§2 \Sigma^* - Ultrapowers

We now consider acceptable \( \mathfrak{U} \) - models \( N = \langle J^A_\mathfrak{U}, B \rangle \) (i.e. \( J^A_\mathfrak{U} \) is acceptable + \( N \) is amenable). Let \( \kappa \leq \delta \) be an uncountable cardinal in \( N \). Then \( N \) is suitable for \( \kappa \) by acceptability. (Since if \( a \in \kappa \), \( a \notin N \), then \( \kappa \) is p.m. closed and \( \langle J^A_\kappa, a \rangle \) is amenable. Hence \( J^a_\kappa = \bigcup_{\nu < \kappa} J^a_\nu \in \text{Def}(J^A_\kappa, a) < J^A_\delta \).)

Call \( F \) an \textit{extender at} \( \kappa, \lambda \) on \( N \) iff \( F \) is an extender on \( \tau(\lambda) \cup N \) of length \( \lambda \). For each \( F \) we have defined the ultrapower \( \pi : N \rightarrow^F M \).

Making use of fine structure, we now define the so-called \( \Sigma^* - \text{ultrapower} \) \( \pi : N \rightarrow^* M \).

The intention is that \( \pi \) will
be $\Sigma_0^{(m)}$-preserving whenever $\omega_1^m > \kappa$.

Under certain conditions (e.g., if $F$ is weakly amenable) it will be $\Sigma_1^{(m)}$-preserving for $\omega_1^m > \kappa$.

If $F$ is both weakly amenable and $\Sigma_1$-amenable it will in fact be $\Sigma^*$-preserving.

An keeping with our earlier definition we first define:

**Def.** Let $N$ be an acceptable $\omega$-model, $\pi : N \rightarrow M$ be

(a) $M$ transitive
(b) $\pi : N \rightarrow \Sigma_0^{(m)} M$ for $\omega_1^m > \kappa$ when $\kappa = \text{crit} (\pi)$
(c) $F = \{ \lambda \cap \pi (x) \mid x \in \mathcal{P}(\kappa) \cap M \}$, where $\kappa < \lambda \leq \pi (\kappa)$, $\lambda$ is p.r. closed and
(d) $N$ = the closure of $\text{rng}(\pi)$ under $\Sigma_0$-foms, under $\Sigma^*$-foms for $\omega_1^{m+1} > \kappa$. 

Two immediate conclusions are:

Lemma 1.1 Let \( \omega^N \leq \kappa \). Then \( \Pi : N \rightarrow M \) if and only if \( \Pi : N \rightarrow^* M \).

Lemma 1.2 Let \( \omega^N > \kappa \). \( \exists \) \( \Pi : N \rightarrow^* M \), then \( \Pi : N \rightarrow \sum^*_N M \).

For the simple ultra-power \( \Pi : N \rightarrow M \), we had \( \mathcal{M} = \{ \pi(f) \mid f \in N \ \& \ f : \kappa \rightarrow N \ \& \ \kappa < \} \).

This is no longer true, in general, for the \( \star \)-ultra-power \( \mathcal{M} \), but we can formulate a similar condition.

Set: \( \Gamma = \Gamma_0(\kappa, N) = \{ \Pi \} \).

The set of \( f : \kappa \rightarrow N \) such that \( f \in N \ \& \ f \in \mathcal{M} \).

Let \( f : \kappa \rightarrow N \) be a good \( \Sigma^0_1 \) map, where \( \omega^{\kappa+1} > \kappa \).

Let \( f : \kappa \rightarrow N \) be a good \( \Sigma^0_1 \) map and let \( y = F(x) \).

Let \( y = G(x) \) be functionally absolute \( \Sigma^0_1 \) definition with the parameter \( p \), if necessary. Since \( \omega^N > \kappa \),
The $\Pi_0^{\text{stars}}$ statements $i \kappa = \text{dom} G$, $\kappa = \text{dom} F$, $\forall \alpha < \kappa. F_\alpha (\nu) = G_\nu (\nu)$ hold in $\mathcal{N}$. Hence the corresponding statements hold of $(\Pi_1, \pi(q))$ in $M - i.e. y = F_\pi(q)(x), y = G_\pi(q)(x)$ define the same $\Sigma_1^m (M)$ map of $\pi(q)$ to $M$.

This means we can extend $\pi$ to elements of $\Gamma$ by setting $\pi(f) = \text{that } \pi'(\nu)$ defined by $F_\pi'(\nu)$, where $y = F_\pi(q)(x)$ is a functionally absolute good def. of $f$ in the parameter $p$. Clearly then:

Lemma 1.3 $M = \{ \pi(f)(\nu) \mid f \in \Gamma, \nu < \kappa \}^3$.

We also have for Theorem 1:

Lemma 1.4 $M \in \mathcal{V} (\pi(f_1)(\nu_1), \ldots, \pi(f_n)(\nu_n)) \iff$

$\exists \pi \in F (\{ \nu_1 \mid \mathcal{N} \in \mathcal{V} (f_1(\nu_1), \ldots, f_n(\nu_n)) \}^{\omega_1})$.

If $\mathcal{V} \in \Sigma_0^{\omega_1}$ for $\omega_1 > \kappa$. 
Thus, if $\pi: N \rightarrow^* M$, then $\mu, \nu$ are uniquely determined by the extender $F$. The question, whether to a given $F$ such $\pi, M$ exist, can be answered as before by an "ultra-power construction". It will again turn out that $\pi, M$ exist iff a certain "term model" $\mathbb{D}$ is well-founded.

Let $N = \langle J^A, B \rangle$ be an acceptable $J$-model. Let $\kappa < \lambda$ and let $F$ be an extender at $\kappa, \lambda$ on $N$. Define a term model $\mathbb{D} = \mathbb{D}^*(N, F)$ by $\mathbb{D} = \langle D, \equiv, \llcorner, \lrcorner, \widetilde{A}, \widetilde{B} \rangle$, where

$D = \{ \langle d, f \rangle : f \in \Gamma, \alpha < \lambda \}$, where $\Gamma = \Gamma(m, N)$,

$\langle d, f \rangle \equiv \langle \beta, \gamma \rangle \iff \langle d, \beta \rangle \in F(\langle \beta, \gamma \rangle) = \langle \gamma, \varepsilon \rangle$,

$\widetilde{A} \langle d, f \rangle \iff d \in F(\langle \varepsilon, f \rangle)$,

$\widetilde{B} \iff B$. 


For \( \Sigma_0 \) formulae, recall:

**Lemma 2.1** Let \( \langle a_i, f_i \rangle \in D \ (i = 1, \ldots, m) \),
\( D = \varphi(\langle a_1, f_1 \rangle, \ldots, \langle a_m, f_m \rangle) \) \( \iff \)
\( \langle \bar{d} \in F(\langle \bar{x} \rangle \mid N \in \varphi(f_n(\bar{x}_n), \ldots, f_m(\bar{x}_m)) \rangle, \)
if \( \varphi \in \Sigma_0 \).

The proof is by induction on \( \varphi \)
making use of the following:

**Lemma 2.2** Let \( R(y_m, x_{d_1}, \ldots, x_{d_r}) \) be
\( \Sigma_1^{(m)}(N) \), where \( \varphi^m > \alpha \) and \( \sum_{i=1}^r d_i \leq m \),

Let \( \nu = m + 1 \), \( \varphi^{\nu} > \alpha \). Let \( f_1, \ldots, f_n \)
be good \( \Sigma_1^{(m)}(N) \) maps, where \( f_i \)
\( i = 1, \ldots, n \), \( i \in \text{dom}(f_i) = \alpha \). There
is a good \( \Sigma_1^{(\nu)}(N) \) map \( g \in \text{dom} \).

\( \forall y_m R(y_m, f_n(\bar{x}_n), \ldots, f_m(\bar{x}_m)) \iff \)
\( \iff R(g(\langle \bar{x}_n, \ldots, \bar{x}_m \rangle), f_n(\bar{x}_n), \ldots, f_m(\bar{x}_m)) \)

for all \( \bar{x}_n, \ldots, \bar{x}_m < \alpha \).

(Note: \( m = j = \ldots = j_n = 0 \) in the case used
in the proof of Lemma 2.1.)
proof of Lemma 2.2

By [MO] §1.2 Lemma 5.3, there is a $\Sigma_1^{(m)}$ function $F$ to $H^m$ such that:

$$V y^m \mathcal{R}(y^m, x_1, \ldots, x_n) \iff \mathcal{R}(F(x_1, \ldots, x_n), x_1, \ldots, x_n).$$

Set $G(n) = F(f_1((n)_{\omega}^0), \ldots, f_m((n)_{\omega}^{n-1})), n \geq 1,$ where $n = \langle (n)_{\omega}^0, \ldots, (n)_{\omega}^{n-1} \rangle.$ Then $G$ is a good $\Sigma_1^{(m)}$ function to $H^m$ and dom $G \subseteq \kappa.$ Hence dom $(G) \in N,$ since $\omega^{m+1} > \kappa.$ Let $d = \text{dom}(G) \in N$ and set $g(n) = \begin{cases} G(n) & \text{if } n \in d \\ 0 & \text{if } n \not\in d \end{cases}$ for $n < \kappa.$ Then $g : \kappa \rightarrow H^m$ is a good $\Sigma_1^{(m)}$ function. [To see this, set $F'(x_1, \ldots, x_n, z) = \begin{cases} F(x_1, \ldots, x_n) & \text{if } z \in d \\ 0 & \text{if not, } \end{cases}$ Then $F' \in \Sigma_1^{(m)}$ to $H^m$ and $g(n) = F'(f_1((n)_{\omega}^0), \ldots, f_m((n)_{\omega}^{n-1}), n).$]

The conclusion is immediate.

QED (Lemma 2.2)
At is obvious that if \( \pi : N \to [M] \)
Then \( \langle d, f \rangle \bar{\in} \langle \beta, g \rangle \iff \pi(f)(d) \in \pi(g)(\beta) \)
for \( \langle d, f \rangle, \langle \beta, g \rangle \in D \) and that
The map \( \langle d, f \rangle \mapsto \pi(f)(d) \) is onto \( M \).
Therefore

**Lemma 2.3**  If \( \bar{\in} \) is not well-founded, then there are no
\( \pi, M \) s.t. \( \pi : N \to [M] \).

From now on assume that \( \bar{\in} \)
is well-founded. We shall show
that \( \pi, M \) exist. By Lemma 2.2 \( \bar{\in} \)
is an equality relation for
\( D \) and satisfies extensionality.
It follows that there is a structure preserving map \([ \cdot ] : ID \to M \)
on to
a transitive \( M \) s.t.
\[ [x] \in [y] \iff x \bar{\in} y \]
\[ = \sim \]
Define \( \pi : N \to M \) by \( \pi(x) = [\xi_0, \text{crit}_x] \), where \( \text{crit}_x \) is the constant function on \( x \). By Theorem we conclude: \( \pi : N \to \mathfrak{E}_0 M \).

Lemma 2.4 Let \( \overline{H} = H^m_N \), \( H = U \pi^n \overline{H} \), where \( p^m_N = \min \{ p^m_N \mid \omega p^m_N > \kappa \} \).

Then \( \pi \mid \overline{H} : \overline{H} \to H \).

Proof.
By the def. of \( H \), whenever \( f \in H \) and \( \text{ng}(f) \subset \overline{H} \), then \( f \in H \).
Hence \( H = \{ [<a, \eta>] \mid \eta \in H, g : \kappa \to \overline{H}, \delta < \lambda \} \).

The conclusion follows easily.
\( \text{QED} \) (2.4)

In particular it follows that \( \kappa = \text{crit}(\pi), [<a, \text{fg}>] = \overline{\pi}(f)(x) \) for \( f \in \overline{H}, f : n \to \overline{H}, \delta < \lambda \).

We now prove:
Lemma 3

(a) $M$ in an acceptable $\Sigma$-model

(b) $\pi : N \rightarrow \mathcal{E}_0^{(m)} M \text{ if } \omega \mathcal{P}_N^{\mathcal{P}_N} > \kappa$

(c) $\pi : N \rightarrow \mathcal{E}_2^{(m)} M \text{ if } \omega \mathcal{P}_N^{\mathcal{P}_N+1} > \kappa$

(d) Let $\mathcal{F}$ be $\mathcal{E}_0^{(m)}$ for an $n$ s.t. $\omega \mathcal{P}_N^n > \kappa$

or $\mathcal{E}_1^{(m)}$ for an $n$ s.t. $\omega \mathcal{P}_N^{n+1} > \kappa$. Then:

$M \models \mathcal{F}([d_1, f_1], \ldots, [d_m, f_m]) \iff$

$\langle \mathcal{F}, N \models \mathcal{F}(f_1(\bar{x}), \ldots, f_m(\bar{x})) \rangle \in F$.

The proof stretches over several sublemmas. We first verify (b)-(d) in the sense of a "pseudo interpretation" of the $\mathcal{E}_n^{(m)}$ formulae in $M$.

Using this we verify (a). We then show that our pseudo interpretation is sufficiently correct for (b)-(d) to hold.
For $\omega_p^m > n$ set:

$$I_m^N = \begin{cases} \{ f \in \mathcal{P} | \text{any } (f) \in H_N^m \} & \text{if } \omega_p^m > \kappa^N \leq \omega_p^m \text{ in } \mathcal{N} \\ \{ f \in \mathcal{P} | \text{any } (f) \in H_N^m \} & \text{if } \omega_p^m + 1 \leq \kappa < \omega_p^m \text{ in } \mathcal{N} \end{cases}$$

**Remark** $I_m^N = \{ f \in \mathcal{P} | f \in H_N^m \}$ if $\omega_p^m \leq \kappa < \omega_p^m$ in $\mathcal{N}$.

**Remark** By the remarks following the proof of [MO] §1.2 Lemma 5.1, $I_m^N$ may (but need not) be regarded as a set of four to $H^m$ - i.e. defined by $\mathcal{C}(y^n, x)$.

Set: $H_m = \{ [\langle d, f \rangle] | \langle d, f \rangle \in D \land f \in I_m^N \}$. It is obvious that:

1. $H_m$ is transitive.

We say that $M \models \varphi(x^i)$ in the pseudo interpretation if $\varphi$ holds when $x^i$ is taken as ranging over $H^i$ for $\omega_p^i > \kappa$. We then get a pseudo-

For Theorem 1:
Lemma 3.1 Let \( \varphi \) be a \( \Sigma_0^m \) formula for an \( n \) r.t. \( \omega^m \uparrow \kappa \) or a \( \Sigma_1^m \) formula for an \( n \) r.t. \( \omega^m \uparrow \kappa \). In the sense of the pseudo interpretation we have:

\[
M \models \varphi([d_1, f_1], \ldots, [d_m, f_m]) \iff
\text{for } z \in F(\exists z \; N \models \varphi(f_1(d_1), \ldots, f_m(d_m)))
\]

proof. By induction on \( m \) and for given \( n \) by induction on \( \varphi \) using Lemma 2.2.

QED (Lemma 3.1)

Cor 3.2 \( \pi \in \text{IN} \rightarrow \Sigma_0^m \text{M for } \omega^m \uparrow \kappa \) in the pseudo interpretation.

Cor 3.3 \( \pi \in \text{IN} \rightarrow \Sigma_2^m \text{M for } \omega^{m+1} \uparrow \kappa \).

Let \( M \models \forall x^m \varphi(x^m, \pi(z)) \), where \( \varphi \in \Pi_1^n \).

Let \( \langle d, q \rangle \in D, q \in \text{IN} \). \( \text{r.t. } M \models \varphi([d, q], \pi(z)) \).

By Lemma 3.1, \( d \in F(\exists z \; N \models \varphi(q(y(3)), z)) \).

Hence \( \forall z \exists y N \models \varphi(q(y(3)), z) \). Hence

\( N \models \forall x^m \varphi(x^m, z) \).

QED (3.3)
By Lemma 2.1, Cor 3.3 we have:

**Cor 3.4** $M$ is an acceptable $J$-model.

**proof:**
At $\omega^1 \leq \kappa$, then $\pi : N \rightarrow \Sigma^M$ cofinally by Lemma 2.4. At $\omega^1 > \kappa$, then $\pi : N \rightarrow \Sigma^M$ in the pseudo interpretation, hence $\pi : N \rightarrow \Sigma^M$ since $H_0 = M$.

QED (3.4)

Set $\omega^m = \text{On} \cap \text{H}_m$.

**Cor 3.5** Let $M = J^A_{\rho^m} \upharpoonright B$. Then $H_m = J^A_{\rho^m}$.

**proof:**
At $\omega^{m+1} > \kappa$, we have:

$\pi ^{\text{H}_m} : J^A_{\rho^m} \rightarrow \langle H_m, A' \cap \text{H}_m \rangle$,

At $\omega^m \leq \kappa < \omega^{m+1}$ in $N$, then by Lemma 2.4 we have:

$\pi ^{\text{H}_m} : J^A_{\rho^m} \rightarrow \langle H_m, A' \cap \text{H}_m \rangle$ cofinally.

QED (3.5)
Thus it remains only to prove:

Lemma 3.5 \( p_m = p_m^+ \) for \( \omega \rho^{m+1} > \kappa \) and \( p_m \leq p_m^+ \) for \( \omega \rho^m > \kappa \).

\textbf{Proof.} By \textit{ind. on} \( n \), \( n=0 \) is immediate, so assume \( n>0 \). We first show: \( p_m \leq p_m^+ \). Let \( A \subseteq \omega \rho^m \) be \( \Sigma_\omega^{(m-1)}(M) \). At \( n \) let \( \text{HN} \) to a show:

\textbf{Claim} \( \langle H_n, A \rangle \) is amenable.

Let \( z \in H_m \). \textbf{Claim} \( z \wedge A \in H_m \).

Let \( A \in A'(x, [eta, f]) \) where \( A' \in \Sigma_\omega^{(m-1)}(M) \). Let \( A' \in \Sigma_\omega^{(m-1)}(N) \) by the same definition. Let \( z = [\beta, g] \) where \( g \in \Gamma_{m}^{+} \). Define \( k: \kappa \rightarrow H_m \) by:

\[ k(\beta) = g((\beta)) \cap \{ x \mid A'(x, f((\beta))) \} \]

Then \( k \in \Gamma_{m}^{+} \). Set \( w = [\lambda, \beta \prec k] \). Then \( w \in H_m \) and by the for

Thm (Lemma 2.1): \( w = z \wedge A \).
To see this note that for \( z_1, z_2, z_3 < n \):
\[
\mathcal{B}_z = \langle z_1, z_2, z_3 \rangle \rightarrow \mathcal{B}(\mathcal{B}_z) = \mathcal{B}(z_3) \cap \mathcal{B}(z_2) \cap \mathcal{B}(z_1).
\]
Henry \( \langle a, b, c \rangle \in \{ \mathcal{B}_z \in \mathcal{X}^3 \mid \mathcal{B}_z = \langle z_1, z_2, z_3 \rangle \} \) is:
\[
\mathcal{B}(\mathcal{B}_z) = \mathcal{B}(z_3) \cap \mathcal{B}(z_2) \cap \mathcal{B}(z_1).
\]

\[ \text{QED (Lemma 3)} \]
As a corollary of the proof of \( \omega \rho^m \leq \omega \rho^m \) for \( \omega \rho^{m+1} > n \), we note:

**Corollary 3.6** Let \( \omega \rho^{m+1} > n \). Then
\[
\pi^m \rho^m \subseteq \rho^m.
\]

**Corollary 3.7** \( \pi : N \to^\ast M \).

**Proof.**

(a1)-(d) in the def. of \( \pi : N \to^\ast M \) are satisfied. We prove (e).

By (a)-(d) we know that \( \pi(f) \) is defined for \( f \in \Gamma \). By Lemma 2.4 we have \([\lambda, \text{id}] = \lambda\) for \( \lambda < \lambda \).

By For Thm 1:

\[
1D = \langle \lambda, f \rangle = \pi \langle \langle \lambda, \text{id} \rangle \rangle
\]

for \( \langle \lambda, f \rangle \in 1D \). Hence \([\lambda, f] = \pi(f) \lambda 0\)

Hence \( M = \{ \pi(f) | \langle \lambda, f \rangle \in 1D \} \),

QED (3.7)

(Cor 3.8, \( [\lambda, f] = \pi(f)(\lambda) \),
We now show that the preservation properties of $\pi$ can be improved if we make stronger assumptions on $N$.

**Lemma 4.1** Let $F$ be weakly amenable. Let $\omega_F^{m+1} \leq N < \omega_F^m$. Then $\pi : \mathcal{N} \rightarrow \Sigma_{m+1}^0, M$ cofinally.

(Recall $\pi : \mathcal{N} \rightarrow \Sigma_{m+1}^0, M$ cofinally means that $\pi$ is $\Sigma_{m+1}^0$ preserving and $\omega_F^m = \sup_N \omega_F^m$.)

**Proof.**

$m = 0$ is immediate. Assume $m > 0$.

**Claim 1.** There is $B \subset \omega_F^m$ s.t. $B \in \Sigma_{m+1}^0(N)$ in the pseudo interpretation and $B \cap \omega_F^{m+1} \notin M$.

Let $B$ be $\Sigma_{m+1}^0(N)$ s.t. $B \cap \omega_F^{m+1} \notin M$. 

Let $B$ be $\Sigma_1^{(m)}(N)$ in $\bar{p} + \text{let } B \text{ with the same } \Sigma_1^{(m)} \text{ def in } p = \pi(\bar{p})$ in the pseudo interpretation. Then $\bar{B} \omega^p_{m+1} = B \omega^p_{m+1} \& M$, since $\#(\bar{a}) \cap N = \#(\bar{a}) \cap M$ and $\bar{N} \geq \omega^p_{m+1}$. QED (Claim 1).

It remains only to show:

Claim 2: There is $D \in \Sigma_1^{(m-1)}(M)$ r.t. $D \omega^p_m$ and $D \& M$.

Let $B$ be as in Claim 1. Then $B$ is $\Sigma_1(\langle H_m, D' \rangle)$ where $D' \in \Sigma_1^{(m-1)}(M)$.

Hence $D' \& M$, since $B \& M$.

Since $H_m = \bigcup_{\bar{p}}$, there is $f$ p.r. in $A$ r.t. f maps $\omega^p_m$ onto $\{x \mid f(x) \in D'\}$. Then $D' \in \Sigma_1^{(m-1)}(M)$ and $D \& M$, since $D' \& M$.

QED (Lemma 4.1)
Claim 1 in the proof of Lemma 4.1 then gives us:

**Cor 4.2** Let \( F \) be weakly amenable and \( \omega \rho^{m+1} \leq \kappa < \omega \rho^m \). Then

\[
\omega \rho^{m+1} \leq \omega \rho^{m+1}_N.
\]

The proof of Lemmas 4.1, 4.2 did not use the full strength of weak amenability but merely the fact that \( \mathcal{P}(\omega \rho^m) \cap N = \mathcal{P}(\omega \rho^m) \cap M \)

where \( \rho = \rho^{m+1}_N \leq \kappa < \omega \rho^m \). Thus:

**Cor 4.3** Let \( \rho = \rho^{m+1}_N \) and \( \omega \rho + N \leq \kappa < \omega \rho^m \). Then the conclusion of 4.2, 4.3 hold.

Proof:

\( \omega \rho + N = \omega \rho + M \), since \( \pi : N \rightarrow \mathbb{M}_1 \).
and \( \pi \cap N = id \). Hence \( \mathcal{P}(wp) \cap N = \mathcal{P}(wp) \cap M \) by acceptability.
QED (Cor. 4.3)

Lemma 4.4: Let \( wp^{m+1} \leq \alpha < wp^m \) in \( N \).

If \( R^m_N \neq \emptyset \), then:

(a) \( \pi : N \to \Sigma_0^m M \) cofinally

(b) \( \pi'' R^m_N \subseteq R^m_M \)

Proof:

\( m = 0 \) is trivial. Let \( m > 0 \).  

Claim: \( M \) is the closure of \( \alpha \) under good \( \Sigma_1^{(m-1)} \) fees.

Proof:

Let \( x \in M \), if \( x = \pi(f) \), then \( \chi(f) > 0 \) and \( f \) is a good \( \Sigma_1^{(m-1)}(N) \) fee. Then \( \pi(x) \in \pi(N) \) and \( \pi(f) \) is a good \( \Sigma_1^{(m-1)}(M) \) fee in a parameter \( \pi(f) \). Let \( q = G(\beta, \pi) \) where \( G \) is a good \( \Sigma_1^{(m-1)}(N) \) fee.
and \( \exists \chi < \omega \rho^m \). Let \( G = \Sigma^0_{ \frac{m-1}{N} } (M) \) by the same functionally absolute definition. Then \( x = \pi (f (x, G (\pi (3)), r) = H (x, \pi (3), r) \), where \( H \) is good, \( x \leq \pi (\lambda) < \omega \rho^m \) and \( \pi (3) < \omega \rho^m \).

QED (Claim 1).

Clearly \( \exists x \in H^k_m \) for \( k < n \).

It follows that \( H^m_{m-1} = h^m_{M^{m-1}, m-1} \).

Hence there is a \( \Sigma^0_{ \frac{m-1}{N} } (M) \) for

\( f \) mapping \( \omega \rho^m \) partially onto \( \omega \rho^{m-1} \). Hence \( \omega \rho^m = \omega \rho^m \).

Since otherwise \( \rho \) holds and \( \forall \langle \nu, \tau \rangle \exists f (\nu) \leq f (\tau) \) \( \exists x \in H^m_\nu \).

But \( \omega \rho^m \) is admissible in \( \tau \phi \), \( \mu \).

\( \omega \rho^m \) hence \( \forall \phi (\rho) < \omega \rho^{m-1} \).

Contrad! This proves (a).
Claim 2. Let \( n < m < 0 \) and the closure of any \( \Sigma_1^{(n)} \) under \( \Sigma_1^{(m)} \) for \( m < n \).

This follows from Claim 1. As it is apparent from the def. of \( \Sigma_1^{(m)} \) for \( m < n \), the \( \Sigma_1^{(m)} \) for \( m < n \) can be characterized as the smallest

Claim 1.

(a) Each \( \Sigma_1^{(i)} \) map to \( H_i \) is good \( \iff \)

\( n \leq i \leq m \).

(b) \( A \vdash G(x_1^i, ..., x_m^i) \) is a \( \Sigma_1^{(i)} \) map to \( H_i \)

\( (i, 1, \ldots, i, m) \) and \( F \) is a good map

to \( H \) \( \iff \)

then \( G(F(x_i)) \) is good.

By induction on \( n \), it follows that for all \( x \) there are

a good \( \Sigma_1^{(m)} \) \( G \) and \( \bar{z} \in H^{(n)} \) such

\( G(x) = G'(x, \bar{z}) \). As particular,

if \( x \in M \), \( x = G(d, x_0, ..., x_{m-1}) \),

\( x < \omega^m \), Then there is
\[-23-\]

\[ z \in H^m_{\mathcal{M}} \text{ s.t. } x = G'(d, \varepsilon, x_0, \ldots, x_{m-1}) \]

(living \( x_0, \ldots, x_{m-1} \in H^m_{\mathcal{M}} \)). But \( \omega \rho^m \) is p.l. closed + \( z = f'(3) \)

for a \( 3 < \omega \rho^m \). Hence \( x = G''(d, 3, x_1, \ldots, x_{m-1}) \) for a \( 3 < \omega \rho^m \)

where \( G'' \) is a good \( \Sigma_1^{(m-1)} \), \( f \subset \mathcal{N} \).

Q.E.D. (Lemma 4.4)

We now investigate the consequences of \( \Sigma_1 \) amenability.

**Lemma 5.1** Let \( F \) be \( \Sigma_1 \)-amenable, \( \frac{\omega \rho^m}{N} \leq \kappa < \omega \rho^m \). Then

\[ \pi \in \mathcal{N} \rightarrow \Sigma_1^{(m)} \text{ weakly essentially finite.} \]

**Proof:**

At \( n = 0 \) this is immediate. Otherwise \( \omega \rho^m > \kappa + \) it follows that \( F \) is

weakly amenable, since if \( \{x_i \mid i \leq \kappa \} \in \mathcal{N}_i \), then \( Y = \{i \mid x_i \in F \}_{\mathcal{N}} \) is \( \Sigma_1^{(n)}(N) \).

and \( Y \subset \mathcal{K} < \omega \rho^m \). Q.E.D. (5.11)
Lemma 5.2 Let $F$ be $\Sigma_1$-amenable.

Let $\omega^{m+1} \leq \pi \leq \omega^m$ in $\mathbb{N}$. Let $B \in \Sigma_1^{(m')} \subseteq \Sigma_1^0(M)$. Then $B \in \Sigma_1^{(m')} \subseteq \Sigma_1^0(M)$.

Proof:

Let $B \in \Sigma_1^{(m')} \subseteq \Sigma_1^0(M)$ in $[\beta, \overline{\alpha}(\beta)] = \overline{\alpha}(\beta)$

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Let $B \in \Sigma_1^{(m')} \subseteq \Sigma_1^0(M) - 24 -

Since $\kappa$ is p.o. closed, we can replace $\kappa$ by $J_{\kappa}^A$ in Lemma 5.2, where $M = \langle J_{\kappa}^A, D \rangle$. (Hence $J_{\kappa}^A = J_{\kappa}^A$, where $N = \langle J_{\kappa}^A, D \rangle$). We then conclude:
Lemma 5.3  Let $F$, $n$ be as above.
Then $\sum_{1}^{(m)}(M) \cap \#(J_{n}^{A}) = \sum_{1}^{(m)}(N) \cap \#(J_{n}^{A})$
where $M = \langle J_{n}^{A}, D \rangle$, $N = \langle J_{n}^{A}, D \rangle$.

Until further notice assume:

(*) $F \in \Sigma_{1}$ - amenable wrt. $N$ and one of the following holds:

(a) $F$ is weakly amenable
(b) $wp^{+} < \kappa$ where $\wp = \wp_{N}^{m}$, $\wp_{N}^{m} \leq \kappa$.

(Thus $\#(wp) \cap N = \#(wp) \cap M$, where $\wp = \wp_{N}^{m}$, $\wp_{N}^{m} \leq \kappa$. Note that (a) holds whenever $\wp_{N}^{m} \leq \kappa$.)

Lemma 6.1  Assume (*). Let $wp^{m} \leq \kappa$. Then

(i) $H_{N}^{m} = H_{M}^{m}$
(ii) $\sum_{1}^{(m)}(N) \cap \#(H_{N}^{m}) = \sum_{1}^{(m)}(M) \cap \#(H_{M}^{m})$

Proof (By induction on $m$).

We first prove (i). It suffices to show $wp^{m} = wp^{m}$, since $wp^{m}$ is a cardinal in $M$ and $H_{M}^{m} = J_{n}^{A} \cap M$
when $M = \langle J_{n}^{A}, D \rangle$ and similarly for $N$.
where \( N = \langle J, \overline{A}, \overline{B} \rangle \) and \( J^A = J^\overline{A} \). Let \( m = h + 1 \). Then \( \rho^m \geq \rho^m_N\), since there is \( B \subseteq \omega \rho^m_N \) i.t. \( B \subseteq \Sigma_1^m(N) \) and \( B \subseteq N \). Hence \( B \subseteq \Sigma_1^m(M) \), since \( \pi : N \rightarrow \Sigma_1^m(M) \) and \( \pi \upharpoonright N = \text{id} \). But \( B \subseteq M \), \( \pi \upharpoonright \Sigma_1^m(M) = \pi \upharpoonright N \) \( \rho = \rho^m_N \). We show: \( \rho^m_N \leq \rho^m_M \). Suppose not.

Then there is \( B \subseteq \omega \rho^m_M \), \( B \subseteq \Sigma_1^m(M) \) i.t. \( B \subseteq M \). But then \( B \subseteq \Sigma_1^m(N) \) by Lemma 5.2 if \( \kappa < \omega \rho^m_N \) and otherwise by the induction hypothesis.

But \( B \subseteq N \) since \( \rho \upharpoonright \kappa \downarrow N \subseteq M \).

Contradiction! This proves (i).

To prove (ii) let \( H = H^m_N = H^m_M \) and let \( B \subseteq H \). \( B \subseteq \Sigma_1^m(M) \) i.t. \( B \subseteq \Sigma_1^m(H, D) \) for a \( D \subseteq \Sigma_1^m(M) \). But this is equivalent to saying \( B \subseteq \Sigma_1^m(H, D) \) for a \( D \subseteq \Sigma_1^m(N) \) (by 5.2 or the induction hypothesis), which is in turn equivalent to: \( B \subseteq \Sigma_1^m(N) \).

QED (Lemma 6.1)
Cor 6.2 \[ \pi : N \xrightarrow{\cong} M \]

Proof: We show \( \pi : N \xrightarrow{\cong} M \). For \( \wp^m \geq n \)

For \( \wp^m > n \) we use Lemma 5.1. For \( \wp^m < n \)

employ induction on \( n \), using Lemma 6.1(a).

QED (Cor 6.2)

Cor 6.3 \[ \pi^* P^m \subseteq P^m \text{ for } \wp^m \leq n. \]

Proof:

Let \( \tilde{p} \in P^m, p = \pi(\tilde{p}) \). Assume \( H \in \text{Horn} \) to show

Claim: \( A_{P^h} \not\subseteq M \) for \( m \geq h \geq 1 \).

We proceed by induction on \( m - h \). At \( m = h \), then \( A_{P^h} = A_{P^h} \), since

\[ \pi : N \xrightarrow{\cong} M, \pi \mid H^m = \text{id}. \]

But

\[ A = A_{P^h} \not\subseteq N; \text{ hence } A \not\subseteq M, \]

\( \pi \mid H \cap M = \pi \mid H \cap N \), where \( H = H^m = H^m \).

Now let it hold for \( h + 1 \leq m \). Then

\[ A_{P^{h+1}} = B \cap H^{h+1}, \text{ where } B \in \Sigma_1 \langle H^h, A_{P^{h+1}} \rangle \text{ in } p. \]

Hence

\[ A_{P^{h+1}} \not\subseteq M, \text{ since otherwise } \langle H^h, A_{P^{h+1}} \rangle \]

\[ \subseteq M. \]

QED (Cor 6.3)
Corollary 6.4 \( \mathcal{P}_N^* \subseteq \mathcal{P}_M^* \)

Proof: Lemma 3 \( \forall \omega \in \omega \); otherwise Corollary 6.3

Corollary 6.5 \( \#(\kappa) \cap \Sigma_1^{(m)}(N) = \#(\kappa) \cap \Sigma_1^{(m)}(M) \).

1. \( \forall \omega^m \uparrow \kappa, \text{ then } \omega^m \uparrow \kappa \) and 2. \( \forall \omega^m \uparrow \kappa, \text{ let } N = \langle J_\alpha^M, J_\alpha^N \rangle, \text{ and } M = \langle J_\alpha^M, J_\alpha^N \rangle \).

Now let \( \omega^m \uparrow \kappa \), \( \kappa \in \text{Induction on } \kappa \text{ we prove:} \)

Claim \( \#(\kappa) \cap \Sigma_1^{(m)}(N) = \#(\kappa) \cap \Sigma_1^{(m)}(M) \)

1. \( \forall \omega^m \uparrow \kappa, \text{ it follows by Lemma 5.2,} \)

Now let \( \omega^m \uparrow \kappa, \text{ let } N = \langle J_\alpha^M, J_\alpha^N \rangle \text{ and } M = \langle J_\alpha^M, J_\alpha^N \rangle \text{ where } \phi \in \Sigma_1^m \text{ and } R_{\phi}(\bar{x}, \bar{z}^m) \iff \langle H, \overrightarrow{R}_{\phi} \rangle \models \phi, \)

where \( \phi \in \Sigma_1^m \) and \( R_{\phi}(\bar{x}, \bar{z}^m) \iff \langle H, \overrightarrow{R}_{\phi} \rangle \models \phi, \)

\( \iff R_{\phi}(\bar{x}, \bar{z}^m) \) and \( R_{\phi} \in \Sigma_1^m \)

relational on \( H \). The claim follows by the induction hypothesis and \( H_N^m = N_M^m \). QED (Cor. 6.5)
Finally we note a stronger form of Lemma 5.

Lemma 7. Let $F$ be $\Sigma_1$-amenable. Let $\omega^m + 1 < \nu < \omega^m$. Let

$$R(x_1, \ldots, x_p, y_1, \ldots, y_q) \in \Sigma_1^{(m)}(N)$$

Let $f_1, \ldots, f_\nu \in \Gamma(n, N)$. Let $d_1, \ldots, d_\nu < \nu$. Let

$$\tilde{R} = \{ \tilde{x} \in \Gamma(n, N) \mid R(\pi(x_1), \ldots, \pi(x_p), \pi(f_1)(d_1), \ldots, \pi(f_\nu)(d_\nu)) \in \Sigma_1^{(m)}(N) \}$$

Moreover, if $p_i \in \Gamma(n, N)$ or $f_i$ is a good $\Sigma_1^{(m)}(N)$ function in $p_i$ (i.e., $i = 1, \ldots, p$) and

$$r \in \text{int.} \ E_{d_1, \ldots, d_\nu} \in \Sigma_1(N) \in \tilde{R},$$

then $\tilde{R}$ is $\Sigma_1^{(m)}(N)$ in $r$, i.e., uniformly in the $\Sigma_1^{(m)}$-class of $R$, the functionally absolute class of $f_i$ from $p_i$ (i.e., $i = 1, \ldots, p$) and the $\Sigma_1$-class of $F_\nu$ from $r$.

The proof is a virtual repetition of that of Lemma 5.2.

(Remark: The proof uses only that $E_{d_1, \ldots, d_\nu} \in \Sigma_1^{(m)}(N)$ in $r$.)
Lemma 8  Let \( \langle N_i \mid i < \theta \rangle, \langle \pi_{ij} \mid i \leq j < \theta \rangle \) be
such that \( N_0 \) is acceptable and:
(a) \( N_i \) is transitive
(b) \( \pi_{ij} : N_i \rightarrow N_j, \) \( \pi_{ij} \circ \pi_{kh} = \pi_{ik}, \) \( \pi_{ii} = \text{id}_i \)
(c) \( \pi_{ii} : N_i \rightarrow N_i \) is the direct limit of
\( \langle N_i \mid i < \xi \rangle, \langle \pi_{ij} \mid i \leq j < \xi \rangle \) for limit \( \xi < \theta \)
(c) \( \pi_{ii+1} : N_i \rightarrow N_{i+1} \) where \( F \) is weakly
amenable and \( \varepsilon_i - \) amenable on \( N_i \).

Then for all \( i < \theta \):
(i) \( N_i \) is acceptable
(ii) \( \pi_{ij} : N_i \rightarrow N_j \) for \( i \leq j \)
(iii) \( \pi_{ij} : P^x_i \rightarrow P^x_j \) for \( i \leq j \)
(iv) Let \( \kappa_i = \text{cof}(F_i). \) If \( \kappa_i \leq \kappa_h \) for
\( i \leq h < i \), then \( \#(\kappa_i) \cap \Sigma^{\omega_1}(N_i) = \#(\kappa_i) \cap \Sigma^{\omega_1}(N_i) \)
for \( m < \omega \).
(v) \( \#(\kappa_h) \leq \omega \) for \( i \leq h < i \), then
\( \pi_{ij} : N_i \rightarrow N_j \) and \( \pi_{ij} : P^m_i \rightarrow P^m_j \)
will also \( \omega^{m+h} \leq \kappa_h \leq \omega^m \) for \( i \leq h < i \),

Then \( \pi_{ij} : N_i \rightarrow N_j \) cofinally,
Lemma 8 is proven by induction on $j$.

Note When $\langle N_i, \pi_i \rangle, \langle \pi_{i'} \mid i \leq i' < \theta \rangle$ are as in (a1, (b), $\lim (\Theta)$, and $\langle N_i \rangle, \langle \pi_{i'} \rangle$ has a well-founded direct limit, then we often write:

$$N_i \langle \pi_i \mid i < \theta \rangle = \lim_{i \leq i' < \theta} (N_i, \pi_{i'})$$

to indicate that $N_i \langle \pi_i \rangle$ is the transitive direct limit of $\langle N_i \rangle, \langle \pi_{i'} \rangle$. 