§ 3 Aterability (A longer sketch)

We now prove a somewhat more general case of § 1 Thm 1, which we believe contains all the new ideas needed for a full proof. We again let \( N \) be an arbitrary robust premouse satisfying ZFC\(^+\). We let \( Y = \langle \langle p_i \rangle, \langle v_i \rangle, \langle \pi_i \rangle, T \rangle \) be a countable putative iteration of a countable \( P_0 \), where \( \delta_0 : P_0 \leq N \).

We prove:

**Main Claim** One of the following holds:

(a) \( \delta(Y) = i + 1 \) and there is \( \delta : P_i \leq N \) s.t.
\[ \delta_{P_i} = \delta_0. \]

(b) \( Y \) has a maximal branch \( b \), which is of limit length, and there is \( \delta : P_b \leq N \) with \( \delta_{P_b} = \delta_0. \)

We shall suppose (b) to fail and prove (a). We again follow Steel's proof closely.

**Def.** Fix \( m^* : \delta(Y) \xrightarrow{\mathcal{L}} 2 \) s.t.
\[ m(i) = \min \{ m^*(j) \mid i \leq j \}. \]

**Def.** \( i \) survives at \( j \) (i surv \( j \)) iff
\[ i \leq j, m(i) = m^*(j) \text{ and } m(l) \geq m(i) \text{ for } l \in (i, j). \]

The following facts were established by Steel:
Fact 1
(a) \( m(i) = m(i') \land i < i' \) \( \rightarrow \) \( i \leq i' \)
(b) Let \( b \) be a branch of limit length in \( T \).
    \( b \) is maximal \( \leftrightarrow \) \( \sup m^n b = \omega \).

Fact 2
Let \( i \) survive and \( l \in (i,i') \) but
    \( \ell \notin (i,i') \). Then \( m(i') < m(\ell) \).

Fact 3
(a) \( i \) survives \( \leftrightarrow i \) survives.
(b) \( i \) survives \( \land \ i < k < i' \) \( \rightarrow \) \( i \) survives.
(c) Let \( b \) be a branch of limit length,
    \( b \) is maximal iff for all \( i \in b \) there is \( j \in b \)
    s.t. \( i < j \) and \( i \) does not survive at \( j \).

We are assuming that (b) fails in the
Main Claim. As before, this says
that a certain relation \( R \) is well
founded.

Def. \( D = \{ (i, j) \mid j < i, P_j < N \land \delta_{P_j} = \delta \} \)

Ref. \( D^2 \) is then defined by:
\( (j, j') R (i, i') \leftrightarrow (i \leq j' \land i \) does not
    survive at \( j' \) and \( \delta_{P_{i'}} = \delta \).

As before, we set:

Def. \( r(z) = \) the rank of \( z \) in \( R \);
    \( r = r(<0, \delta>) \).
Def. Let $i \leq j \leq lh(Y)$.
\[ c(i, j) = \{ h \mid j < h < lh(Y) \land h \text{ is a successor ordinal} \land T(h) \leq i \land T(h) \text{ survives at } h \} \]

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Fact 4
(a) $i \leq i' \rightarrow c(i, i') \subset c(i', i)$
(b) $i \leq i' \rightarrow c(i) \subseteq c(i', i)$
(c) $c(i, i')$ is finite (in fact, if $h, k \in c(i, i')$ and $h < k$, then $T(k) < T(h)$ and $m(k) < m(h)$.)

Def. Let $i < s \leq lh(Z)$
i is a break point at $s$ if whenever $i < h \leq s$ s.t. $T(h) \leq i$, then $T(h)$ does not survive at $h$. (In other words, $c(i, i) \upharpoonright s = \emptyset$.)

Following Steel, we define the concept of enlargement, which will play a central role in the proof.

Def. The standard world $\overline{W}$ is defined as before. The standard enhanced world is now $\overline{W} = \langle \overline{W}_r + w, \theta, a \rangle$, where $r, \theta, a, \overline{W}$ are defined as before. The concept of world and enhanced world are defined as before.
Let $1 \leq \delta \leq k(h)$. By an enlargement of $\mathcal{Y}$ with $\mathcal{Y}$, $\mathcal{S}_0$, $\mathcal{N}$, we mean a sequence $E = \langle E_i | i < \delta \rangle$, such that $E_i = \langle W_i, N_i, S_i \rangle$ and:

(a) $W_i = \langle W_i, \theta_i, o_i \rangle$ is an enhanced world.

(b) $N_i, S_i \subseteq W_i$, $S_i : p_i < N_i$, where

\[
\langle W_i, N_i, S_i \rangle \equiv \langle W, N, S, e \rangle.
\]

(c) $\langle W_i, N_i, S_i \rangle \equiv \langle W, N, S, e \rangle$ for $h < i$.

(d) $S_i \cap S_h = S_h$ and $E_i = E_h$, where

\[
\mathcal{C}_i = \sup S_i \cap \mathcal{S}_i \quad \text{and} \quad E_i = E_h.
\]

(e) $p_i = \text{dom}(\theta_i) \geq \omega i + 1 (i, \tilde{r})$, where $\tilde{r}$ is defined in $W_i$ from $N_i, S_i, o_i$ and $\tilde{r}_i = r_i (\langle i, S_i \rangle)$.

(f) $C^{E_h} = \mathcal{Q}(S_h \cap S_h, T_h, e)$ in $W_h < h \rightarrow

\begin{align*}
\langle & \mathcal{C}^{E_i} = \mathcal{Q}(S_i \cap S_i, T_i, e) \rangle \text{ in } W_i \quad \text{for } h < i, \\
\text{where} & T_i = \text{the complete theory of } \langle W_i, N_i, S_i, e \rangle
\end{align*}

and $T_i = \langle T_h | h \leq i \rangle$.

We shall need, however, a stronger version of (f).

In order to formulate this, we first define:

\[ D_h \quad \text{for } h \leq i \text{ set:} \]

\[ S^h, o_i = \text{The set of } E_i \text{ formulas } \varphi \text{ at } \]

\[ C^{E_i} = \mathcal{Q}(S_i \cap S_i, T_i, e), \langle S^h, o_i | i < m \rangle \text{ in } W_i. \]

Set: $S^i = S^i_m$.

Then (f) means: $S^i_o = S^i_o$. We strengthen

\[ \text{the } \text{to:} \]

(g) $S^h, o_i = S^h$ for $h \leq i$, $m \leq 1 (i, \tilde{r})$. 

We again work in $V[G]$, where $G$ collapses the standard enhanced world $W$ to $\omega$. We assume (b1) of the Main Claim to be false in $V$ and prove:

**Thm 1** Let $\delta < \text{lh}(\gamma)$. Then $\gamma^{\delta+1}$ has an enlargement $\mathcal{E}$. Moreover, if $\gamma < \delta$ is a breakpoint at $\delta$ and $\mathcal{E}$ an enlargement of $\gamma^{\delta+1}$, then $\mathcal{E}$ can be so chosen that $\mathcal{E}[i+1] = \mathcal{E}$ and $W_i^\mathcal{E} \cap \omega = W_i^{\mathcal{E}} \cap \omega$. From this we prove (a), thus showing the Main Claim to be true:

**Case 1** $\text{lh}(\gamma) = \delta + 1$

Then $\gamma$ has an enlargement $\mathcal{E}$. Then $W_{\delta+1}^\mathcal{E} = \delta + 1$, $P_{\delta+1} \preceq N_{\delta+1}$. But

$\langle W_{\delta+1}^\mathcal{E}, N_{\delta+1}, \delta, P_{\delta+1} \rangle \equiv \langle W_i, N_i, \delta_i \rangle$

Hence there is $\delta \in \overline{W}$ s.t. $\delta_{\overline{\delta}} = \delta_i$, QED

**Case 2** $\text{lh}(\gamma)$ is a limit ordinal $\theta$.

Define $m_{\delta} < \omega$, $j < \theta$ (l<ω) by:

$m_{0} = 0$, $m_{i+1} = \min \{ m^{\ast}(j) | j > j_{i} \}$

Then $\delta_{\theta}$ is a breakpoint in $\theta$, hence in $\theta$. Applying Thm 1 we define a sequence $E_{\delta} (l \leq \omega)$ s.t. $E_{\delta}$ is an enlargement of $\gamma^{\delta+1}$ and $\mathcal{E}$.

$W_i^{E_{\delta}} \cap \omega = W_i^{\mathcal{E}} \cap \omega$ for all $i$. Contr! QED (Main Claim)
At remains to prove Thm 1.
We first show:

**Lemma 2** Let $\mathcal{E}$ be an enlargement of $\mathcal{E}$ at $i$. Let $\gamma = T(i+1)$, where $\gamma$ does not survive at $i+1$. There is an enlargement $\mathcal{E}$ of $\mathcal{E}$ at $i+1$, $\mathcal{E}(i+1) = E$ and $\mathcal{E}(i+1)$ is admissible.

We imitate the construction in §2. As before there is $\pi : \mathcal{X} \rightarrow \mathcal{X}(i+1)$ of $\mathcal{X}$ satisfying (1)(a) of §1.

We again define $\sigma : \mathcal{P} \rightarrow \mathcal{P}$ by $\sigma(\pi, i+1, (4)(i+1)) = \delta \gamma (x)(i+1))$ and observe that $\sigma \in \mathcal{W}_\gamma$. Since $\gamma$ does not survive at $i+1$, we have:

$\langle \pi, i+1, \sigma \rangle \mathcal{R}_\gamma \langle \gamma, \tilde{\mathcal{X}} \rangle \mathcal{R}_\gamma$, where $\mathcal{R}_\gamma$ is defined in $\mathcal{W}_\gamma$, $\tilde{\mathcal{X}}$ from $\mathcal{N}_\gamma$, and $\mathcal{R}$ was defined in $\mathcal{W}$ from $\mathcal{N}$, $\mathcal{X}_0$.

We again let $\bar{\pi} = \pi_\gamma \langle \langle \pi, i+1, \sigma \rangle \rangle \mathcal{R}_\gamma$, and pick $x \geq \bar{\pi} = \Theta_{\bar{\pi}} \mathcal{W}_\gamma$ s.t. $x \in \mathcal{W}_\gamma$ and $C_x \in \mathcal{E}$ is admissible. The theory $T = \mathcal{T}_{\bar{\pi}}$, in the infinitary language of $C_x$, is as before and we observe that $\mathcal{T}_{\bar{\pi}}$ is consistent. Note that for each $z < \delta_x (u_z) = \delta_x (k_z)$ the
statement:
\[ \forall \bar{\tau} > 3 \forall \alpha (\bar{\varepsilon} < \bar{\tau} < \alpha \land C_{\bar{\varepsilon}, \bar{\tau}} \text{ is admissible} \land T_{\bar{\varepsilon}, \alpha} \text{ is consistent}) \]
holds in \( W_\eta \). This has the form:
(1) \( C_{\bar{\varepsilon}, \infty} \models \varphi(\bar{\tau}, \eta, \delta_0, \hat{T}_0, \mathbf{e}) \) \( \mu \in W_\eta \)
where \( \varphi \in \eta + T_0 = \text{the complete theory of } \langle \bar{W}_\eta, N_\eta, \delta, \mathbf{e} \rangle \).
But just as before:
(2) \( C_{\bar{\varepsilon}, \infty} \models C_{\bar{\varepsilon}, \delta_0(\eta, \delta_0)} \models C_{\bar{\varepsilon}, \infty} \) in \( W_\eta \).
Since (1) holds for all \( \bar{\tau} < \delta_0(\eta, \delta_0) \)
we conclude:
(3) \( C_{\bar{\varepsilon}, \delta_0(\eta, \delta_0)} \models \bigwedge \exists \bar{\tau} \varphi(\bar{\tau}, \eta, \delta_0, \hat{T}_0, \mathbf{e}) \).
Hence there are arbitrarily large \( \bar{\tau} < \delta_0(\eta, \delta_0) \) such that for an \( \alpha < \delta_0(\eta, \delta_0) \),
\( C_{\bar{\varepsilon}, \alpha} \) is admissible and \( T_{\bar{\varepsilon}, \alpha} \) is consistent. The same is obviously true in \( W_\xi \), since:
\[ (C_{\bar{\varepsilon}, \delta_0(\eta, \delta_0)}^{W_\eta})^W_\eta = (C_{\bar{\varepsilon}, \delta_0(\eta, \delta_0)}^{W_\eta})^W_\xi \]
Using the definition of \( S_m = S_{\xi m} \) and the fact that \( W_\xi \) is a \( \mathcal{ZF} \)-model, it is easily seen that:
4) \( C^{E_i}_{c_i,\infty} \models \forall \tau \forall m < \omega \forall \phi \in \Sigma^\prime_{c_i} \)
   \( C^{E_i}_{c_i,\tau} \models \phi(\delta^{c_i}_\tau \delta^{c_i}_m \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle) \)

Hence, by (1)(b) of § 2:

5) \( C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \models \forall \tau \forall m < \omega \forall \phi \in \Sigma^\prime_{c_i} \)
   \( C^{E_i}_{c_i,\tau} \models \phi(\delta^{c_i}_\tau \delta^{c_i}_m \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle) \)

Since the \( \phi \in \Sigma^\prime_{c_i} \) are \( \Sigma^\prime_{1} \) formulae, it is clear that if (5) holds for some \( \tau < \delta^{c_i}_\tau(m_i) \), then for all larger \( \tau \) also. Hence:

6) \( C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \models \) There are \( \delta, \tau \in \Delta, C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \)
   admisible \( \delta < \tau < \Delta \) \( \wedge T_{\tau,i} \) \( \in \) consistent
   \( \wedge \forall m < \omega \forall \phi \in \Sigma^\prime_{c_i} \)
   \( C^{E_i}_{c_i,\tau} \models \phi(\delta^{c_i}_\tau \delta^{c_i}_m \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle) \).

This statement has the form:

7) \( C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \models \psi(\delta^{c_i}_\tau \delta^{c_i}_m \tau_{\tau,i} \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle) \)

where \( \psi \in \Sigma^\prime_{1} \).

Hence, by (1)(b) of § 2, we have in \( W_{c_i} \):

8) \( C^{E_i}_{c_i,\tau} \models \psi(\delta^{c_i}_\tau \delta^{c_i}_m \tau_{\tau,i} \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle) \).

This says that there is a \( \Delta \) \( \in W_{c_i} \) \( \ni \)
   \( C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \) admisible and there is
   \( T \in (C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)}) \) \( \ni \)
   \( \tau_{\tau,i} \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle \) \( \notin \)
   \( \psi \).

This says that there is a \( \Delta \) \( \in W_{c_i} \) \( \ni \)
   \( C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)} \) admisible and there is
   \( T \in (C^{E_i}_{c_i,\tau,\delta^{c_i}_\tau(m_i)}) \) \( \ni \)
   \( \tau_{\tau,i} \text{it}_i \text{it}_o \varepsilon_1 \langle s^{c_i}_\tau \mid l < m \rangle \).

Where:

\( \psi \in \Sigma^\prime_{1} \).
Note, however, that if $\varphi \in S_m$, then $C_{c,1}^{E_i} \models \varphi (\delta; \delta, i, e, \langle \xi \rangle_{\xi < i} \leq n)$. And hence:

$$C_{c,1}^{E_i} \models \varphi (\delta; \delta, i, e, \langle \xi \rangle_{\xi < i} \leq n),$$

since $\varphi$ is $\Sigma_i$. By induction on $m$, we then get in $W_i$:

(9) $S_m^i$ has the same definition in $C_{c,1}^{E_i}$ as in $C_{c,1}^{E_i}$.

Now let $M$ be a model of $\mathcal{T}_{c,1}$. Set:

$$W_{i+1} = W_i, \quad N_{i+1} = N_i, \quad \delta_{i+1} = \delta_i^i.$$

(a) Let follow as before in §2.

By (9) we have:

$$S_m^i = S_m^{i+1}, \quad (\text{for } \alpha < \omega).$$

From this (g) follows. QED (Lemma 2)

A corresponding to Lemma 2, we now prove:

Lemma 3 Let $E$ be an enlargement of $Y(i+1)$.

Let $h = T(i+1)$ survive at $i+1$. There is an enlargement $E$ of $Y(i+2)$ at $i+1$.

a) $E|h = E|h$.

b) $W_{i+1}^{E} = W_{h}^{E}$.

c) $\delta_{h} = \delta_{i+1}^{E}$. 


proof of Lemma 3.
We first note:

\[ (10) \forall j, \forall i : c(j, i) = c(h, i) \quad \text{for } h \leq j \leq i. \]

**proof.** Otherwise there is \( k \geq i, j \in (h, i] \) s.t.
\( j = T(h+1) \) and \( j \) survives at \( k+1 \). Hence \( k \geq i \), since \( T(h+1) = h + h = T(i+1) \). Hence \( j < i+1 < k+1 \) and \( m(i+1) < m(j) \). Hence \( j \)
does not survive at \( k+1 \). Contr. \( \sqcup \)

But \( \exists (h, i] \geq 1, \forall j \in (h, i] \quad \text{hence } i+1 \notin c(h, i] \). From now on let:

\( (11) \forall j, \forall i : c(j, i) = m+1 \quad \text{for } h \leq j \leq i. \)

Define \( g : \lambda_i \rightarrow \delta_i \) \( h \), \( g(n) = \delta_h \) \( ; \quad \) and \( \sigma : P_{i+1} \rightarrow \mathbb{N} \), exactly as before.

\( (\text{Hence } \sigma \upharpoonright \lambda_i = g, \sigma(\lambda_{i+1}) = \delta_h.) \)

We shall form an enlargement \( \mathcal{E}' \) of \( \mathcal{E}(i+2) \) s.t.
\( \mathcal{E}' / h = \mathcal{E} / h, W_{i+1} = W_h, S_{i+1} = \sigma. \)

This means, however, that we must redefine \( W_i', \sigma_i' \), for \( h \leq i \leq i, \) s.e.

we need: \( \delta_i = \delta_i', \gamma_i = \sigma_i' \gamma_i = g \gamma_i \),

whereas \( c_i' = \sup \gamma_i \), \( \tau_i' = c_i \geq c_h = \delta_i(h,c) = \sup \sigma_i \gamma_i \).

Set: \( \overline{\tau} = \overline{\omega_\tau} + n \) (where \( \tau = \omega_\tau + m+1 \))

Let \( \tau = \omega_\tau (W_\tau) \overline{\tau} \). Let \( \rho > \tau, \rho \in W_\rho \),

s.t. \( \mathcal{E}_\rho \in \mathcal{D} \) is admissible.

Let \( T = T_{\rho, \delta} \) be the theory in the infinitary language of \( \mathcal{E}_\rho \)
consisting of:

\[ \text{Predicates:} \]
\[ \text{Constants:} \ x, (x \in \mathbb{E}_D^L), \ W, N, \delta \]

Axioms: (A) as before, and
(B) \( \psi, \varphi, \delta \in \overset{\circ}{W} \) and \( \langle \overset{\circ}{W}, \overset{\circ}{N}, \delta, e \rangle \vdash T_e \),
where \( T_e \) is the complete theory of \( \langle \overset{\circ}{W}, \overset{\circ}{N}, \delta, e \rangle \).

(2i) \( \delta \overset{\circ}{\times} \delta = \delta \overset{\circ}{\times} \delta \)

(2ii) \( pW = w + m \), where \( r^i \) is defined in \( W \)

from \( N, \overset{\circ}{s}_{o, e} \) as \( r \) was defined in \( \overset{\circ}{W} \)

from \( N, \overset{\circ}{s}_{o} \) and \( r = r^i(\langle \overset{\circ}{s}, \delta \rangle) \) in \( W \).

Then \( T_{\overset{\circ}{W}} \) is consistent, since

\( \langle W, (W, W) \overset{\circ}{\times} + m, N, \delta, e \rangle \) in a model, Since

\( T \equiv \overset{\circ}{W} \), it follows that there are arbitrarily large \( t, \varphi \in \overset{\circ}{W} \), not, for some \( d > 2 \),

\( d \in \overset{\circ}{W} \), \( CE_D \) admmissible, and

\( T_{\overset{\circ}{W}} \) is consistent (cf. the argument in the proof of Lemma 2). In particular,
we can pick \( d \) large enough that

\( \forall \varphi, \psi \in S_m \ CE_D \vdash \varphi(\delta_{(e, e), t, \varphi}, \psi_{(e, e), t, \varphi}, \langle \delta_{(e, e), t, \varphi}, \psi_{(e, e), t, \varphi} \rangle) \)

for all \( m \leq n = 1c(d, i) + 1 \).

(Note that \( (CE_D)_{\overset{\circ}{W}, e} \varphi \sim (CE_D)_{\overset{\circ}{W}, e} \varphi \)).
Hence in $\overline{W}$:

\[(13) \quad C_{E_l}^{E_l} = \mathcal{V}_d \mathcal{V}_e (C_{E_l}^{E_L} \text{ is admissible}) \]

\[\delta_l < \varepsilon < d \land \exists \delta_l \text{ in common with } \eta \]

\[\Delta m \leq m \land \phi \in \mathcal{S}_m^l \quad C_{E_l}^{E_l} \vdash \varphi(\delta_l_\phi, t_e, \varepsilon, <s_k^l | k \leq m>)\]

This has the form:

\[(14) \quad C_{E_l}^{E_l} \vdash \psi(\delta_l_\phi, t_e, \varepsilon, <s_k^l | k \leq m>)\]

(Note that $\delta_l (l_1) = T_{\delta_l}$.)

But since $s_k^l = s_k^l_\eta$; for $k \leq m+1$, we conclude:

\[(15) \quad C_{E_l}^{E_l} \vdash \psi(\delta_l_\phi_1, t_e, \varepsilon, <s_k^l | k \leq m>)\]

But then by $(11b)$ of $\S 1$:

\[(16) \quad C_{E_l}^{E_l} \vdash \psi(\delta_l_\phi_1, t_e, \varepsilon, <s_k^l | k \leq m>)\]

We note that:

\[(17) \quad \text{If } \varphi \notin \mathcal{S}_l \text{ and } \psi \in \mathcal{S}_l, \text{ then } C_{E_l}^{E_l} \vdash \varphi(\delta_l_\phi_1, t_e, \varepsilon, <s_k^l | k \leq m>)\]

for $m \leq n$, since $s_m^l = s_m^l_\eta$ for $m \leq m+1$.

By $(11b)$ of $\S 1$ it follows that:

\[(18) \quad \text{If } \varphi \notin \mathcal{S}_m^l \text{ and } \varphi \in \mathcal{S}_l, \text{ then } C_{E_l}^{E_l} \vdash \varphi(\delta_l_\phi_1, t_e, \varepsilon, <s_k^l | k \leq m>)\]

for $m \leq n$.

By $(16)$ there are $\delta_l, \delta_1 \in \mathcal{S}_{E_l}^{E_l} \text{ s.t.}$.
\(19\) \(z < z' < z, \ C_{E'}^{\bar{z}, z} \ n\)

\[\forall \varphi \in S_m^l \ C_{E'}^{\bar{z}, z} \ = \ \varphi \left( g^{\bar{z}, z'} \ n, \ t_{z', z} \ e, \ \left< S_k^{\bar{z}, z} \ | \ k < \infty \right> \right)\]

for \(m \leq n\), and the theory \(\bar{\zeta} = \bar{\zeta}_{\bar{\zeta}, \varphi}\) is consistent in the infinitary language of \(C_{E'}^{\bar{z}, z}\), where \(\bar{\zeta}_{\bar{\zeta}, \varphi}\) is like \(\bar{\zeta}_{\varphi}\) except that in (B)(iii) we replace \(g_{\bar{\zeta}, \varphi}\) by \(g_{\bar{\zeta}, \varphi}\).

By (18) we have:

\[20\] \(A \ \forall \varphi \in S_m^l \text{ and } \varphi \in \exists_{z, z'} \therefore \]

\[C_{E'}^{\bar{z}, z} = \exists_{z, z'} \left< S_k^{\bar{z}, z} \ | \ k < \infty \right> \]

for \(m \leq n\).

Recall that:

\[\left( C_{E'}^{\bar{z}, z} \right)_{\sigma(\varphi)} = \left( C_{E}^{\bar{z}, z} \right)_{\sigma(\varphi)} \leq \left( C_{E}^{\bar{z}, z} \right)_{\sigma(\varphi)} \leq \left( C_{E}^{\bar{z}, z} \right)_{\sigma(\varphi)} \]

Hence by (19), (20):

\[21\] \(\left< S_k^{\bar{z}, z} \ | \ m \leq n \right> \text{ has the same definition in } C_{E}^{\bar{z}, z} \text{ as in } C_{E}^{\bar{z}, z} \text{ in the parameters} \sigma_{\bar{z}, z} = g_{\bar{z}, z}, t_{\bar{z}, z}, e, \ e.

Now let \(\bar{\zeta}_{\bar{z}, z}\) be a good model of \(\bar{\zeta}_{\bar{z}, z}\) for \(h \leq l \leq i, S\geq + 1\):

\[W_{l} = W_{l}^{\bar{z}, z}, \ N_{l} = N_{l}^{\bar{z}, z}, \ \delta_{l} = \delta_{l}^{\bar{z}, z}, \]

\(\text{for } h \leq l \leq i\). For \(l < h\), set:

\[< W_{l}^{\bar{z}, z}, N_{l}^{\bar{z}, z}, \delta_{l}^{\bar{z}, z} > = < W_{l}^{\bar{z}, z}, N_{l}^{\bar{z}, z}, \delta_{l}^{\bar{z}, z} >, \]

Finally set:

\[W_{l + 1} = W_{l}^{\bar{z}, z}, \ N_{l + 1} = N_{l}^{\bar{z}, z}, \ \delta_{l + 1}^{\bar{z}, z} = \delta_{l}^{\bar{z}, z}. \]
It is easily verified that

\[ \text{IF} = \langle \langle w'_2, n'_i, s'_i \rangle \mid i < i' + 2 \rangle \]

is an enlargement of \( D_{i' + 2} \).

We use (2.11) to show: \( s_m = s_{m + 1} \) for \( h < \ell \leq i' \), \( m < m \geq c(h, i' + 1) \).

C.E.D. (Lemma 3)

This proof shows more than we have stated: For \( h < \ell \leq i' \) set: \( \bar{z}_h = \\{ z \in \text{definable in } W_h \} \) from \( t_h \) and hence \( \bar{z}_h \in W_h \). Clearly:

\( \bar{z}_h \) is definable in \( W_h \). But there are only countable sets, hence \( \bar{z}_h \), \( h, \bar{z} \in \text{C}^{E_h}_{h, s_h}(x_i) \).

We note:

**Corollary 3.1** Let \( \bar{z} = \langle \bar{z}_h \mid h < \ell \leq i' \rangle \) be as above. Then \( \bar{z} = \bar{z}(\bar{z}, t_i; \sigma) \) is a consistent theory in \( \text{C}^E_{h, s_h}(x_i) \), where:

\( \bar{T} \) is as follows:

**Predicate** \( \bar{z} \)

**Constants** \( W, N, \).

**Axioms** (A) \( ZFC \), \( \forall x \in x \Rightarrow \exists \bar{x} \bar{x} = \bar{x} \) for all \( x \),

\( \overline{W} = \langle \overline{w}_h \mid h \leq i' \rangle \), \( \overline{W} \cap [\overline{t}_i]_\omega = [\overline{t}_i]_\omega \),

\( \overline{N} = \langle \overline{n}_n \mid h < \ell \leq i' \rangle \), \( \overline{d} = \langle \overline{d}_h \mid h < \ell \leq i' \rangle \).

\( \overline{W} \) is an enhanced world, \( \text{On} \cap \overline{W}_0 = \overline{t}_0 \)
$-15-$

(B) (i) $N^l_\delta, \delta^l \in W^l, \delta^l : P^l \preceq N^l_\delta,$

\[ \langle \delta^l \rangle, N^l_\delta, \delta^l, \pi \rangle \models \Delta_i \langle \delta \rangle, \text{ moreover} \]

\[ \langle \delta^l \rangle, N^l_\delta, \delta^l, \pi \rangle \models \Delta_i \langle \delta \rangle \text{ for } \gamma \leq \delta \]

(ii) $\delta^l \cap N^l_\delta = \delta^l \cap \pi \delta^l$

(iii) $p^l \delta^l = \omega \cdot x^l + m \quad (m = \text{l.c.}(n, i) + 1)$.

Proof.

$CE, \delta, \theta \in \text{admissible and}$

\[ \langle H, \langle W^l \mid h \leq l \leq i \rangle, \langle N^l \mid h \leq l \leq i \rangle \rangle, \]

\[ \langle \delta^l \mid h \leq l \leq i \rangle \rangle \text{ is a model of } \Delta_i, \text{ when} \]

$W^l \in \text{as in Lemma 3, } \langle \Delta_i \rangle \text{ is chosen} \]

as above and $H = H[T, \alpha], \text{ where } \alpha > \delta^l (\mu_i)$

is regular in $\mathcal{V}[\alpha]$. \( \Box \).

(CE) (Cor 3, 1)

$\frac{\langle \tilde{e} \rangle}{\langle \tilde{e} \rangle} = \langle \tilde{e} \rangle \text{ is any sequence}$

of sub-sets of $\tilde{e}$ and $\tilde{e} = T_e \text{ for } l \leq h,$

then $\langle \tilde{e} \rangle$ is the same theory

on $CE, \delta, \theta \in \text{admissible}$

with $\tilde{e}$ in place of $\tilde{e}$.

Finally, if $\tilde{e} \in W^l$ is any

function and $\tilde{e} = T_e \text{ for } l \leq h,$

and we let $\tilde{e} \text{ be the}$

above theory on $CE, \delta, \theta \in \text{admissible},$ then

$\tilde{e} = \text{sup } \tilde{e} \Rightarrow \tilde{e}, \tilde{e} \check{\in} \text{ make}$
Lemma 4. Let \( E \) be an enlargement of \( \mathcal{M} \), \( E = \{ \langle w_0, N_0, E_0 \rangle \mid l \leq i \} \).

There exist \( \tilde{\sigma} \in \mathcal{W}_h \), \( \tilde{z} \in C_{\tilde{\sigma}, h}(l, i) \), \( \tilde{t} \in C_{\tilde{\sigma}, h}(m, i) \) such that:

(i) \( \tilde{\sigma} : p_{i+1} \leq N_h \) and \( \tilde{\sigma} \cdot t_{h, i+1} = \tilde{z} \).

(ii) \( \tilde{z} = \{ z_0 \mid h \leq l \leq i \} \).

(iii) \( \tilde{t} = \{ t_0 \mid l \leq i \} \) such that \( \tilde{t}(h+1) = t_h \) and \( \tilde{t}_l \in \omega \) for \( h < l \leq i \).

(iv) \( T(\tilde{z}; \tilde{t}; \tilde{\sigma}) \) is consistent in the infinitary language of \( C_{\tilde{\sigma}, h}(h, i) \), where \( \tilde{\sigma} = \sup \sigma_{\tilde{\mu}_i}^e \).

(v) Set \( S_m^l = \) the set of \( \Sigma_1 \) formulas \( \varphi \) such that \( C_{\tilde{\sigma}, h}^E \varphi \subseteq \{ \langle 0^m, \tilde{t}_l, (h+1), e, (S_{h}^l)_{\overline{k}} \rangle \mid m \leq n \rangle \).

Set \( S_m^l = \) the set of \( \Sigma_1 \) formulas \( \varphi \) such that \( C_{\tilde{\sigma}, h}^E \varphi \subseteq \{ \langle 0^m, \tilde{t}_l, (h+1), e, (S_{h}^l)_{\overline{k}} \rangle \mid m \leq n \rangle \).

Then \( S_m^l = S_m^l \) for \( m \leq n \), \( h \leq l \leq i \).
We call \( \langle \sigma, \overline{t}, \overline{\sigma} \rangle \) satisfying Lemma 4 an **enlarger** w.r.t. \( E = E \cup (i+1) \). It is clear that if \( \langle \sigma, \overline{t}, \overline{\sigma} \rangle \) is an enlarger and we set \( E \cup \overline{h} = E \cup h \).

\[ W^F_l = \overline{W}^F_l, \quad N^F_{i+1} = \overline{N}^F_l, \quad I^F_l = \overline{I}^F_l \quad (h \leq l \leq i) \]

where \( \overline{D} \) is a good model of \( \overline{T} (\overline{t}, \overline{\sigma}, i) \), and

\[ W^F_{i+1} = W_h, \quad N^F_{i+1} = N_h, \quad I^F_{i+1} = \overline{\sigma} \]

Then \( E \) is an enlargement of \( Y \cup (i+1) \) with \( \overline{t} = t^F_i \).

We call any such \( E \) an enlargement given by \( \langle \sigma, \overline{t}, \overline{\sigma} \rangle \).

It is clear, however, that being an enlarger w.r.t. \( E \) depends only on \( t_h \) and is expressible in \( \langle \overline{W}_h, \overline{N}_h, \overline{I}_h \rangle \) in \( T_\overline{h} \). For any \( t^* = \langle T^*_l \mid l \leq h \rangle \) w.r.t. \( T^*_l \subseteq \omega \) for \( l \leq h \), we define \( \langle \overline{t}, \overline{\sigma} \rangle \) is an **enlarger** w.r.t. \( Y \cup (i+1) \) w.r.t. \( t^* \) if the above holds. Thus Lemma 4 can be
Corollary 4.1 Let $E$ be an enlargement of $Y_l(i+1)$. Let $t^* = t^E_h$. Then

$\langle \overline{W}_h, N_h, \delta_h \rangle \subseteq \text{There is an enlargement of } Y_l(i+2) \text{ wrt. } t^*.$

Moreover, if $\langle \overline{t}', \overline{t}, \sigma \rangle$ is such an enlargement, then it gives rise to an enlargement $\overline{E}$ of $Y_l(i+2)$ with

$\overline{E} | h = E | h$, $\overline{t}' = t'_E$, $\langle W_{h+1}^E, N_{h+1}^E, \delta_{h+1}^E \rangle = \langle W_h, N_h, \sigma \rangle$.

We now apply this machinery to prove Thm 1. We proceed by induction on $\delta < \ell h(l)$.

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Case 1 $\delta = 0$. Then $\langle \langle w, N, \delta \rangle \rangle$ is an enlargement of $Y_l(i)$.

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Case 2 $\delta = 1' + 1$. Let $h = T(1' + 1)$

Case 2.1 $h$ does not survive at $1' + 1$.

Let $E$ be an enlargement of $Y_l(1' + 1)$. At $i$ is a breakpoint at $1' + 1$, then either $i = i_0$ or $i < i_0$ by the induction hypothesis, we may assume $W_i^E \cap O_n = W_i \cap O_n$.

By Lemma 2 we can then extend $E$ to be an enlargement $\overline{E}$ of $Y_l(1' + 2)$.

---

\[ QED \ (\text{Case 2.1}) \]
Case 2. Let $h$ survive at $j+1$.
Let $E$ be an enlargement of $Y(j' + 1)$.
At $i' = j$ in a breakpoint at $j+1$, then $h > i'$. Hence we can assume:
$W_{h 1} E 1 O 0 n < W_{i'} E 1 O 0 n$.
By Lemma 3, $E | h$ extends to an enlargement $E'$ of $Y(j' + 2) $ sat:
$W_{h 1} E' 1 O 0 n < W_{i'} E' 1 O 0 n$.

Q.E.D (Case 2)

Case 3. Let $i_0 \leq \delta$ and $i$ survive at $\delta$.
Choose $i_0 \leq \delta$ sat $i$ survives at $\delta$.
Let $E$ be an enlargement of $Y(j_0' + 1)$. At $i' = \delta$ in a breakpoint at $\delta$,
we may assume $i_0$ chosen large enough that $i < i'$. Hence by the
ind. hyp. we may assume:
$W_{i_0'} E 1 O 0 n < W_{i'} E 1 O 0 n$.
We extend $E | i_0$ to an enlargement $E'$
of $Y(\delta + 1)$ sat $W_{\delta'} E' = W_{i_0'} E$. We

We proceed as follows: For $j < i_0$ we construct enlargements $E_j$
of $Y(j' + 1)$ sat. $E_j | h = E_h | h$ for $h < j$ and $W_{i'} E_j = W_{i_0'} E$, $N_{i'} E_j = N_{i_0'} E$. 

For $j_i = i \circ \omega_i$, let $E_0 = E$. Now let $j = k + 1$, $h = T(i')$. We know that $E_h$ extends to an enlargement of $Y_{i'}$, since $h$ is a breakpoint at $j$. By Cor. 4.1, there is in $W_h$ an enlarger $\langle \tilde{t}, \tilde{e}, \sigma \rangle$ of $Y_{i'}$ into $t$. We let $\langle \tilde{t}_h, \tilde{e}_h, \sigma_h \rangle$ be the least such, in the sense of $L_{\gamma_0}[a \omega_n \gamma_0]$, where $\gamma_0 = 0 \in \gamma(W_h)$. We recall that $C_{\gamma_0} \subseteq L_{\gamma_0}[a \omega_n \gamma_0]$, where $\langle \tilde{t}, \tilde{e}, \sigma \rangle \subseteq C_{\gamma_0}$.

Let $E_i$ be an enlargement given by $\langle \tilde{t}_h, \tilde{e}_h, \sigma_h \rangle$. Then $W_i^{E_i} = W_h^{E_h} = W_{i_0}^{E_0}$, $N_i^{E_i} = N_h^{E_h} = N_{i_0}^{E_0}$, and $\tilde{t} = t_i^{E_i}$.

Now let $j = \gamma_j$, $\lim_j (\gamma_j)$. Then

$E' = \bigcup_{j \leq \gamma} E_i$ is an enlargement of $Y_{\gamma_j}$. We extend this to $Y_{\gamma_j}$ by setting $W_{\gamma_j}^{E_j} = W_{i_0}^{E_0}$,

$N_{\gamma_j}^{E_j} = N_{i_0}^{E_0}$, and $\delta_{\gamma_j}^{E_j} = \sigma_{i_0}^{E_0}$,

where $\sigma : P_{\gamma_j} \subseteq N = N_{\gamma_j}$ is defined by $\sigma(b_{\gamma_j}) = \delta_{j}^{E_j}$ for $i_{j} \leq i \leq \gamma_j$. 


At it is easily verified that $E_3$ is an enlargement, as soon as we have verified that $0 \in W_3 = W_3^{E_3}$.

Since $W_3$ is a $ZFC^*$ model, this will follow from: $\langle \delta^1, l^0, l, \leq, \leq \rangle \in W_3$, where $\delta^1 = \delta^{E_1}$. But $\delta_{l+1} = \delta^1(l+1)$ and $\delta^1$ is defined canonically from $\langle \delta^1, l^0, l, \leq, \leq \rangle$ for $\text{fin}(\lambda)$. Hence $\langle \delta^1, l^0, l, \leq, \leq \rangle$ is recursively definable in $\langle W_3, W_3, E_3, l^0, A \rangle$ from $E_3$.

QED (Thm 1)