \) iff
\( \forall \mu \in \Sigma_1(M) \) for \( \delta < \lambda \).
Note. Either of the properties:
weakly amenable and $E_1$ amenable
can hold without the other. Both can fail.

If $\pi : M \rightarrow N$ weakly, it follows easily that:

(a) $\#(\alpha) \backslash M \subseteq N$, since $X = \alpha \cap \pi^{-1}(X)$
for $X \in \#(\alpha) \backslash M$.

(b) $\#(\alpha) \backslash N \subseteq M$ iff $F$ is weakly amenable.

As $F$ is an extender on $U$ of length $\lambda$, then whenever $\pi : M \rightarrow N$ weakly,
the ordinals $\lt \pi(\alpha)$ in $N$ will all have the form $[\alpha, f]$, where $\alpha < \lambda$
and $f \in U^2$, $f : \alpha \rightarrow \alpha$. Moreover, every such $[\alpha, f]$ is an ordinal
$\lt \pi(\alpha)$ in $N$. Hence $\lambda = \pi(\alpha)$,
i.e. $[\alpha, f] \subseteq \lambda$ for all such $f$. Thus
we define:
Def. Let $F$ be an extender on $U$ of length $\lambda$. $F$ is whole if whenever $f \in U^2$, $f : \kappa \rightarrow \kappa$, $\kappa < \lambda$
Then there is $\beta < \lambda$ such that
$$<d, \beta> \in F(\exists \delta < \kappa \exists \gamma > 1 f(\delta) = \gamma 3)$$

By the above remarks it is obvious that:

**Lemma 5** Let $\pi : M \rightarrow N$ weakly, where $F$ is at $\kappa, \lambda$ (i.e., $\kappa = \text{crit}(F)$ $\pi(\kappa) = \lambda$), $F$ is whole in $H$ $\lambda = \pi(\kappa)$

Hence $\pi(\kappa) \in \text{wfc}(N) \iff F$ is whole.

We also note:

**Fact** Let $U, U'$ be suitable for $\kappa$, where $U' \subseteq U$. Let $F$ be an extender on $U$. Then $F|^U$ is an extender on $U'$.

Def. $F$ is an extender on $M$ at $\kappa, \lambda$ iff $F$ is an extender on $\pi(\kappa)^M$ of length $\lambda$, where $M$ is suitable for $\kappa$. 
Def \( N = \text{Ult}(M, F) \) iiff \( \forall \pi (\pi : M \to N) \)

Def \( N = \langle J^A_x, F \rangle \) is coherent iiff

iiff \( J^A_x \) is acceptable, \( F \) is a whole extender on \( J^A_x \) for an \( \bar{x} \) s.t. \( \kappa = \text{crit}(F) \) is the largest cardinal in \( J^A_{\bar{x}} \), and

and \( J^A_x = \text{Ult}(J^A_{\bar{x}}, F) \),

Lemma 6 Let \( N = \langle J^A_x, F \rangle \) be coherent

Then:

(a) \( N \) is amenable

(b) Let \( \bar{x} = \text{the least } \bar{x} \) s.t. \( \text{dom}(F) = \#(\kappa) \cap J^A_{\bar{x}} \). Then \( \kappa \)
    \( \text{is the largest cardinal in } J^A_{\bar{x}} \).

(c) Let \( \lambda = F(\kappa) \). Then \( \lambda \) is the largest cardinal in \( N \).

(d) Let \( \bar{y} < \bar{x} \) s.t. \( \kappa \) is the largest cardinal in \( J^A_{\bar{y}} \). Let \( \omega_{\bar{y}} = \sup_{\omega \in J^A_{\bar{y}}} F(\omega) \).
    Set \( \bar{F} = F|_{J^A_{\bar{y}}} \). Then \( \langle J^A_{\bar{y}}, \bar{F} \rangle \)
    is coherent.
\[ \mu f. \]

(a) is immediate by Lemma 4

(b) and (c) are trivial. We prove (d)

Set \( \overline{\pi} = \pi \cap J^A_\beta \). Then:

1. \( \overline{\pi} : J^A_\beta \rightarrow J^A_\beta \) is finally

2. \( \pi = v + (\overline{\pi} \Gamma) \), \( \chi = \overline{\pi}(\pi) = \overline{F}(\pi) \)

3. \( \overline{F} = \overline{\pi} \Gamma \overline{\pi}(\pi) \circ J^A_\beta \)

It remains only to show:

4. \( J^A_\beta = \text{the } \Sigma \text{ closure of } \sigma \gamma(\overline{\pi}) \cup \chi. \)

Proof.

Let \( x \in J^A_\beta \), \( x \in \overline{\pi}(\chi) \). Let \( f \in J^A_\beta \), \( f : \pi \rightarrow x \). Then \( \overline{\pi}(f \mid \chi : \pi \rightarrow \overline{\pi}(\chi) \) and

\[ x = \overline{\pi}(f \mid \chi(x) \text{ for all } x < \chi. \]

Q.E.D. (Lemma 6)
Lemma: There is a \( \phi \)-formula \( \varphi \) such that if \( N = \langle j^A_{a}, E \rangle \) is acceptable, then \( N \) is coherent if \( N \models \varphi \).

Proof:

(\( \varphi \) is the statement that there are arbitrarily large \( \lambda \) such that:

(a) \( FnS^A_3 \) is a fan and there is exactly one pair of ordinals \( \langle \kappa, \lambda \rangle \in FnS^A_3 \).

(b) \( j^A_{\kappa} \models ZF^- \), \( j^A_{\lambda} \models ZF^- \).

(c) \( A \models \langle a, b \rangle \in FnS^A_3 \), then \( a < \kappa \) and \( b < \lambda \).

(d) Let \( \langle \langle a_1, b_1 \rangle, \ldots, \langle a_m, b_m \rangle \rangle \in S^A_3 \) such that \( \langle a_i, b_i \rangle \in E \) for \( i = 1, \ldots, m \).

Let \( a \) be \( \Sigma_1(\langle j^A_{\kappa}, \varphi \rangle) \) in the parameter \( a \) and let \( b \) be \( \Sigma_1(\langle j^A_{\lambda}, \varphi \rangle) \) in the same parameter. Then \( \langle a, b \rangle \in E \).

(\( \exists \) There are \( \exists \lambda, \exists \gamma > \exists \alpha \) such that

\( \text{dom}(FnS_3^A) \) \quad \text{and} \quad \text{dom}(FnS_3^A) \)
(f) There is $\bar{a} \subseteq 3 \cap (\beta < \gamma, \beta < \gamma)$, $f \in S^A_3$, $f : \kappa \rightarrow \kappa$ and $\bar{a} \in \text{dom}(F_{\kappa}S^A_3)$ where $\bar{a} = \{ \langle \alpha, \beta \rangle < \kappa \mid f(\alpha) = \beta \}$. Then there is $\alpha \in S^A_3$ s.t. $\langle \bar{a}, \alpha \rangle \in \mathcal{F}$ and $\langle \beta, \delta \rangle \in \alpha$.

(g) There are $\bar{a}, \bar{f}, f, \kappa, \beta$.

(i) $f : \kappa \rightarrow S^A_3$, $f : \lambda \rightarrow S^A_3$.

(ii) $\forall \bar{a} = \{ \langle x, \beta \rangle \mid f(x) \in f(\beta) \}^3$, $\alpha$ and $\bar{a} = \{ \langle x, \beta \rangle \mid f(x) \in f(\beta) \}^3$, then $\langle \bar{a}, \alpha \rangle \in \mathcal{F}$.

(iii) $\forall \bar{a} = \{ \langle x, \beta \rangle \mid f(x) \in A \}^3$, $\alpha = \{ \langle x, f(x) \rangle \in A \}$.

Then $\langle \bar{a}, \alpha \rangle \in \mathcal{F}$.

(a1) - (e) guarantee that $\mathcal{F}$ is an extender at some $\kappa_1 \times \gamma$ on $\Pi^{\kappa_1}_{\kappa_1} \gamma J^A_{\kappa_1}$ where $\kappa$ is the largest cardinal in $J^A_{\kappa_1}$.

(f) guarantees that if $\bar{\gamma} : J^A_{\kappa_1} \rightarrow M$ weakly, then $\gamma = \bar{\gamma}(\kappa_1)$.

(g) Then guarantees that $M = J^A$.

It is clear that (a1) - (g) hold for all coherent $N$.

QED (Lemma 71)
Let $\mathcal{N} = \langle \mathcal{J}_d^\mathcal{A}, \mathcal{F} \rangle$ be coherent. Let $\vec{x}$ be least $\alpha$ such that $\operatorname{dom}(\mathcal{F}) \cap \mathcal{A} = \mathcal{A}(\alpha) \cap \mathcal{J}_d^\mathcal{A}$. Clearly we have $\mathcal{F}$ is weakly amenable if and only if $\vec{x} = \kappa + N$, where

$$\vec{x} = \kappa + N \iff \mathcal{A}(\kappa) \cap N \subset \mathcal{J}_d^\mathcal{A}.$$ 

Thus, if $\vec{\beta}$, $\vec{\beta}'$, $\vec{\mathcal{F}}$ are as in Lemma 6 (d), then $\vec{\mathcal{F}}$ is not weakly amenable.
As an example of the use of $\Sigma_1$-amenability, we prove the following lemma. The argumentation used will be of great importance later in connection with the so-called $\star$-ultra products.

**Lemma** 8. Let $F$ be $\Sigma_1$-amenable with $\bar{M}$, let $\pi : \bar{M} \to M$, let $\kappa = \text{u} \text{it}(F)$. Then

$$\kappa(\kappa) \wedge \Sigma_1(M) \subset \Sigma_1(\bar{M}),$$

**Proof:**

Let $A$ be $\Sigma_1(M)$, $A \subset \kappa$. Then

$$A \in \text{V} \text{e} \text{R}(z, \exists, \forall),$$

where $\text{R} \in \Sigma_0(M)$, $\forall \in \pi(f)(\alpha)$, $f \in \bar{M}$, $f : \kappa \to \bar{M}$ and $\alpha < \text{length}(F)$.

Hence:

$$A \in \text{V} \text{e} \text{R}(z, \exists, \forall) \text{R}(\pi(M), \exists, \pi(f)(\alpha)).$$

$$\text{V} \text{e} \text{R}(\pi(M), \exists, \pi(f)(\alpha)) \subset \Sigma_0(M).$$
Let \( \overline{P} \) have the same \( \Sigma_0 \) def. over \( \overline{M} \). Then:

\[ A^3 \leftrightarrow \forall u \in \overline{M} \ v \in F(\{5 \mid \overline{P}(u, \overline{z}, f(s))\}) \]

\[ \Sigma_1(\overline{M}) \text{ in } d, f, x, \text{ where} \]

\[ \overline{F} \cup \Sigma_1(\overline{M}) \text{ in } x. \]

QED (Lemma 8).

The same proof shows:

**Lemma 9** Let \( F, \overline{M}, M \) be as above. Let \( \overline{R}(x_1, \ldots, x_p, y_1, \ldots, y_q) \) be \( \Sigma_1(M) \). Let \( f_1, \ldots, f_q \in \overline{M} \) s.t.

\[ f : n \rightarrow \overline{M} \text{, where } n = \text{rest} \{ F \}. \]

Let \( x_1, \ldots, x_p < n = \text{length} (F) \). Then:

\[ \overline{R} = \{ \overline{z} \mid \overline{R}(\pi(x_1), \ldots, \pi(x_p), \pi(f_1(x_1), \ldots, \pi(f_q(x_1))) \}
\]

\[ \in \Sigma_1(\overline{M}) \text{ in } f_1, \ldots, f_q, x_1, \ldots, x_p, \]

and \( x, \) where \( F(x_1, \ldots, x_p) \) is \( \Sigma_1(\overline{M}) \) in \( x \) (uniformly in the \( \Sigma_1 \) def. of \( F \) and the \( \Sigma_1 \) def. of \( F_F \) from \( M \)).