§1 Remarks on the gap two problem

Consider a first order language $L$ with predicates $E, A, B$ and axioms:

1. $\exists C^+ A$ in an infinite cardinal $+$
2. $B = A^+$ in the largest cardinal $\kappa$.

By a $(\kappa, \kappa^+)$-model of $L$ we understand a model

$$\mathcal{M} = \langle M, \mathcal{L}_M, \mathcal{A}_M, \mathcal{B}_M, \mathcal{E}_M, \cdots \rangle$$

such that $\mathcal{M} = \kappa^+$ and $\mathcal{A}_M = \kappa$.

Note The usual notion of $(\kappa, \kappa^+)$-model requires only that $\mathcal{M} = \kappa^+$ and $\mathcal{A}_M = \kappa$.

If we added a predicate $F$ and the axiom $F: \forall \mathcal{M} \in \mathcal{V}$, then the two notions would become equivalent for models of Th is theory.

Note If $\mathcal{M}$ is a $(\kappa, \kappa^+)$-model, then $\mathcal{B}_M = \kappa^+$. To see this note that, for any $z$, $\mathcal{M}$, letting $\preceq_M$ be the natural ordering of $\mathcal{M}$, if $z \in \mathcal{M}$, then for all $x \in \mathcal{B}_M$,

$$\exists z \exists x \exists z <_{\mathcal{M}} x^3 \leq \kappa$$

(Th is is because either $\exists z \exists x <_{\mathcal{M}} x^3 = \emptyset$ or else...
Let \( M \models f : A \rightarrow x \). Let \( M \models f : A \rightarrow x \).
Set \( \bar{f} = \{ z \mid \forall y \exists z = f(y) \} \). Then
\( f : A \rightarrow \{ z \mid \exists x \leq \omega \ x \} \). Hence
\( B = \{ z \mid \exists x \leq \omega \ x \} \) has cardinality
\( \leq \omega^+ \). Suppose \( \bar{B} \leq \omega \). Then, by the
above argument, for each \( x \in \text{On}_\omega \), we have
\( \{ z \mid \exists x \leq \omega \} \leq \omega \). But
\( \text{On}_\omega = \bigcup \{ z \mid \exists x \leq \omega \} \). Hence
\( \text{On}_\omega \leq \omega^+ < \omega^{++} \). Contrad!

\textbf{Lemma 1} Let \( M \) be a \((\omega; \omega^+)\) model,
set \( b \) be an initial segment of
\( \langle \text{On}_\omega, \leq \rangle \) sat. \( \text{cf}(b) = \omega^+ \). Then
\( b \) has a supremum in \( \langle \text{On}_\omega, \leq \rangle \).
(i.e. There is \( z \in \text{On}_\omega \) sat. \( z \leq \langle \rangle \).\( \langle \rangle \text{ sat. } \forall x \in b \ x \leq \langle \rangle \)
for all \( u \in \text{On}_\omega \).
1. of Lemma 1. Suppose not.

Since \( \{ z \mid z <_\alpha x^3 \} \subseteq \tau^+ \) for \( x \in \Omega_{\alpha^+} \) and \( \Omega_{\alpha^+} = \tau^{++} \), we have:

\[ \text{cf}(\Omega_{\alpha^+}) = \tau^{++} \text{ in } <\alpha. \]

Hence \( b \) is a proper segment of \( \Omega_{\alpha^+} \), let \( x \in \Omega_{\alpha^+} \setminus b \). Let \( f \in \mathcal{N} \) s.t.

\[ \mathcal{M} \models f : B \xrightarrow{\text{ult}} x. \]

Then \( \tilde{f} = \{ \langle u, v \rangle \mid \mathcal{N} \models u = f^{|\mathcal{M}|_u} \} \) is a map of \( B \) onto \( \tilde{x} = \{ z \mid z <_\alpha x^3 \} \). For \( z \in B \) let \( \mathcal{M} \models f_z = f^{|\mathcal{M}|_z} \) and let \( \tilde{f}_z : \tilde{z} \xrightarrow{\text{ult}} \tilde{x} \) have the obvious definition.

Then \( \tilde{f}_z : \tilde{z} \xrightarrow{\text{ult}} \tilde{x} \). Since \( x = \text{sup} b \) in \( <\alpha \), there \( u \in \tilde{x} \setminus b \). Hence \( \text{there is } z \in B \) s.t. \( u \in \text{rng}(\tilde{f}_z) \).

However:

(1) \( \text{rng}(\tilde{f}_z) \cap b \) is bounded in \( b \),

since \( \text{rng}(\tilde{f}_z) = \tilde{z} \leq \tau \) and

\[ \text{cf}(b) = \tau^+ \text{ in } <\alpha. \]

Pick \( d \in b \) s.t. \( \text{rng}(\tilde{f}_z) \cap b \subset d \),
Then there is a unique $q \in \text{On}$ with

\[ q \in \operatorname{ran} (f_z) \mid u > d ^ 3, \]

so follow immediately that

\[ q = \min \{ q \in \operatorname{ran} (f_z) \mid u > d ^ 3 \}. \]

(Note that def. of $q$ does not depend on $d$; it would be the same for any $d' \in b$ if $t. \operatorname{ran} (f_z) \mid b < d ^ 2$.)

Obviously

\[ q_2 < q_1 \in B \rightarrow q_1 , q_2 , \in M. \]

We now define $\langle n_\mu \mid u \in B \backslash \{ \} \rangle$. Let

\[ M = \{ n_\mu \in \omega \mid \text{as follows} \} \]

Pick a $d = \lim _{n_\mu} \in b$ s.t. $\operatorname{ran} (f) < d _n$.

Working in $M$, define a map

\[ q_\mu : \{ z, u \} \rightarrow \text{On} \]

by

\[ q_\mu (w) = \min \{ q \in \operatorname{ran} (f_w) \mid u > d ^ 3 \}. \]

Then in fact $M \models q = q (w)$ for

\[ z < w \leq u. \]

An $M$ let $A = A_\mu = \{ q (w) \mid z \leq w \leq u \}.$

Then $M \models A$ is finite, in which

\[ A \subseteq \omega. \]
\[ q(w) \leq q(w') \text{ for } w' \leq w. \]

Let \( \mathcal{M} \models m_u = z \).

(5) \( \mathcal{M} \models m_u \leq w \); hence \( m_u \in A \).

(6) \( z \leq m \leq u \in B \)

\[ z \leq m \leq u \in B \rightarrow \]

\[ \rightarrow m_u \leq m_u', \]

since \( \mathcal{M} \models q^u = q^u' \) [\( z, u \)] and hence \( \mathcal{M} \models a_u \leq a_u' \).

(7) Let \( z \leq u \in B \). There is \( u' \in B \)

\[ \forall u, \mathcal{M} \models a_u \neq a_u', \]

(hence \( m_u \leq m_u' \)).

\( m_u \leq m_u' \)

\[ q_u = q(w) = \min_a \text{ in } \mathcal{M}, \]

\[ q_u = q(w) = \min_a \text{ in } \mathcal{M}. \]

Let \( p \in o_m \setminus b \) s.t. \( p \leq q_u \).

(Then there must exist \( a_u \), since otherwise \( q_u = \sup b \) in \( \leq o_m \).)

Then \( \mathcal{M} \models \forall u' \in B \) p.eq \( q_u \).

Let \( p \text{ e.q. } q_u \), where \( u \leq u' \).

Then \( q_u \leq p \leq q_u' \).

Hence \( \mathcal{M} \models a_u = \sup \{ q_u \} \neq \sup \{ q^u \} = a_u' \).

Hence \( m_u' \geq m_u \). \( \Box \)
Now select \( <z_3, 13 < z^+ > \) so that:
\[
z_3 < B \text{ and } z_0 = z_1 \\
z_3 \leq z_3 \text{ and } m_{z_3} > m_{z_3} \\
z_3 \leq 3 \text{ and } z_3 \geq z_3 \text{ for all } 3 < \lambda
\]
(Lim \( \lambda \)).

Then \( m_{z_3} < m_{z_3} \), for \( 3 < 3' \).

Hence \( <m_{z_3}, 13 < z^+ > \) injects \( \mathbb{Z}^+ \) into \( A \), where \( \mathbb{Z} = \mathbb{Z} < \mathbb{Z}^+ \).

Contradiction! QED (Lemma11)
Now let $L^*$ be $L$ together with a new predicate $C$ and the additional axiom:

$$C = \{ C \lambda \mid B \subset \lambda \land \text{Lim}(\lambda) \}$$

is a $\square_B$-sequence.

We shall give a model of set theory satisfying $\text{GC} + \Delta^+$, i.e.,

$L^*$ has no $(\omega, \omega_2)$-model.

(We shall ensure that $\square_{\omega_2}$ will hold for $\omega > \omega_1$, hence $(\kappa, \kappa^+) \not\rightarrow (\omega, \omega_2)$ for $\kappa > \omega_1$.) Since $\Delta^+$ holds, there will be a Kurepa tree in the model. This will show that the gap 2 conjecture can fail at $(\omega, \omega_2)$ even in the presence of a Kurepa tree.

The absence of a $(\omega, \omega_2)$-model for $L^*$ means, of course, that $\square$ fails. Since $\square$ holds whenever $\omega_2$ is not Mahlo in $L$, we shall force over a ground model containing a Mahlo cardinal $\kappa$. The forcing has two stages. In the first stage
we do ordinary collapsing to turn $\kappa$ into $\omega_2$. The resulting generic extension has neither a Kunen tree nor an $(\omega, \omega_2)$-model for $L^*$. We then force to reinstate the principle $\Diamond^+$, but without adding a $(\omega, \omega_2)$-model of $L^*$.

In the following let $N$ be a countable transitive model of $ZFC + GCH + \Diamond^+$ and there is a Mahlo cardinal.
Let $\kappa$ be a Mahlo cardinal in $N$.
Let $S = \langle S_\alpha \| \alpha < \omega_1 \rangle$ be a fixed $\Diamond^+$-sequence in $N$. In the first stage of our forcing we use the normal conditions for collapsing to make $\kappa$ become $\omega_2$.

Def. Let $\omega_i \leq \tau < \kappa$.

$C^\tau = \{ \text{the set of maps } p \text{ with} \}

\text{dom}(p) \subseteq \omega \times [\tau, \kappa), \bar{p} \leq \omega_1 \text{ and}

p(i, \nu) < \nu \text{ for } <i, \nu> \in \text{dom}(p)

p \leq q \iff p \supseteq q \text{ for } p, q \in P^\tau.$

We also set $C^\omega = C^\omega_{\omega_1}$. 

The properties of this forcing are well known:

(a) $\mathbb{C}^\kappa_\xi$ is $\omega_1$-distributive.

(b) (Assume $\text{GC}^{+}$) If $\mu > \omega_1$ is regular, then $\mathbb{M}^\kappa_\xi$ satifies the $\mu$-CC (i.e., every antichain has cardinality $< \mu$).

(c) $\mathbb{C}^\kappa_\xi \models \forall \bar{x} \in \bar{x}_0: \bar{x}_0 \leq \omega_1$ ($\xi < \mu$)

At follows that if $\mu > \omega^N_1$ is regular in $N$ and $G \in \mathbb{C}^\mu_\xi$ - generic over $N$, then $\omega^N_1 = \omega^N_1$ and $\omega^N_2 = \kappa$.

We also note that (a), (b) are satisfied by $\mathbb{C}^\kappa_\xi \times \mathbb{C}^\mu_\xi$, and that $\mathbb{C}^\kappa_\xi = \mathbb{C}^\mu_\xi \times \mathbb{C}^{\omega_1}_\xi$ for $\omega_1 \leq \xi < \kappa < \mu$.

We force with $\mathbb{C}^\kappa_\xi$, where $\kappa$ is a Mahlo cardinal in $N$. It is known that $\square$ then becomes false in the resulting model. We improve this to:

**Lemma 2** Let $\kappa$ be Mahlo in $N$ and let $G$ be $\mathbb{C}^\kappa$ - generic over $N$. Then $L^G$ has no $(\omega, \omega_2)$ - model in $N[G]$. 
prof. of Lemma 2.
Suppose not. Let \( M \) be an \((\omega, \omega_1)\) - model of \( L^\kappa \) in \( N[G] \). Let
\[
M = \langle \omega, \mathcal{L}, A, B, C, \ldots \rangle.
\]
We can assume w.l.o.g. that \( \omega \in \kappa \) and hence that \( |\omega| < \kappa \) in \( N[G] \).
Let \( M = M^G \kappa \) and let \( \theta > \kappa \) be regular in \( N \) s.t. \( \omega \in H_\theta \) in \( N \).
Set \( H = H_\theta \). Since \( G \in \mathcal{G} \) - generic over the \( \mathcal{ZFC} \) - model \( H \) and the
above properties of \( M \) are absolute in \( H[G] \), there is a \( p \in G \) s.t.
\[ p \Vdash (\omega \in \kappa \land |\omega| < \kappa). \]
Fix Skolem functions for \( H \) and set:
\[ X_\alpha = \text{the smallest } X \subseteq H \text{ s.t. } \exists \nu \exists \gamma \exists \xi \exists X \]
\[ X \subseteq X_\alpha \text{ and } X_\alpha \models p, \nu, \gamma, \xi \exists X \]
for \( \alpha < \kappa \). Set:
\[ C = \{ \alpha < \kappa \mid \exists \eta X_\alpha \models \xi \exists \gamma \}. \]
Then \( C \) is club in \( \kappa \). By Mahlo's
there is a regular \( \tau \in C \). Let
\[ X = X_\tau \text{ and set: } \sigma : H \leftrightarrow X, \]
where \( \sigma \) is transitive.
Then $\sigma : \overline{H} < H$, $\tau = \sigma \cap (\sigma)$, $\sigma (\overline{e}) = \overline{e}$, and $\sigma (\overline{C^2}) = C^2$. Clearly:

(2) $\varphi \models \overline{C^2} \sigma (t_1, \ldots, t_n) \iff$

$\varphi \models C^2 \sigma (t_1, \ldots, t_n)$,

for $\varphi \in C^2$, $t_1, \ldots, t_n \in \overline{H}$, since $\sigma (\overline{e}) = \overline{e}$.

Let $\overline{G} = \overline{G} \cap C^2$. By (2):

(3) There is a unique $\overline{\sigma} : \overline{H} \subseteq \overline{H}[\overline{G}]$ defined by $\overline{\sigma}(\overline{x}) = \overline{x}$.

Hence:

(4) $\varphi \models \overline{G} \iff \overline{\sigma}$, where

$\overline{\sigma}(\overline{x}) = \overline{\sigma}(\overline{x}^\overline{G}) = \sigma(\overline{x})^\overline{G} = \overline{\sigma}(\overline{x})$ for $\overline{x} \in \overline{H}$.

In particular:

(5) $\overline{\sigma} \models \overline{G} = \overline{id}$.

By (3) we have $\overline{\sigma} \models \overline{G} \iff \overline{G}$, but

$\overline{\sigma} \models \overline{G} = \overline{id}$ by (5). Hence

(6) $\overline{G} \models \overline{G}$.

Let $\overline{e} = \langle \overline{10}1, \overline{e}, \overline{A}, \overline{B}, \overline{C}, \overline{\cdots} \rangle$.

(7) $\overline{e}$ is an end extension of $\overline{e}$.

Let $x \in \overline{e}$. Claim $\overline{e} \models \overline{x}^3 = \overline{\overline{e}} \models \overline{x}^3$.

The case $\overline{e} \models x = \emptyset$ is trivial by (6).

So assume $\overline{e} \models x \neq \emptyset$. There is
A 2-

\( f : B \to B \quad \text{in} \quad A \models \tau \).

Then

\( f : B \to B \quad \text{is} \quad \in \quad A \models \tau \).

Let \( g \in A \models \tau \).

Then

\( g : \in \quad A \models \tau \).

The same proof shows:

(8) \( A \models \tau \).

Thus \( \text{On} \) in an initial segment of \( \text{On} \) in \( N[\alpha] \). Moreover:

(9.1) \( \text{cf}(\text{On} \models \tau ) = \omega_1 \) in \( N[\alpha] \).

Hence \( \text{there is} \ u \in \text{On} \models \tau \).
Note: Without the axiom:
\[ C = \{ C_\lambda \mid \lambda \in \text{Lim}(\alpha) \} \text{ is a } \square_B \text{-sequence.} \]

as meaning:
(a) \( C_\lambda \) is cut-in \( \lambda \) and \( \sigma \tau \lambda (C_\lambda) \subseteq B \)
(b) If \( \gamma \in C_\lambda \) and \( \text{Lim}(\gamma) \), then \( C_\gamma = \gamma \cap C_\lambda \).

(This is the usual formulation of \( \square_B \) with the additional condition that, whenever \( \bar{\gamma} \in C_\lambda \) is a successor point in \( C_\lambda \), then \( \bar{\gamma} \) is a successor ordinal.

It is trivial that if there is a \( \square_B \)-sequence, then there is one with the additional condition.)

Def: For \( \kappa \in \text{On}_{\nu} \), let \( \Omega \) be \( \text{Lim}(\kappa) \) set:
\[ C_\kappa = \{ \gamma \mid \nu \gamma = \gamma \in C_\kappa \} \]
In particular set: \( D = C_\kappa \), where \( \nu = \sup \text{On}_{\nu} \).
Set \( \Gamma = \{ \kappa \in \text{On}_{\nu} \mid \nu \kappa = \text{Lim}(\kappa) \} \)
and \( \bar{\kappa} = \{ \text{Lim}(\kappa) \} \) for \( \kappa \in \text{On}_{\nu} \).

Then \( D \) has the properties:
(10) (a) \( D \cap \Gamma \) is cofinal in \( \text{On}_{\nu} \)
(b) If \( \kappa \in D \cap \Gamma \), then \( D \cap \bar{\kappa} = C_\kappa \).
Def any $D \subseteq \mathcal{O}_\omega$ satisfying (10) is called devilish.

We have shown that there is a devilish set in $N[G]$, but $N[G] = N[G^-][G^-]$ where $G' = G \cap C^\omega$ is $C^\omega$-generic over $N[G^-]$. Hence there is $q \in G'$ such that

$$q \not\in N[G^-]$$

(there is a devilish set).

Now let $G_0 \times G_1$ be $C_i^\omega \times C^\omega$-generic over $N[G^-]$ with $q \in G_i$ ($i = 0, 1$). We know that $C_i^\omega \times C^\omega$ is $\omega_1$-distributive. Hence $\omega_1$ is absolute in $N[G^-][G_0 \times G_1] = N[G^-][G_0][G_1]$. Let $D_i \subseteq N[G^-][G_i]$ be devilish. We derive a contradiction.

We first note:

(12) $\cap_i D_0 \cap D_1$ is bounded in $\mathcal{O}_\omega$.

**Proof.**

Suppose not. Then

$$D_0 = D_1 = \bigcup_{x \in \Pi D_0 \cap D_1} C_x.$$ 

Hence

$$D = D_i \subseteq N[G^-][G_i]$$

for $i = 0, 1$.

By the product lemma, $D \subseteq N[G^-]$. But $\overline{D} \subseteq \mathcal{O}_\omega (C_x)$, hence

$$\overline{D} \subseteq C_x \subseteq A$$

for $x \in \Pi \cap D$.

By this we get:
Claim. $\text{cf}(D) \leq \omega_1$ in $\langle \omega_7 \rangle$ (in $\mathbb{N}^\omega$).

**Proof.** Suppose not. Let $\langle x_i : i < \omega_2 \rangle$ be a monotone sequence in $D$, then $\langle x_i : i < \omega_2 \rangle$ is bounded in $\text{On}_{\omega_1}$ and there is $\omega \in \text{On}_{\omega_1}$ such $\omega = \sup x_i$ in $\langle x_i \rangle$ by Lemma 1. But then $x_i \in D \cap \omega = C_\omega$ for $i < \omega_1$. Hence $C_\omega \geq \omega_1$, where $\omega \in \text{P}_D$. Contradiction! QED.

Since $D$ is unbounded in $\text{On}_{\omega_1}$, it follows that $\text{cf}(\text{On}_{\omega_1}) = \omega_1$ in $\langle \omega_7 \rangle$. But in $\mathbb{N}^\omega$, we have $\text{cf}(\text{On}_{\omega_1}) = 2 = \omega_2$.

**Contradiction!** QED (12).

Now let $x_0 \in \text{On}_{\omega_1}$ such that $(D_0 \cap D_1 \cap \Gamma) \setminus x_0 = \emptyset$.

In $\mathbb{N}$ we define for each $z \in D_0 \cap \Gamma$ an $m_z$ such that $\overline{D_1} = m_z \in \omega$ (hence $m_z \in A$). We do this as follows:

Pick a $z' \in D_1 \cap \Gamma$ such that $z <_{\omega_1} z'$.

Arguing in $\mathbb{N}$, there is a sequence $\langle z^\omega(i) : i < m_z \rangle$ defined by:

$z^\omega(0) = \text{the least } r \in C_z \setminus x_0$

$z^\omega(i+1) = \text{the least } r \in C_{z^\omega(i)} \setminus x_0$

$\forall z \in C_z \cap C_{z^\omega(i)} \setminus x_0 \setminus C_{z^\omega(i+1)}$.

Then $\overline{D_1} = m_z < \omega_1$, since otherwise
\[ \omega_i \leq m_z = \omega \quad \text{and there is } u > x_i \]
\[ \omega_i \geq u = \sup_i x^2(i), \quad \text{Then } u \in \Gamma \cap D_0 \cap D_1, \]
where \( \omega > x_0 \). Continue!

It is obvious that the definition of \( m_z \) does not depend on the \( C_{z'} \) chosen, since \( C_{z'} = \omega\cap D_1 \).

Also:
\[(13) \quad z < z', \quad \text{in } (\Gamma \cap D_0) \rightarrow m_z \leq m_{z'}, \]
Moreover:
\[(14) \quad \text{If } z \in (\Gamma \cap D_0), \quad \text{there is } z' > z \quad \text{in } \Gamma \cap D_0 \quad \text{s.t.} \quad m_z < m_{z'}. \]

**Proof:**
Choose \( w \in D_1 \cap \Gamma \) s.t. \( w > x_0, z \),
Choose \( z' \in D_0 \cap \Gamma \) s.t. \( z' > w \),
Then \( \omega_i \geq m_{z'} \geq m_z \). Hence
\( \omega_i \geq m_z \geq m_{z'} \). QED (14)

Now choose \( z_3 \in D_0 \cap \Gamma \) (\( z < \omega_1 \)) s.t.
\[ z < \omega_i \quad \text{and } m_{z_3} \leq n_{z_3}, \]
and \( z > \omega_z \) for all \( z < \lambda \)
(see (11)). Then \( m_{z_3} \leq m_{z_3'} \)
for \( z < z' < \omega_1 \). Hence
\( m_{z_3} \) \( \leq \omega_1 \) injects \( \omega_1 \).
int into $\overline{A_{x_0}}$, where $\overline{A_{x_0}} = \omega$. Contr!

QED (Lemma 2)

Our main result is:

Lemma 3 Let $G$ be $\mathcal{C}^\kappa$-generic over $N$. There is a generic extension $N[G][H]$ of $N[G]$ s.t. in $N[G][H]$

(a) $\kappa = \omega_2$; (b) $\Diamond^+$ holds;

(c) $L^*$ has no $(\omega, \omega_2)$-model.

The proof depends on the construction of forcing conditions $IP \in N[G]$ s.t.

(a)-(c) are forced. The actual construction of $IP$ will be given in §2. Here we list the salient properties of $IP$ and derive Lemma 3 from them.

Since GCH holds in $N$, there is an $A_0 \in N$ s.t. $A_0 \subseteq \kappa$ and $L[\mathcal{A}_0] = \mathcal{H}_\kappa$

for all cardinals $\tau$ s.t. $\omega \leq \tau \leq \kappa$.

Let $G$ be $\mathcal{C}^\sigma$-generic over $N$. Set

$A_1 = \{ \langle \mu, \gamma, i \rangle | V(p \in G \gamma(p, i)) = \mu \}$

$A = \{ \langle \langle \xi, i \rangle | \xi = 0 \lor \mu \in A_0 \lor i = 1 \lor \mu \in A_1 \}$

Let $\mathcal{I}$ be regular in $N$ s.t. $\omega \leq \tau \leq \kappa$.

Let $G = G \cap \mathcal{C}^\tau$. Then $G$ is $\mathcal{C}^\tau$-

- generic over $N$ and
\[ \text{and } N[A] = N[A_{f \cap A}] \text{. Moreover } H_\xi = L_\xi[A] \text{ in } N[G] \text{. In particular } L_\xi[A] = H_\xi \text{ in } N[G] \text{,} \]

Moreover, we shall define a set of conditions IP \subseteq N[G] with the following properties:

(A) IP \subseteq L_\xi[A] \text{ and } L_\xi[A] \text{-definable,} \]

(B) Each \( p \in IP \) is a function \( \eta : \text{dom}(p) \rightarrow \omega \) in a countable subset of \( \eta \). Moreover, \( \eta \leq \xi \in IP \iff \eta \supseteq p \text{ for } p, \eta \in IP \).

(C) IP is \( \omega_1 \)-distributive.

Def. Let \( \xi < \omega_1 \). Set \( IP_\xi = \{ p \in IP : \exists \eta \leq \xi \text{ in } IP \} \)

with \( \eta \leq p \in IP_\xi \rightarrow p \supseteq \eta \).

(D) \( IP \subseteq L_\xi[A] \)

(E) Let \( p \in IP \), \( \xi < \omega_1 \), \( q \in IP_\xi \) a.t. \( q \leq p \in IP \)

in \( IP \). Then \( q \cup p \in IP_\xi \). (Hence \( q \cup p \leq p \in IP )\)

(F) Let \( \langle L_\xi[A], A \rangle \) \( < \langle L_\xi[A], A \rangle \)

a.t. \( \text{cf}(\xi) = \omega_1 \). Then \( IP_\xi \subseteq L_\xi[A] \).

*Note. The actual definition of IP given in \( \S 2 \) refers to a \( \Box \)-sequence \( S = \langle S_\xi : \xi < \omega_1 \rangle \). We have assumed that \( \Box \)-hitting in \( N \). At \( \tau \) is known that any \( \Box \)-sequence in \( N \) remain a \( \Box \)-sequence in \( N[G] \). Hence we may take: \( S = \text{the } L_\xi[A_\tau] \text{-least } \Box \text{-sequence.} \)

By (D), (E) a standard proof gives:

(7) IP satisfies the \( \omega_2 \)-cc.
Let $X$ be a maximal antichain in $\mathbb{P}$. Define $<\bar{z}_i | i \leq \omega_1>$ by $\bar{z}_0 = \emptyset$. Given $\bar{z}_i$, select for each $p \in \mathbb{P} \bar{z}_i$ a $q \in X$ s.t. $q_p$ is compatible with $p$. Let $\bar{z}_{i+1}$ be the least $\bar{z} \supseteq \bar{z}_i$ s.t. $q_p \in \mathbb{P} \bar{z}_i$ for all $p \in \mathbb{P} \bar{z}_i$. For limit $X \subseteq \omega_1$ s.t. $\bar{z}_X = \sup_{\bar{z}_i < X} \bar{z}_i$.

Claim: $X \subseteq \mathbb{P} \bar{\omega}_1$ (hence $\bar{X} \subseteq \omega_1$).

Suppose not. Let $p \in X \setminus \mathbb{P} \bar{\omega}_1$. Then $p \uplus \omega_1 \in \mathbb{P} \bar{z}_i$ for some $i < \omega_1$.

Since $\text{dom}(p)$ is countable. By our construction there is $q \in \mathbb{P} \bar{z}_{i+1}$ s.t. $q \in X$ and $q$ is compatible with $p \uplus \omega_1$. But then there is $q' \in \mathbb{P} \bar{z}_{i+1}$ s.t. $q' \leq q \uplus \omega_1$.

Set $p' = p \uplus q'$. Then $p' \in \mathbb{P}$ and $p' \leq q', q$. Contradiction! QED (1).

A further property of $\mathbb{P}$ is:

($G$) $\mathbb{P} \vdash \Box^+$. 
Now let $\tau \leq \mu$ and let $\bar{H}$ be $1P^\tau$-generic over $N[G]$. Define $1P^\tau_H$ by:

$$1P^\tau_H = \{ p \in 1P \mid p \upharpoonright \tau \in \bar{H} \}.$$

By (E), (D) a standard proof gives:

(2) $H \cup 1P_\tau$-generic over $N[G]$ if

$$H = H \cap 1P^\tau \cup 1P^\tau - \text{generic over } N[G] \text{ and } H \cup 1P^\tau_H - \text{generic over } N[G][\bar{H}] \quad (\tau \leq \omega).$$

Proof:

$\rightarrow$ If $\bar{H} \cup 1P^\tau$-generic, then if $\Delta \in N[G]$ is dense in $1P^\tau$, then $\Delta^* = \{ p \in 1P \mid p \upharpoonright \tau \in \Delta \}$ is dense in $1P$. Now let $\Delta \in N[G][\bar{H}]$ be dense in $1P^\tau_H$. Claim: $H \cap \Delta \neq \emptyset$.

Let $\Delta = \Delta^\bar{H}$, where $\Delta \in N[G]$. Suppose w.l.o.g. that $1P^\tau \models \Delta$ is dense in $1P^\tau_H$, where $\bar{H}$ is the canonical term for $\bar{H}$.

(Hence $1P^\tau \models (H \cup 1P^\tau$-generic) and $\phi \models \varphi \in \bar{H}$ for $\phi \in 1P^\tau$. ) Set $\bar{\Delta} = \{ p \in 1P \mid p \upharpoonright \tau \models \varphi \in \Delta \}$. At

Claim: $\bar{\Delta}$ is dense in $1P$. 

We also need:

(H) Let $\mathcal{H}$ be $\mathbf{IP}^\infty$-generic over $N[G]$. Then $\mathbf{IP}_H^\infty \times \mathbf{IP}_H^{\mathcal{H}}$ is $\text{w}_0$-distributive in $N[G][\mathcal{H}]$. 

\textbf{Claim 1:} Let $p \leq p'$, let $\mathcal{H}$ be $\mathbf{IP}^\infty$-generic over $N$. Then $p \in \mathbf{IP}_H^\infty$. Let $p \leq p'$, let $p \in \Delta = \Delta^{\mathcal{H}}$. Pick $q \in \mathcal{H}$ s.t. $q \leq p \mathcal{V}$, $p \mathcal{V}$ and $q \vdash p' \in \Delta$, set $p'' = q \uplus p'$. Then $p'' \leq p'$, $p'' \in \Delta$, QED (→)

Let $\Delta$ be dense in $\mathbf{IP}_H$. 

\textbf{Claim 2:} $\Delta \cap \mathcal{H} \neq \emptyset$.

It suffices to show: $\Delta \cap \mathbf{IP}_H^\infty$ is dense in $\mathbf{IP}_H^\infty$. Let $p \in \mathbf{IP}_H^\infty$. Then $\Delta^* = \{ p' \in \mathbf{IP}_H^\infty | p \leq p' \mathcal{V} \in \Delta \}$ is dense in $\{ q \in \mathbf{IP}_H^\infty | q \leq p \mathcal{V} \}$ by (E). Hence there is $q \in \Delta^* \cap \mathcal{H}$. Hence there is $p' \leq p$, $p' \in \Delta$, s.t. $p \mathcal{V} = q$. Hence $p' \in \Delta \cap \mathcal{H}$, QED (2)
At suffices to prove:

Sublemma 3.1 Let $H$ be $IP$-generic over $N' = N[G]$. There is no $(\omega, \omega_2)$-model for $L^s$ in $N'[H]$.

Proof: Suppose not.

By a good model of $L^s$, let us understand an $(\omega, \omega_2)$-model whose elements are ordinals $< \omega_2$.

We can assume, w.l.o.g., that there is a good model in $N'[H]$.

But then there is a good model in $H_\theta | N'[H] = H_\theta | N'[H] = H_\theta | N[G][H]^*$, where $\Theta > \alpha$ is regular in $N'$.

G is then $\mathcal{C}_\alpha$-generic over $M = H_\Theta$ and $H$ is $IP$-generic over $M' = M[G]$.

Thus, there is a $g \in H$ s.t. $\forall H^* \forall^{I_P^M} (\text{there is a good model})$.

Hence there is $\forall \in G$ s.t.

(3) $\forall H^* \forall^{I_P^M} (\text{there is a good model}),$

where $\mathcal{C}_\alpha$ is the canonical term for $C$ (in particular $\forall \in \mathcal{C}_\alpha$-gen. over $N'$ and $\forall H^* \forall \in \mathcal{C}_\alpha$ for $\forall \in \mathcal{C}_\alpha$).
An $N$ we now define for $\alpha \leq \kappa$:
\[ X_\alpha = \text{The smallest } X < H_\alpha \text{ s.t.}
\alpha \in \{i, j\} \]
Set $C = \{ \alpha \mid \alpha = X_\alpha \cap \kappa \}$, Then $C \cap \kappa$ is unbounded in $\kappa$ and there is $\zeta \in C$ which is regular. Set $X = X_\zeta$ and let $\sigma : \bar{M} \to X$, where $\bar{M}$ is transitive. Then $\sigma : \bar{M} \preceq M$, $\tau = \text{crit} (\sigma)$, $\tau (\bar{\zeta}) = \kappa$.
Hence $\sigma \cap L^*_\zeta [A_\zeta] = \text{id}$, where $L^*_\zeta [A_\zeta] = H^N_{\zeta}$. As before,
\[ \sigma^{< \zeta} (C^\zeta) = C^{< \zeta} \]. Since
\[ (\forall \bar{t} \in \bar{M} \quad \bar{t} \in \prod_{n \in \omega} \bar{t}_n) \to (\forall \bar{t} \in \bar{M} \quad \bar{t} \in \prod_{n \in \omega} \bar{t}_n) \]
$\sigma$ extends to a $\bar{\sigma} : \bar{M} [\bar{G}] \preceq M [G] = M'$ (where $\bar{G} = \bar{G} \cap C^\zeta$), defined by:
\[ \bar{\sigma} (t \bar{G}) = \sigma (t) \bar{G}, \quad \text{Set } \bar{M}' = \bar{M} [\bar{G}] \],
Then $\bar{\sigma} (A \cap \tau) = A$ and $\bar{\sigma} \cap L^*_\zeta [A] = \text{id}$, where $L^*_\zeta [A] = H^N_{\zeta} [\bar{G}]$. At follows that $\bar{\sigma} (1P^\zeta) = 1P$. Since:
\[ (\forall \bar{t} \in \bar{M}' \quad \bar{t} \in \prod_{n \in \omega} \bar{t}_n) \to (\forall \bar{t} \in \bar{M}' \quad \bar{t} \in \prod_{n \in \omega} \bar{t}_n) \]
$\bar{\sigma}$ extends to a $\sigma^* : \bar{M}' [\bar{H}] \preceq M' [H]$ defined by $\sigma^* (t \bar{H}) = \bar{\sigma} (t \bar{H})$ (where $\bar{H} = H \cap 1P^\zeta$).
Since (3) holds, where $p \in \bar{G}$, $q \in \bar{H}$, there is $\bar{\eta} \in M'[\bar{H}]$, which is a good model (in $M'[\bar{H}]$, hence in $N'[\bar{H}]$). Thus $\sigma^* (\bar{\eta}) = \eta$ is a good model in $N'[\bar{H}]$.

At follow exactly as before that:
(6) $E_n$ is an end extension of $\bar{\eta}$;

\[ A_{\bar{\eta}} = A_{\bar{\eta}} \quad \text{and} \quad B_{\bar{\eta}} = B_{\bar{\eta}}. \]

As before, this implies:
(7) There is $D \subseteq N'[\bar{H}]$ which is a devilish set for $\bar{\eta}$.

But $\overline{H}$ is $1P$-generic over $N' = N[\overline{G}][\overline{G}']$, where $\overline{G} = G \cap C^\infty$, $G' = G \cap C_{\infty}^\infty$. Since $C_{\infty}^\infty$, $1P \subseteq C_{\infty}^\infty$, it follows that $G'$ is $C_{\infty}^\infty$-generic over $N[\overline{G}][\overline{H}]$. We can then repeat the argument in the proof of Lemma 2 ((47)–(44)) to show:
(8) There is no devilish set in $N'[\overline{H}]$.

As before, however, there is a devilish set in $N'[\bar{H}] = N'[\bar{H}][\bar{H}]$,
where \( H \in \text{IP}_H \) - generic over \( N'[\bar{H}] \).

Hence there is \( q \in H \) such that

\[ q \upharpoonright H \text{ is a devilish set.} \]

Let \( H_0 \times H_1 \) be \( \text{IP}_H \times \text{IP}_H \) - generic over \( N'[\bar{H}] \) with \( q \in H_i \), \( i=0,1 \).

Let \( D_i \in N'[H_i] \) be devilish.

Since \( \text{IP}_H \times \text{IP}_H \) is \( \omega \) - distributive, we obtain a contradiction exactly as before, arguing in \( N'[H][H_0 \times H_1] \).

QED (Lemma 3)