§ 6 Non-Unique Henkons
(by Martin Zeman)

In the following suppose $M$ to be a premouse and $M' \upharpoonright \omega$ to be a direct normal iteration of $M$ of limit length $\lambda$. Let $b, b'$ be distinct branches cofinal in $\omega$ s.t. $M_b, M_{b'}$ both are well founded (hence can be taken as transitive), where

$$\langle M_b, \langle \pi_{i,b} \rangle \rangle = \lim_{i < \gamma \in \varepsilon} (M_{i,b}, \pi_{i,j})$$

and similarly for $M_{b'}, \pi_{i,b'}$.

Clearly, if $\pi \subseteq b \cap b'$ then $b \cap (\pi + 1) \cap b' \cap (\pi + 1) = \phi$.

We may assume w.l.o.g. that $b - x, b' - x$ have no truncation point.

We now recall some definitions:

**Def.** Let $N$ be transitive + $\pi$-closed.

$N$ is **strong** in $N$ iff there are arbitrary large $\lambda \in N$ s.t.

a) $\lambda$ is a cardinal in $N$ and $H^N_{\lambda} \in N$.

b) There is an extender $E \in N$ at $\kappa, \lambda$ s.t. $F : N \rightarrow N'$ extends and $H^N_{\kappa} = H^N_{\lambda}$

**Def.** Let $N$ be as above, where $\langle N, A \rangle$ is extendible, $A \subseteq \text{On}$. Let $F$ be an extender at $\kappa, \lambda$.

$F$ is coherent with $A$ over $N$ iff $\pi : N \rightarrow N'$ exists and $F(\pi(x)) = A \cap x$.

**Def.** Let $\langle N, A \rangle$ be as above. $\lambda \in N$ is **$A$-strong** in $N$ iff there are arbitrarily large $\lambda \in N$ s.t. some $F \in N$ is at $\kappa, \lambda$ and is coherent with $A$.?
Note. Let $N = \mathcal{J}_\alpha^A$ be acceptable, where $A$ need not be a set of ordinals. We define the coherency of $F$ on $A$ over $N$ as before, except that we require $\mathcal{E}(A \cap |\mathcal{J}_\alpha^A|) = A \cap |\mathcal{J}_\alpha^A|$ instead of $F(\text{An} \kappa) = A \cap \kappa$.

(Note. Let $N = \mathcal{J}_\alpha^A$ be acceptable s.t. there are arbitrarily large $\tau < \kappa$ s.t. $\tau$ is a cardinal in $N$. If $\kappa$ is $A$-strong in $N$, then $\kappa$ is $A$-strong in $N$.)

Def. Let $N$ be as above, where $N = H^M_{\kappa^1}, \kappa \in M$. $\kappa$ is Woodin in $M$ iff for every $A \in M$, $A \subset \kappa$, there is $\lambda \in N$ which is $A$-strong in $N$.

(Note. Assuming that $N$ has a well-ordering in $M$, this is equivalent to the usual definition of Woodinness. In particular $\langle N, A \rangle \models ZFC$ whenever $A \in M$, $A \subset \kappa$.)

\[ \cdots \cdots \]

Returning to our structures $M, Y$, we define:

\[ \xi_0 = \text{min} \{ b - (\omega + 1) \} \]
\[ \xi_{2i+1} = \text{min} \{ b' - \xi_{2i} \} \]
\[ \xi_{2i+2} = \text{min} \{ b - \xi_{2i+1} \} \]

Then $\langle \xi_i : i < \omega \rangle$ is monotone and:

\[ T(\xi_{2i+1}) < \xi_{2i} < \xi_{2i+1} \]
\[ T(\xi_{2i+2}) < \xi_{2i+1} < \xi_{2i+2} \]

Hence:

(1) \[ T(\xi_{i+1}) < \xi_i < \xi_{i+1} \]
Clearly each \( \xi_n \) is a successor ordinal since it is a successor in its brand. Let:

\[
\begin{align*}
\text{Def} \quad \xi_n &= \xi_{n-1} \quad \xi_n^* := T(\xi_n)
\end{align*}
\]

By (1) we conclude:

(2) \( \kappa \gamma_{m+1} < \kappa \xi_{m+1}^* \leq \kappa \gamma_m \leq \kappa \gamma_{m+2} \)

(\( \kappa \gamma_m \leq \kappa \gamma_{m+2} \) \( \text{since } (\gamma_m + 1) \times (\gamma_{m+2} + 1) \))

We have:

(3) \[
\begin{array}{ccc}
\pi_{\xi_n^*} & : & M_{\xi_n^*} \\
\downarrow & {} & \downarrow \\
E_{\gamma_m} & : & M_{\xi_n^*}
\end{array}
\]

Since \( \xi_n^* \) is not a truncation point.

(4) \( \sup_{\mathfrak{m}} \xi_n = \lambda \),

since otherwise \( \sup_{\mathfrak{m}} \xi_n \in (b \cap b') - \lambda \).

(5) \( \tau_{\gamma_m} = \kappa \gamma_m \cup \gamma_{\mu_n} \) is a cardinal in \( M_{\xi_n^*} \),

since otherwise \( \xi_n^* \) would be a truncation point.

But \( \tau_{\gamma_m} < \kappa \xi_n^* \) and \( \kappa \xi_n^* \) is a limit cardinal in \( M_{\gamma} \) for \( \gamma > \kappa \xi_n^* \). Hence:

(6) \( \tau_{\gamma_m} \) is a cardinal in \( M_{\gamma} \) for \( \gamma > \kappa \xi_n^* \)

\[
\begin{align*}
\text{Def} \quad \mathcal{F} &:= \mathop{\sup}_n \kappa_{\gamma_n} = \mathop{\sup}_n \gamma_{\mu_n} \\
\mathcal{N} &:= \mathcal{F} = \mathop{\bigcup}_n \mathcal{E}_{\gamma_{\mu_n}} = \mathop{\bigcup}_n \mathcal{E}_{\gamma_{\mu_n}}
\end{align*}
\]
Lemma 1. There are arbitrarily large \( \tau \) which are strong in \( N \).

Proof.

For all but finitely many \( i < \omega \) we construct a sequence of extenders \( \langle G_i^n \mid n < \omega \rangle \) s.t.

1. \( G_i^n \in N \) and \( N \) is extendible by \( G_i^n \)
2. \( \text{crit}(G_i^n) = \kappa_i^+ \), where \( \kappa_i \leq \kappa_i < \kappa_{i+2} \)
3. \( \text{lh}(G_i^n) = \kappa_{i+n+2} \)
4. \( G_i^n = G_i^n \upharpoonright \kappa_{i+n+2} \) for \( m \leq n \)
5. \( G_i^n \) is coherent w.r.t. \( E \)

This will prove the lemma.

We first specify the set \( ND \) on which \( \langle G_i^n \rangle \) is not defined.

**Def.** \( ND \) is the set of those \( i < \omega \) for which

\[
e \in (E_{M_i}^{\kappa_{i+2}}) \in \prod_{\tau^+} \kappa_{i+2}, \kappa_{i+2}
\]

Claim 1.1 \( ND \) is finite.

**Proof.** Suppose not. Then there are \( i, j \in ND \) s.t.

\( i < j \) and \( ij \) are both odd or even.

Then \( \xi_{i+2} < \tau, \xi_{j+2} \) and \( \xi_{i+2}, \xi_{j+2} : M_{\xi_{i+2}} \rightarrow M_{\xi_{j+2}} \)

Also \( \text{crit}(\pi_{\xi_{i+2}, \xi_{j+2}}) \geq \kappa_{i+2} \geq \kappa_{j+2} \).

Define \( \bar{E} = E_{M_{\xi_{i+2}}}^{\kappa_{i+2}} \). Then \( \text{crit}(\bar{E}) \geq \kappa_{i+2} \).

But then \( \bar{E} \leq \bar{E}_{M_{\xi_{i+2}}}^{\kappa_{i+2}} \), thus \( \text{crit}(\bar{E}) \geq \kappa_{j+2} \), which means that \( j \notin ND \). Contradiction.

\( \square \) Claim 1.1.

Before proceeding further we prove some general lemmas on normal ultrapowers \( U = \langle (M_i) \mid i < \lambda \rangle \), \( \lambda \).


Claim 1.2. Let \( \bar{\nu} = \nu \epsilon (E_{\nu}) \prec \kappa \prec \lambda_\beta \), \( E_{\nu}^{M_\gamma} \mid \kappa \notin M_\gamma \), \( \kappa \) and \( \gamma \) are regular.  

a) \( \nu_\gamma = h \epsilon (M_\gamma) \)  
b) \( \aleph_\gamma^M \leq \kappa \)

Proof:  
a) Immediate  
b) \( \xi < \kappa, \xi \epsilon \bar{\nu} \) & \( \xi \epsilon \kappa \) & \( \xi \epsilon E_{\nu_\gamma}^{M_\gamma} \)  

Thus \( \xi \epsilon \Xi_\gamma (M_\gamma) \) subset of \( J_\kappa^{M_\gamma} \).

\( \blacksquare \) Claim 1.2

Claim 1.3. Let \( \eta, \chi, \kappa \) be as above. Let \( \beta + 1 \leq \gamma \) s.t. \( \kappa < \lambda_\beta \). Then  
a) \( E_{\nu_\gamma} (\pi_{\beta+1} \gamma) \prec \kappa \) for \( \beta + 1 \prec \gamma \)  
b) \( \pi_{\beta+1} \gamma \) is total

Proof:  
a) Let \( \delta \) be s.t. \( \beta + 1 = T (\delta + 1) \). Then \( \beta + 1 \) is the least \( \xi \) with \( \lambda_\xi \prec \kappa \).  

Thus, \( \kappa < \lambda_\delta \prec \kappa \).  

b) Suppose not. Let \( \xi \) be the least point of truncation.  

Then \( \pi_{\xi} : M_\xi^* \rightarrow M_\gamma \) with \( E_{\nu_\gamma} (\pi_{\xi} \gamma) \prec \kappa \). \( \xi \)

Thus, \( \aleph_\gamma^M \leq \kappa \) since \( E_{\nu_\gamma}^{M_\gamma} \kappa = E_{\nu_\gamma}^{M_\gamma} \kappa \). But \( \lambda_\beta \)  

\( \in M_\gamma^* \), thus it is collapsed in \( M_\gamma^* \), but \( \lambda_\beta \) must 

be a cardinal in \( M_\xi^* \). Contradiction.

\( \blacksquare \) Claim 1.3

\( \ast \) 12. \( M_\xi^* \) is the truncate of \( M_\xi \), i.e. \( M_\xi^* = M_\xi \upharpoonright \gamma \gamma \), where \( \xi = T (\delta \)
Def. Let \( p, \lambda \) be as above. We define \( p^{\ast} = p^{\ast}(\lambda) \leq p \) as follows:

\[ p^{\ast} = \text{that } \beta \text{ st. } \beta + 1 \leq p \text{ and } k_\beta \leq \lambda \leq k_\beta. \]

If there is no such \( \beta \) or \( p, \lambda \) are not as in Claim 2, then \( p^{\ast} \) is undefined.

Def. Let \( p, \lambda \) be as above. We define \( p^{h} = p^{h}(\lambda) \) inductively by:

\[ p^{0} = p, \quad p^{h+1} = (p^{h})^{\ast} \]

(Hence \( p^{h}, \lambda \) are as in Claim 1.2 of \( p^{h+1} \) is defined.)

Let \( m = m(p, \lambda) \) be the largest \( h \) st. \( p^{h} \) is defined.

Def. Let \( p, \lambda \) be as above. Set

\[ \delta = \delta(p, \lambda) = \text{the least } \delta \text{ st. } \lambda < \lambda_{\delta}. \]

Claim 1.4. Let \( p, \lambda \) be as above. Let \( m = m(p, \lambda), \quad \bar{p} = p^{m}(\lambda). \)

Then one of the following holds:

a) \( E_{p}^{m} \downarrow \lambda \in M_{\bar{p}} \)

b) \( \delta = \delta(p, \lambda) \leq \bar{p} \)

Proof. Suppose not. Then both a), b) fail. In particular,

\[ \delta(p, \lambda) \not\leq \bar{p}, \text{ thus } \delta(p, \lambda) < \bar{p}. \]

Let \( \beta \) be least \( s.t. \beta + 1 \leq \bar{p} \) and \( \lambda < k_\beta. \) Let \( \delta := T(\beta + 1). \)

Then \( \lambda < k_\beta \) since \( p^{\ast} \) is not defined. But then \( \lambda < \lambda_{\delta} \) since \( k_\beta < \lambda_{\delta}. \) Thus \( \delta(p, \lambda) < \delta, \) in fact \( \delta(p, \lambda) < \delta \) by the failure of b).

Thus either

\[ \delta = \delta + 1 \]

or \( \delta \) is limit, in which case we pick any \( \gamma \in \left[ \delta, \delta + 1 \right) \)

s.t. \( \gamma + 1 \leq \bar{p} \)

In both cases \( \lambda < \lambda_{\gamma} \), \( \gamma + 1 \leq \bar{p} \) and \( \gamma < \beta \),

which is a contradiction with the definition of \( \beta. \)

\[ \square \text{ Claim 1.4} \]
Claim 1.5 Let i \(\notin \text{ND} \). Set \( \gamma^i := \gamma^i + 1 \) and \( \kappa^i := \kappa^i + 2 \).

Define \( \gamma^h = \gamma^h(k) \) as above. Let \( m := m(\gamma^i, \kappa^i) \), \( \bar{\gamma} := \gamma^m \). Then \( \mathcal{E}_{\bar{\gamma}}^{f|k} \in \mathcal{M}^f_{\bar{\gamma}} \).

It is enough to prove:

Claim 1.6 We take the assumptions of Claim 1.5.

Let \( \gamma^h \) be defined. Set \( \kappa_m := \epsilon(\mathcal{E}_{\gamma^h}^m) \).

If \( \mathcal{E}_{\gamma^h}^m | k \notin \mathcal{M}^h_{\gamma^h} \) then:

a) \( \gamma^{h+1} \) is defined.

b) \( \kappa_m > \kappa_{\gamma^h} \).

Proof By induction. Suppose \( \gamma^{h+1} \) is not defined.

Then \( \delta(k) \leq \gamma^h \) by Claim 4.

Case 1 \( \delta(k) = \gamma^h \)

But \( \delta(k) = \gamma^h \). Thus \( \kappa = \gamma^h \).

and \( \kappa > \kappa_{\gamma^h} > \kappa_{\gamma^h-1} > \ldots > \kappa_{\gamma^h} = \kappa_{\gamma^h+1} \) \( \ast \)

By Claim 3, \( \pi_{\gamma^h+1} \) is total and \( \epsilon(\pi_{\gamma^h+1} \gamma^h) \geq \lambda_{\gamma^h} \).

Thus, \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\gamma^h+1}^m) \) and \( \mathcal{E}_{\gamma^h+1}^m(k) \in \mathcal{M}^h_{\gamma^h+1} \).

Now \( \kappa \leq \kappa_{\gamma^h} \) since we assume that \( \gamma^{h+1} \) is not defined.

Thus, \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\gamma^h+1}^m) \) and \( \mathcal{E}_{\gamma^h+1}^m \in \mathcal{M}^h_{\gamma^h+1} \).

i.e. \( \kappa_{\gamma^h} \leq \kappa \). But \( \mathcal{E}_{\gamma^h+1}^m(k) \in \mathcal{M}^h_{\gamma^h+1} \). Thus \( \gamma^{h+1} = \mathcal{E}_{\gamma^h+1}^m(k) \).

Thus \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\gamma^h+1}^m) = \epsilon(\mathcal{E}_{\gamma^h+1}^m) \). Thus \( \kappa \notin \text{ND} \)

as in Case 1. Contradiction. \( \ast \) Case 2

Case 2 \( \delta(k) < \gamma^h \)

Let \( \beta \) be st. \( \gamma^h = \delta(k) = \tau(\beta+1) \) and \( \beta+1 \leq \gamma^h \).

By Claim 3, \( \pi_{\beta+1} \gamma^h \) is total and \( \epsilon(\pi_{\beta+1} \gamma^h) > \lambda_{\gamma^h} \).

Thus, \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\beta+1}^m) \) and \( \mathcal{E}_{\beta+1}^m(k) \in \mathcal{M}^h_{\gamma^h} \).

Now \( \kappa \leq \kappa_{\gamma^h} \) since we assume that \( \gamma^{h+1} \) is not defined.

Thus, \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\beta+1}^m) \) and \( \mathcal{E}_{\beta+1}^m \in \mathcal{M}^h_{\gamma^h} \).

i.e. \( \kappa_{\gamma^h} \leq \kappa \). But \( \mathcal{E}_{\beta+1}^m(k) \in \mathcal{M}^h_{\gamma^h} \). Thus \( \gamma^{h+1} = \mathcal{E}_{\beta+1}^m(k) \).

Thus \( \kappa_{\gamma^h} = \epsilon(\mathcal{E}_{\beta+1}^m) = \epsilon(\mathcal{E}_{\beta+1}^m) \). Thus \( \kappa \notin \text{ND} \)

as in Case 1. Contradiction. \( \ast \) Case 2

# a)
Proof of b) Set $\beta = \gamma^{\kappa+1}$, $\varepsilon_0 = \tau(\beta+1)$. Then again by Claim we have $\pi_{\beta+1}, \gamma^\kappa$ is total and $\nu(\pi_{\beta+1}, \gamma^\kappa) > \lambda_{\beta} > \kappa$, thus $\kappa^\kappa = \nu(\pi_{\beta+1}, \gamma^\kappa)$. But $\kappa^\kappa < \lambda_{\beta} = \pi_{\varepsilon_0+1}(\kappa^\kappa)$, whence $\kappa^\kappa = \nu(\pi_{\varepsilon_0+1}, \gamma^\kappa)$.

Thus, $\kappa^\kappa < \kappa^\kappa < \kappa^\kappa$.

Claim 1.5

We now define the sequence $\langle G_i^\kappa, i \in \omega \rangle$ for $i \in \text{ND}$.

Def Let $i \in \text{ND}$, $\gamma_i = \gamma_i^{i+1}$, $\kappa_i = \kappa_i^{i+2}$, $\nu_i = \nu(\gamma_i, \kappa_i)$, and $\bar{\nu}_i = \bar{\nu}_i$. Set

$$G_i^\kappa = E_{\bar{\nu}_i}^{M_{\bar{\nu}_i}} \kappa_i$$

Claim 1.7 $G_i^\kappa \in M_{\bar{\nu}_i+2}$.

In fact, $G_i^\kappa \in \check{E}_{\bar{\nu}_i+2}^{M_{\bar{\nu}_i+2}}$ and $\check{E}_{\bar{\nu}_i+2}^{M_{\bar{\nu}_i+2}} = (G_i^\kappa, \bar{\nu}_i)$, are complete.

Proof We know $G_i^\kappa \in M_{\bar{\nu}_i}$ and $\bar{\nu}_i = \delta(\kappa) \leq \bar{\nu}_i$ (since $\kappa < \bar{\nu}_i$).

Case 1 $\check{\nu}_i+2 = \delta(\kappa) = \bar{\nu}_i$.

Then $G_i^\kappa \in M_{\bar{\nu}_i+2}$ follows immediately. Now, we know that $G_i^\kappa$ is extendable by a subset of $\kappa$, $\lambda_{\bar{\nu}_i+2}$ is a cardinal in $M_{\bar{\nu}_i+2}$ and $\check{\nu}_i+2 = \check{\nu}_i+2 = \lambda_{\bar{\nu}_i+2}$. Thus $\kappa^+ \lambda_{\bar{\nu}_i+2} < 1$ = $\lambda^+ \bar{\nu}_i+2 < \lambda^+ \check{\nu}_i+2$ so $G_i^\kappa \in \check{E}_{\bar{\nu}_i+2}^{M_{\bar{\nu}_i+2}}$.

Case 2 $\check{\nu}_i+2 = \delta(\kappa) < \bar{\nu}_i$.

Thus, $\lambda^+ \check{\nu}_i+2$ is a cardinal in $M_{\bar{\nu}_i}$ and $G_i^\kappa \in M_{\bar{\nu}_i}$. By acceptability, $G_i^\kappa \in \check{E}_{\bar{\nu}_i+2}^{M_{\bar{\nu}_i}}$.

As to completeness, we know that $\check{E}_{\bar{\nu}_i+2}$ is extendable by $E_{\bar{\nu}_i}^{M_{\bar{\nu}_i}}$, so $\check{E}_{\bar{\nu}_i+2}$ is so, since $\lambda \nu_i < \nu_i$ and $\nu_i \leq \bar{\nu}_i$. 

\[ \]
Fact then $\mathcal{E}^{\mathcal{M}_{i+2}}_{i+2}$ is separable by $G_i^c$, which is an initial segment of $\mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}}$. Since $G_i^c \in \mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}}$ and $\mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}} \models \exists F$, the rest of the claim follows.

Claim 1.8 $G_i^c$ has the properties (a) - (e).

Proof (a) We know $\mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}} \cap \mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}} = \mathcal{E}^{\mathcal{M}_{i+2}}_{\bar{\tau}}$, so $G_i^c \in \mathcal{N}$. Since $\mathcal{A}^{\mathcal{M}_{i+2}}_{\bar{\tau}}$ is a cardinal in $\mathcal{N}$, we get $\mathcal{N} \models (G_i^c \text{ is $\omega$-complete})$, so $\mathcal{N}$ is extendible by $G_i^c$.

(b) - (d) are trivial.

Proof of (e): Let $\pi : \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \to \mathcal{N}'$, $\bar{\pi} : \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \to \mathcal{N}''$, where $\bar{\tau}^+ = (\bar{\tau}^+)^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}$. Then $\pi(a) \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} = \bar{\pi}(a) \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}$ for any $a \in \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}$, $a \in \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}$ by a standard argument. In particular,

$$\pi\left(E^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}\right) \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} = \bar{\pi}\left(E^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}\right) \cap \mathcal{E}^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} = E^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \cap \mathcal{N}.$$ 

The rest follows from the fact that $E^{\mathcal{M}_{i+1}}_{\bar{\tau}^+} \models E^{\mathcal{M}_{i+1}}_{\bar{\tau}^+}$.

Claim 1.9 $G_i^{n+1}$ has the properties (a) - (e).

Moreover, $G_i^{n+1} \in \mathcal{E}^{\mathcal{M}_{i+n+2}}_{\bar{\tau}}$ and is $\omega$-complete inside it.
Proof (a) we know $G_{i+1}^{u+1} \in M_{\xi_i+u+2}$ and is codeable there by a subset of $\kappa_{\xi_i+u+3} < \lambda_{\xi_i+u+2} \leq \lambda_{\xi_i+u+2}$. Both the latter are cardinals in $M_{\xi_i+u+2}$, so $G_{i+1}^{u+1} \in \bigcup_{\xi_{i+u+2}^H} \lambda_{\xi_{i+u+2}^H}$. Then $G_{i+1}^{u+1} \in N$ follows from: $E^{M_{\xi_i+u+2}} \cap \lambda_{\xi_i+u+2} = E^{\bigcup_{\xi_{i+u+2}^H} \lambda_{\xi_{i+u+2}^H}}$.

The second part of Claim 9 follows from the fact that

i) $\bigcup_{\xi_{i+u+2}^H} \lambda_{\xi_{i+u+2}^H} = \left( G_i^u \cap \omega \right.$-complete $\left.) \right.$

where $\lambda_{\xi_i+u+2} = \pi_{\xi_i+u+2}^\xi_i, \xi_i+u+2 = \lambda_{\xi_i}^{\pi_{\xi_i+u+2}}$

ii) $E^{M_{\xi_i+u+2}} \cap \lambda_{\xi_i+u+2} = E^{M_{\xi_i+u+3}} \cap \lambda_{\xi_i+u+3}$

and from acceptability.

Using all of this, we get $N \models (G_{i+1}^{u+1} \cap \omega)$-complete.

Since $E^{M_{\xi_i+u+2}} \cap \lambda_{\xi_i+u+3} = E \cap \lambda_{\xi_i+u+3}$ and $\lambda_{\xi_i+u+3}$ is a cardinal in $N$, so $N$ is extendible by $G_{i+1}^{u+1}$.

(e) We recall the following two facts, which we will use in the proof:

\[
\{ \begin{array}{l}
E \cap \lambda_{\xi_i+u+2} = E^{M_{\xi_i+u+2}} \cap \lambda_{\xi_i+u+2} \\
E \cap \lambda_{\xi_i+u+3} = E^{M_{\xi_i+u+3}} \cap \lambda_{\xi_i+u+3}
\end{array} \]

By the induction hypothesis, if $\pi_i^N$ is defined by

$\pi_i^N : \mathcal{E}_{\tilde{\xi}_i} \rightarrow N_i^N$, then $E \cap \lambda_{\xi_i+u+2} = E \cap \lambda_{\xi_i+u+3}$.

Let $\tilde{\pi}_i^N : \mathcal{E}_{\tilde{\xi}_i} \rightarrow \tilde{\tilde{N}}_i^N$. Then $E \cap \lambda_{\xi_i+u+2} = E \cap \lambda_{\xi_i+u+3}$.

Thus, as in Claim 8, $E^{N_i^{\tilde{\tilde{N}}}} \cap \lambda_{\xi_i+u+3} = E^{\tilde{\tilde{N}}_i^N} \cap \lambda_{\xi_i+u+3} = E \cap \lambda_{\xi_i+u+3}$ and $\pi_i^{N_i^{\tilde{\tilde{N}}}}(a) \cap \mathcal{E}_{\tilde{\xi}_i} = \tilde{\tilde{\pi}}_i^N(a) \cap \mathcal{E}_{\tilde{\xi}_i}$ for $a \in \mathcal{E}_{\tilde{\xi}_i}, a \in \mathcal{E}_{\tilde{\xi}_i}^N$. Then:

\[ \pi_{\xi_i+2}^\times (E \cap \kappa_i^\times) \cap J_{\kappa_i \gamma_i+2}^E = \pi_{\xi_i}^\times (E \cap \kappa_i^\times) \cap J_{\kappa_i \gamma_i+2}^E = \]
\[ = \pi_{\xi_i}^\times (E \cap \kappa_i^\times) \cap \bigcap_{\gamma_i \gamma_i+2} J_{\kappa_i \gamma_i+3}^E = \]
\[ = \pi_{\xi_i+2}^\times (E \cap \kappa_i^\times) \cap \bigcap_{\gamma_i \gamma_i+2} J_{\kappa_i \gamma_i+3}^E = \]
\[ = \pi_{\xi_i+2}^\times (E \cap \kappa_i^\times) \cap \bigcap_{\gamma_i \gamma_i+3} J_{\kappa_i \gamma_i+3}^E = \]

We used the identities \((\gamma)\) to derive the third and the last equality.

Thus proves lemma 1.

In what follows we shall make extensive use of the sequences \(\langle G_i^\mu ; \text{new} \rangle\), \(\mu \neq \text{ND}\) defined above.

**Def.** \(\alpha = \operatorname{sup} \left( \text{def} (M_\alpha), \text{def} (M_\beta) \right)\)

\(Q = J_\alpha^E\)

(Note. \(E \cap \kappa^\times \cap J_\alpha^E\), hence \(E_{\kappa,\nu}^\mu = \emptyset\) for \(\delta \preceq \nu < \alpha\))

**Lemma 2.** If \(\delta \subseteq \delta^\#\), then \(\delta\) is Woodin in \(Q\).

**Proof.** \(\delta = \operatorname{sup} \{ \kappa_i^\times ; \text{new} \} \), thus \(\delta\) is a cardinal in \(Q\) and \(N = H_\delta^Q\) by acceptability.

We prove that if \(A \subseteq Q\), \(A \cap \delta\) then all but finitely many \(\kappa_i^\times\) are \(A\)-strong in \(N\).

Let \(A \subseteq Q\), \(A \subseteq \delta\). Pick \(i\) big enough so that
\[ A \in \text{rng} (\pi_{\gamma_i+2}^\times) \cap \text{rng} (\pi_{\gamma_i+2}^{\#}) \] for \(\gamma \geq \xi_i+2\), \(\gamma' \geq \xi_i+1\).

For \(\gamma \in b_{b_1}^b\) let \(A_\gamma\) be s.t. \(\pi_{\gamma_i+2}^\times (A_\gamma) = A\) and \(\text{rng} (\pi_{\gamma_i+2}^\times (A_\gamma)) = A\).
Claim 2.1  \(^{\widetilde{G}_i^0}\) is coherent not \(A\).

Proof  We remind the definition of \(\widetilde{G}_i^0\). \(\widetilde{G}_i^0 = \mathcal{E}_{\Omega_{y_i^0}}^{M_0} k_{y_{i+2}}\), 
where \(y_i^0 = y_i^{\nu_i}\) and \(\nu = \nu_i (y_{i+1}, y_{i+2})\). We prove the claim by induction on \(\nu_i\), \(0 \leq \nu_i \leq \omega\), for \(\mathcal{E}_{\Omega_{y_i^0}}^{M_0} k_{y_{i+2}}\).

Case 1  \(\nu_i = 0\)

Then \(y_i^0 = y_{i+1}\), thus \((\mathcal{E}_{\Omega_{y_i^0}}^{M_0} k_{y_{i+2}})(A \cap k_{y_i^0}) = (\mathcal{E}_{\Omega_{y_{i+1}}}^{M_0} k_{y_{i+2}})(A \cap k_{y_{i+1}}) = \pi_{\Omega_{y_{i+1}}}^{\Omega_{y_{i+2}}}(A_{\Omega_{y_{i+1}}} \cap k_{y_{i+2}}) = (A \cap k_{y_i^0} \cap k_{y_{i+2}}) = (A \cap k_{y_{i+1}}) \cap k_{y_{i+2}} = A \cap k_{y_{i+2}} \) \(\square\) Case 1

Case 2  Suppose that the claim holds for \(\mathcal{E}_{\Omega_{y_i^0}}^{M_0} k_{y_{i+2}}\)
and that \(y_i^{\nu_i+1}\) is defined. Set \(\beta_i = y_i^{\nu_i+1}\), \(\xi_i = T(\beta_i+1)\).

\(M_{\beta_i}^* = M_{\xi_i}^* \parallel y_i^{\beta_i}\). Then \(\pi_{\beta_i+1, y_i^{\beta_i}}\) is total and \(e_\pi(\pi_{\beta_i+1, y_i^{\beta_i}}) > \lambda_{\beta_i}\)
by Claim 3, so we have:

\begin{equation}
(8) \quad \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}} (A \cap k_{y_{i+2}}) = \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i}} (A \cap k_{y_{i+2}}) = A \cap k_{y_{i+2}}
\end{equation}

Moreover, \(\pi_{\Omega_{y_i^0}}^{\Omega_{y_{i+2}}} : M_{\beta_i}^* \rightarrow \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}}\) and \(\lambda_{\beta_i} = \text{ev}(\mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i}}) > k_{y_i^0}\).

Thus

\begin{equation}
(9) \quad \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}} (A \cap k_{y_{i+2}}) = \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i}} (A \cap k_{y_{i+2}}) = A \cap k_{y_{i+2}}
\end{equation}

Putting all of this together, we get:

\begin{equation}
(\mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i}} (A \cap k_{y_{i+2}})) (A \cap k_{y_{i+2}}) = \pi_{\Omega_{y_i^0}}^{\Omega_{y_{i+2}}} (A \cap k_{y_{i+2}}) = \pi_{\Omega_{y_i^0}}^{\Omega_{y_{i+2}}} (A \cap k_{y_{i+2}})
\end{equation}

\begin{equation}
= \pi_{\Omega_{y_i^0}}^{\Omega_{y_{i+2}}} (\mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}} (A \cap k_{y_{i+2}})) = \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}} (A \cap k_{y_{i+2}})
\end{equation}

\begin{equation}
= \mathcal{E}_{\Omega_{y_i^0}}^{M_{\beta_i+1}} (A \cap k_{y_{i+2}}) \cap k_{y_{i+2}} = A \cap k_{y_{i+2}}
\end{equation}

We used (8) to derive the second and (9) to derive the third equality. This proves Claim 2.1. \(\square\) Case 2
Claim 2.2. \( G_i^m \) is coherent in \( A \) over \( N \).

Proof. By induction on \( n \).

\( n = 0 \). This was done in Claim 2.1.

\( n = m + 1 \). We know that \( G_i^m \in M_{\xi_{i+m+2}} \) and

\[
\begin{align*}
A \cap \xi_{i+m+2} & = A_{\xi_{i+m+2}} \cap \xi_{i+m+2} \\
A \cap \gamma_{i+m+2} & = A_{\xi_{i+m+2}} \cap \gamma_{i+m+2}
\end{align*}
\]

Thus \( G_i^{m+1} \left( A \cap \xi_i^* \right) = G_i^m \left( A \cap \xi_i^* \right) \cap \xi_{i+m+3} = \)

\[
= \pi_{\xi_{i+m+2}} \xi_{i+m+2} \left( G_i^m \left( A \cap \xi_i^* \right) \right) \cap \xi_{i+m+3} =
\]

\[
= \pi_{\xi_{i+m+2}} \xi_{i+m+2} \left( A \cap \xi_{i+m+2} \right) \cap \xi_{i+m+3} =
\]

\[
= \pi_{\xi_{i+m+2}} \xi_{i+m+2} \left( A_{\xi_{i+m+2}} \cap \xi_{i+m+2} \right) \cap \xi_{i+m+3} =
\]

\[
= A_{\xi_{i+m+2}} \cap \gamma_{i+m+2} \cap \xi_{i+m+3} = A \cap \gamma_{i+m+3}.
\]

We used the identities (10) to derive the fifth and the last equalities

\[ \Box \] Claim 2.2.

This proves Lemma 2.

\[ \Box \] Lemma 2.
Def. Let \( M \) be a pm. We define \( \delta(M) \) by
\[
\delta(M) := \cup \{ E^\gamma \mid E^\gamma \text{ is \mu-complete} \}.
\]
(Hence, if \( M \) is active, i.e. \( E_{\text{om}}^M \neq \emptyset \), then \( \delta(M) = E_{\text{om}}^M + 1 \).)

Fact. Let \( \gamma = \langle \langle \xi_i \rangle \rangle \), \( T \) be a smooth iteration.

Let \( \xi := T(i+1) \). Then \( \gamma \leq \delta(M, \xi, \gamma) \).
(This holds since \( \gamma < \lambda^\xi \) and \( E^M_{\text{om}} \neq \emptyset \).

(Or "one-small")

Def. A pm \( M \) is called basic iff there is no \( \mu \leq \delta(M) \)

such that for some \( \delta < \mu \) the following holds:

\[
(*) \quad E^M_{\mu} \neq \emptyset \text{ and } \exists \gamma \in \text{Woodin}_\mu.
\]

Clearly any good iterate of a basic mouse is basic.

Lemma 3. Let \( M, Y, \xi, \chi, \theta, \lambda, \mu, \nu \) be as above, where \( M \) is basic.

Then \( \theta = M_\theta \) or \( \theta = M_\theta' \).
Moreover, if \( \lambda < \delta(M) \), then \( M_\lambda = \theta = M_\theta \).

Proof. We prove the first part of the lemma, the second part follows easily from this proof.

Suppose \( M_\theta \neq \theta \neq M_\theta' \) and w.l.o.g. \( \lambda = \lambda_\theta(M_\lambda) \).
Then there is a \( \gamma \) s.t. \( \gamma \leq \delta \leq \lambda \) and \( E^M_{\text{om}} \neq \emptyset \). Since \( \delta \)

is a limit cardinal in \( M_\lambda \), \( \lambda \leq \gamma \). Pick \( \gamma \) to be the least possible. Then \( J^M_\mu = \delta \in \text{Woodin}_\mu \). CONTRADICTION! (QED) (Lemma 3)

(*) follows from (*) that \( J^M_\mu \models V_{\delta} \prec \kappa \), \( \delta \in \text{Woodin}_\mu \),
where \( \kappa = \text{crit}(E^M_{\text{om}}) \). To see this, let \( \theta = \kappa + \mu^M_{\text{om}} \) and
\[
\pi : J^M_\mu \rightarrow E^M_{\text{om}}.
\]
Then \( \delta \leq \pi(\theta) \). But then \( J^M_\mu \models V_{\delta} \prec \kappa \), \( \delta \in \text{Woodin}_\mu \).

Checking \( J^M_\mu \models V_{\delta} \prec \kappa \), \( \delta \in \text{Woodin}_\mu \),
the same argument: \( J^M_\mu \models V_{\delta} \prec \kappa \), \( \delta \in \text{Woodin}_\mu \).
We now prove an analogue of Lemma 2 which is a generalization in the sense that we allow sets \( A \) which are not elements of \( \mathcal{Q} \) but are in a certain sense definable over \( \mathcal{Q} \).

**Lemma 4**  Let \( M \) be basic. Let \( A \in \mathcal{S} \) be \( \Sigma_0^{(m)}(\mathcal{Q}) \) where \( \omega^m \mathcal{Q} > \mathcal{S} \). Then \( \mathcal{I}^* \models A \) -strong in \( N \) for sufficiently large \( i \in \mathcal{W} \).

**Proof**  Then \( \langle N, A \rangle \) is amenable.

By Lemma 3, we can w.l.o.g. assume \( M_0 = \mathcal{Q} \).

Let \( A \) be \( \Sigma_0^{(m)}(M_0) \) in \( p \). If \( \mathcal{L}(M_0) > \omega = \mathcal{L}(M_0) \), then \( A \in M_0 \). Otherwise, again by Lemma 3, \( M_0 = \mathcal{Q} = M_0 \) so \( A \in \Sigma_0^{(m)}(M_0) \) in \( p \). Thus, let \( p \in M_0 \) be s.t.

\[ p_1 = A \text{ if } A \in M_0, \quad A \text{ is } \Sigma_0^{(m)}(M_0) \text{ in } p. \]

Now pick \( i \in \mathcal{W} \) long enough s.t.

\[ p \in \text{rng}(\pi^{\mathcal{Q}_i}_{\mathcal{Q}_i+1, \mathcal{Q}_i+1}) \quad \text{and} \quad p' \in \text{rng}(\pi^{\mathcal{Q}_i}_{\mathcal{Q}_i+1, \mathcal{Q}_i+1}). \]

**Claim**  \( G_1^{\omega} \) is coherent over \( A \).

**Proof**  The proof is like the proof of Lemma 2. As before, we define the sets \( A_{pq} \), which now need not be elements of \( M_0 \), but are \( \Sigma_0^{(m)}(M_0) \) in \( p_q \) next \( p_{q1} \).
where $\pi_{\gamma_1}^*(p'_1) = p_1$ resp. $\pi_{\gamma_1}^!(p'_1) = p'_1$. As before we observe that

$$A_{\tilde{\gamma}_1}^*(\eta_{\gamma_1+2}) = A_{\tilde{\eta}_1}^*(\eta_{\gamma_1+2}) = A_{\tilde{\eta}_1}^!(\eta_{\gamma_1+2}).$$

The amendments to the proof of lemma 2 are then straightforward.

Recapitulating:

**Corollary 4.1** Let $M$ be basic. Let $A$ be $\Sigma_0^{(\omega)}(Q)$ in $p$, where $w^m_Q \geq \delta$. Let $i \in \text{ND}$ be s.t. for all $\gamma, \gamma' \geq \tilde{\gamma}_1^+$ we have $p \in \text{mg}(\pi_{\tilde{\gamma}_1}) \cap \text{mg}(\pi_{\tilde{\gamma}_1}')$ and $A \in \text{mg}(\pi_{\tilde{\gamma}_1})$ of $M_b = Q$. Then $G_1^\sigma$ is coherent w.r.t. $A$ for $m < \omega$.

**Coroll 4.1.**

Using this we prove:

**Corollary 4.2** Let $M$ be basic and let $F$ be a $\Sigma_1^{(\omega)}(Q)$ partial map to $\delta$. Then $F''k_{\gamma_1} < k_{\gamma_1}$ for sufficiently large $i < \omega$.

**Proof** Pick $i$ big enough s.t. $i \notin \text{ND}$ and for all $\gamma, \gamma' \geq \tilde{\gamma}_1^+$:

- $p \in \text{mg}(\pi_{\tilde{\gamma}_1}) \cap \text{mg}(\pi_{\tilde{\gamma}_1}')$ if $M_b = Q = M_b'$
- $p \in \text{mg}(\pi_{\tilde{\gamma}_1})$ and $F \in \text{mg}(\pi_{\tilde{\gamma}_1})$,

where $F$ is $\Sigma_1^{(\omega)}(Q)$ in $p$.

**Claim** $F''k_{\gamma_{i+1}} < k_i^*$

**Proof** Suppose not. Let $\xi < k_{\gamma_{i+1}}$ be s.t. $\gamma = F'(\xi) > k_i^*$. Let $H(\xi, u, x, y)$ be a $\Sigma_0^{(\omega)}(Q)$ relation s.t.

$$y = F(x) \equiv (\exists z_1^u) H(z_1^u, p, x, y).$$

Let $\beta$ be the least s.t. $(\exists z_1^u, z_2^u \in S_{p^*}) H(z_1^u, p, z_2^u, z_1^u).$ Then $\beta \in \text{mg}(\pi_{\tilde{\gamma}_1})$ for $\gamma > \tilde{\gamma}_1^+$ and $\beta \in \text{mg}(\pi_{\tilde{\gamma}_1})$ for $\gamma > \tilde{\gamma}_1^*$ if $M_b = Q.$
Then \( \{ r \} = \{ \xi < \delta ; (\exists x \in X) H(x, \eta, \pi, \xi, \xi) \} \subseteq \Sigma^m_0(Q) \)
and the conditions of Cor. 3.1. are satisfied, thus \( \{ r \} \) is coherent w.r.t. \( \theta^m_\nu \) for \( m > \omega \). Pick \( m \) big enough st. \( \nu < \kappa_{\nu + m + 2} \). Then
\[
\phi = \{ r \in \nu^* \mid \theta^m_\nu (r \in \nu^* \mid \kappa^*_\xi) = \nu \in \nu^* \mid \kappa^*_\nu + m + 2 = \nu \}
\]

Claim

Thus, let \( F^*_{\xi + 1} \) be a \( \Sigma^m_1(M^*_\xi \mid \xi + 1) \) map on \( P^*_{\xi + 1} \) by the same definition, where \( \pi^*_{\xi + 1} (P^*_{\xi + 1}) = P \) or \( F^*_{\xi + 1} \in M^*_\xi \).

and
\[
\pi^*_{\xi + 1} (F^*_{\xi + 1}) = F, \quad \text{if } F \in M^*_\xi.
\]

Then \( \nu = F (\xi) \in \Lambda^m (\pi^*_{\xi + 1} F) \)

and
\[
\nu < \kappa^*_\xi < \kappa^*_\nu + 2 < \kappa^*_\nu + 1 \leq \pi^*_{\xi + 1} (\pi^*_{\xi + 1} \kappa^*_\nu + 1) \leq \pi^*_{\xi + 1} (\pi^*_{\xi + 1} \kappa^*_\nu + 1).
\]
But \( \kappa^*_\nu + 1 = \pi^*_{\xi + 1} (\pi^*_{\xi + 1} \kappa^*_\nu + 1) \). Thus, \( \nu < \kappa^*_\nu + 1 \).

Cor 4.2.

Recall that a mouse is an iterable premouse.

We then get:

Lemma 5. Let each \( M^*_\xi \) be iterable. Then:

a) \( u^m_{\phi^* Q} \geq \delta \)

b) \( \delta \) is \( \Sigma^m - \)regular in \( Q \) (i.e. \( \delta \) is a \( \Sigma^m - \)partial map to \( \delta \), then \( \sup (f''(\gamma)) < \delta \)

For \( \gamma < \delta \) .

Proof. It is enough to prove a). b) then follows from Cor 4.2.

So suppose a) fails. Let \( m \) be s.t. \( u^m_{\phi^* Q} \geq \delta \) and \( u^{m+1}_{\phi^* Q} < \delta \).

Pick \( p \in P^m_{\phi^* Q} \) and \( A \) which is \( \Sigma^m_1(Q) \) on \( p \) s.t. \( A \cup u^{m+1}_{\phi^* Q} \not\subseteq Q \).

Let \( b = b_{\phi^* Q} \). Then \( b \in \Sigma^m_1(Q) \), so there is \( \xi \in \omega \)

s.t. \( u^{m+1}_{\phi^* Q} \subset \kappa^*_\xi \) and \( b (\kappa^*_\xi) \cap \delta = \kappa^*_\xi \).

Let \( X = b (\kappa^*_\xi) \) and \( \bar{\delta} : \bar{\delta} \mapsto X \), \( \bar{\delta} \) transitive. Then \( \bar{\delta} : \bar{\delta} \mapsto \omega_{m+1} \).

Using downward extensions of embeddings, lemma lift \( \delta \).
to $\sigma: Q' \to \Omega$ where $\sigma(p') = r, p' \in E_{Q'}^{m}$. Then $\kappa_{\gamma} \in Q'$ since otherwise $Q' = \mathcal{F}^{\kappa}_{\gamma}$ and $A \in E_{Q'}^{(m)}(Q')$, thus $A \in Q$, a contradiction. But then $\kappa_{\gamma} = e_{\alpha}(r)$ and $\sigma(\kappa_{\gamma}) > \delta$, so $Q' = \mathcal{F}^{\kappa}_{\gamma}$ where $E' = E_{Q'}^{\kappa_{\gamma}} \in Q$, $\alpha = \delta$. But $\sup \{ \gamma_{1} : E_{Q}^{\gamma_{1}} = 0 \} < \delta$, so pick $r > \kappa_{\gamma}$ st. $E_{Q}^{\gamma} = 0$ and iterate $\mathcal{F}^{\kappa}_{\gamma}$ by $E_{Q}^{\gamma}$ long enough st. the height of the resulting structure $\bar{Q}$ is greater than $\alpha$. Since $A \in E_{Q}^{(m)}(Q', Q) \subseteq Q$. Since $\nu$ is a cardinal in $\bar{Q}$, $A \in \mathcal{F}^{\kappa}_{\gamma} Q \subseteq Q$. Contradiction.

Thus the possible case $\mathcal{F}^{\kappa}_{\gamma} = \mathcal{F}^{\kappa}_{\gamma} \nu$ is smooth.

Thus by the previous lemma, if $\nu$ is a good iteration of a basic mouse above the ultimate projectum, then $\nu$ has at most one cofinal branch. More generally:

**Corollary 6.** Let $M$ be a basic mouse st. $u_{M}^{\omega} \leq r$ for a $\nu$ with $E_{\nu}^{\nu} = \emptyset$. Let $\nu$ be a normal iteration of $M$ of limit length

Then $\nu$ has at most one well-founded branch.

**Proof.** Let $\nu$ be a counterexample of minimal length. Then $\nu$ is an iteration by the unique strategy. Hence each $M_{\nu}$ is a mouse. Let $b, b'$ be distinct branches with $M_{b} = Q$. Then $u_{Q}^{\omega} \leq \nu$.

**Case 1.** There is a truncation point $i + 1 \in b$. Suppose it is the last one, i.e. $T_{i+1,b}$ is total. Let $\kappa_{i} = e_{\nu}(E_{M_{i}}^{M_{i}})$. Then $u_{Q}^{\omega} = u_{M_{i+1}}^{\omega} E_{\kappa_{i}} \leq \kappa_{i} < \delta$. Contradiction.

**Case 2.** $T_{M_{i+1}}$ is total. Since $T_{M_{i+1}}$ is a $\mathcal{Z}^{\ast}$, map, $u_{Q}^{\omega} \leq T(\nu) < \delta$.

Since $E_{Q}^{\nu} = \emptyset$. Contradiction.  

**Corollary 6.1.** If $M$ is a basic mouse st. $u_{Q}^{\omega} \leq \nu$ for a $\nu$ with $E_{M}^{\nu} = \emptyset$ then $M$ is uniquely iterable.  

**Note.** Cor. 6.1 holds if $\nu = On \cap M$, hence $M = \langle \mathcal{F}^{\kappa}_{\gamma}, F \rangle$ with $F = \emptyset$ is always iterable if it is basic + iterable.

Note: It follows that the Dodd-Jensen Lemma holds: if $\nu$ is a basic + iterable.
Corollary 6.2 Let $M$ be a basic mouse s.t. $\text{wp}^M < \omega_\kappa$ and no $\delta \in M$ is Woodin in $M$. Let $\mathcal{I}$ be a normal iteration of $M$ of limit length. Then $\mathcal{I}$ has at most one cofinal well-founded branch.

Proof:

Again take $\mathcal{I}$ as being of minimal length. Let $b, b'$ be distinct branches with $M_b = Q$. Then $\text{wp}^Q \geq \delta$. An Case 1 we get a contradiction exactly as before. An Case 2 we have $\delta \geq \text{wp}^Q < \omega_\kappa$. Hence $\delta \in Q + \delta$ is not Woodin in $Q$.

Contradicted by Lemma 2. QED (6.11)
The Dodel-Jensen Lemma for \( M \) as in Cor. 6, says: If \( N \) is a smooth iterate of \( M \), then:

(a) There is no \( \gamma < \text{ht}(N) \) s.t.
\[
\sigma : M \to \Sigma^* N \langle \gamma \rangle.
\]

(b) Suppose \( \sigma : M \to \Sigma^* N \). Then \( N \)

is a simple iterate of \( M \) and
\[
\pi(\overline{3}) \leq \sigma(\overline{3}) \]

for \( \overline{3} \in M \), where \( \overline{\pi} \) is a smooth iteration map from \( M \) to \( M \). (Hence \( \overline{\pi} = \pi \), \( N \) is the unique smooth iteration map.)

This will suffice for all our applications (in virtually all applications \( N \) will in fact be a normal iterate of \( M \)). Nonetheless, it would be nice to have the same lemma

with “good” in place of “smooth”. This in fact holds, since if \( \sigma : M \to \Sigma^* N \langle \gamma \rangle \),

then \( N \langle \gamma \rangle \) is as in Cor. 6. 1. This fact can

be used to turn a good iteration of \( M \) to \( N \langle \gamma \rangle \) into a good iteration by a good sequence \( \langle \langle M_i \rangle, \langle \gamma_i \rangle, \langle \pi_i \rangle \rangle \)

with the property:
If \( i \) is a normal iteration of \( M_i \), where \( M_i \) is as in Cor 6.1. All branches in such an iteration will be unique and hence the proof of Dodd - Jensen can be carried out. We refrain from giving the proof here, since this strong form of Dodd - Jensen will not be needed.

(Note Since writing this we have been able to show that smoothly iterable premises are, in fact, iterable. This will be proven in §9. The proof makes no use of the theorems in this section.)
\( \Sigma_0 \)-iterations

Now suppose \( \mathcal{S} \) to be a direct normal \( \Sigma_0 \)-iteration of \( M \) with distinct branches \( b, b' \) cofinal in \( \mathcal{S} = \text{length} (\mathcal{S}) \), s.t. \( M_b, M_{b'} \) are well-founded.

We define \( \mathcal{F}, \mathcal{F}_i, \mathcal{F}^+ \) exactly as before. Lemmas 1, 2, 3 go through exactly as before. Lemma 4 goes through in the following form:

**Lemma 4'** Let \( M \) be basic, let \( A \subset \mathcal{S} \) be \( \Sigma_0 (\mathcal{S}) \) where \( \mu^e \geq \mathcal{S} \).

Then \( \mu^e \) is \( A \)-strong in \( N \) for sufficiently large \( i \leq \omega \). (Moreover, the full Lemma 4 holds if \( b \) has a truncation point and \( M \) \( \alpha \).

\[ \Box \text{Lemma 4'} \]

A similar version of Corollary 4.1 holds. We then get:

**Corollary 4.2'** Let \( M \) be basic and let \( F \) be a \( \Sigma_1 (\mathcal{S}) \) partial map to \( \mathcal{S} \). Then \( F'' \mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{S}} \) for sufficiently large \( i \leq \omega \). (Moreover, the full Cor. 4.2 holds if \( b \) has a truncation point and \( M \) \( \alpha \).

\[ \Box \text{Cor. 4.2'} \]

Hence:

**Lemma 5'** Let \( M \) be a basic \( \Sigma_0 \)-mouse. Then

\[ a) \quad \mu^e \geq \mathcal{S} \]

\[ b) \quad \mathcal{S} \text{ is } \Sigma_1 \text{ regular in } \mathcal{S}. \]

(Moreover, the full Lemma 5 holds if \( b \) has a truncation point and \( M \) \( \alpha \).

\[ \Box \text{Lemma 5'} \]

Corollary 6 goes through in the following form:
Corollary 6. Let $M$ be base such that $\nu^M \leq \nu$ for a $\nu$ with $E_\nu \neq 0$. Let $\gamma$ be a smooth $\Sigma_0^1$-iteration of $M$ of limit length. Then $\gamma$ has at most one well-founded cofinal branch.

Corollary 6.1. If $M$ is base and $\Sigma_0^1$-iterable such that $\nu^M \leq \nu$ for a $\nu$ with $E_\nu \neq 0$, then $M$ is uniquely smoothly $\Sigma_0^1$-iterable.

We close this section with a useful property of base premice.

Lemma 7. Let $M$ be a base premouse, $M = \langle J^E_\kappa, F \rangle$ with $F \neq 0$.

Then $\nu^M < \kappa$, where $\kappa$ is the largest cardinal in $M$.

Proof. Suppose not. Let $\kappa = \kappa_\nu(F)$. We prove that $\kappa$ is wooden on $M$. Let $\mathcal{C} = \kappa^+M$. If $\nu < \kappa$, then $F \nu \in J^E_\kappa$ since $F \nu$ is a subset of $J^E_\nu$, and hence $\nu' = \nu^M < \kappa$ and $\Sigma_1(M)$ in $\nu$.

$\langle x, y \rangle \in F \nu \iff (\exists \{z \}) (\langle x, z \rangle \in F \wedge y = 2\nu z)$

Claim. $J^E_\kappa$ is extendible by $F \nu$.

Proof. Suppose not. Let $\mathcal{D}$ be the ultrapower and let $\mathcal{D}$ be $\mathcal{D}$. But since $J^E_\kappa = 2\mathcal{D}$ and $F \nu \in J^E_\kappa$, $\mathcal{D}$ thinks it is not extendible by $F \nu$, so there is an $E_\nu$-decreasing sequence $\langle \langle \alpha_i, t_i \rangle, i \in \omega \rangle \in J^E_\kappa$, where $\alpha_i < \nu$ and $t_i : \kappa \rightarrow J^E_\kappa$. Pick $\lambda < \kappa$ such that $t_i \in J^E_\lambda$ and let $\lambda = \Sigma_{\omega}^1$-hull of $\{t_i, i \in \omega \} \cup \{\kappa+1\}$ in $J^E_\kappa$ and $\tau : \omega \rightarrow \lambda$, where $\omega$ is transitive. Then $x_i \tau_i, \omega \in J^E_\kappa$ and $J^E_\kappa = \kappa$, thus $\omega \in J^E_\kappa$. Let $\tau(t_i) = t_i$ for $i \in \omega$. 

We close this section with a useful property of base premice.
Then \( \text{dom} (\bar{t}_1) = \kappa \). Now, since the sequence \( \langle \bar{x}_i, t_1 \rangle \in \omega \) is \( E_\kappa \) - decreasing, we have:
\[
\langle x_{i+1}, t_{i+1} \rangle \in E_{\kappa} \langle x_i, t_i \rangle \iff
\]
\[
\langle \pi(t_{i+1}) \rangle, t_{i+1}(\gamma) \in t_i(\gamma) \rangle \in F_{\langle x_{i+1}, x_i \rangle} \iff
\]
\[
\langle \pi(t_{i+1}) \rangle, t_{i+1}(\gamma) \in t_i(\gamma) \rangle \in F_{\langle x_{i+1}, x_i \rangle} \iff
\]
\[
\langle x_{i+1}, t_{i+1} \rangle \in E_{\kappa} \langle x_i, t_i \rangle ,
\]
since the sets on the left sides of the second and third proposition are equal by the elementarity of \( \sigma \).

This means that \( J^E_{\kappa} \) is not extendible by \( F | \kappa \).

Contradiction (since \( J^E_{\kappa} \) is extendible by \( F \)).

Claim.

Now let \( \pi : J^E_{\kappa} \rightarrow J^E_{\pi(\kappa)} \). If \( \pi(a) \in \pi(a) \cap J^E_{\pi(\kappa)} \),
then \( J^E_{\pi(\kappa)} \models (F | \pi(\kappa) \text{ is coherent w.r.t. } \pi(a)) \) for all \( \nu, \lambda : (F | \pi(\kappa) \cap \nu = \pi(\pi(a) \cap \nu) \cap \nu = \pi(a) \cap \nu \)

Thus \( J^E_{\kappa} \models (3\mu)(\mu \in \pi(a) - \text{strong}) \).

But then \( J^E_{\kappa} \models (3\mu)(\mu \text{ is } \pi(a) - \text{strong}) \), so \( \kappa \) is Woodin
in \( J^E_{\kappa} \) and thus \( J^E_{\kappa} \models (2\text{FC} + \kappa \text{ is Woodin}) \). Contradiction.

Hence the initial segment condition is vacuously true

for basic premises.

Corollary 4.1 \( \text{Let } M = \langle J^E_{\kappa}, F \rangle \text{ be as above. There is no } \lambda < \kappa \text{ s.t. } \langle J^E_{\lambda}, F | \lambda \rangle \text{ is a premise.} \)

Proof. Suppose not. Let \( \lambda < \kappa \text{ be s.t. } M = \langle J^E_{\lambda}, F | \lambda \rangle \text{ is a pm} \).

Let \( \lambda \) be the largest cardinal in \( J^E_{\kappa} \). Then \( E^M_{\lambda} \neq 0 \).

Thus \( \text{cf}(\mu)^M \lambda < \kappa \). Thus \( \kappa \) is collapsed on \( M \). Let \( \mu \) be the size of \( \lambda \) in \( M \). Then \( \mu \leq \kappa < \mu^+ \).
But \( \langle \mu, \mu^+ M \rangle \in \mathbb{F} \left( \{ \langle \xi, \eta \rangle \}, \xi^+ M = \xi \eta \right) = \mathbb{F} \left( \{ \langle \xi, \eta \rangle \}, \eta \xi^+ = \xi \eta \right) \cap \lambda, \quad [\xi \neq \mu^+ M] \)

which means \( \mu^+ \eta = \mu^+ M < \lambda \). Contradiction.

\( \Box \) Cor 7.1