§2 Forcing to obtain a $\Diamond^+$-sequence from a $\Diamond$-sequence.

Let $N$ be a transitive model of
\[ ZFC + 2^\omega = \omega_1 + 2^{\omega_1} = \omega_2 + \Diamond. \]

Let $S = \langle S_\alpha | \alpha < \omega_1 \rangle$ be a fixed $\Diamond$-sequence.

For $\alpha < \omega_1$ let $c_\alpha : \omega \rightarrow \alpha + 1$, where $c_\alpha = \langle c_{\alpha \xi} | \xi < \omega_1 \rangle \in N$. For $\alpha < \omega_1$ set
\[ M_\alpha = L_{\beta_\alpha} [S_{\alpha \beta_\alpha + 1}, c_{\alpha \beta_\alpha + 1}], \]
where $\beta_\alpha$ is the least $\beta > \alpha$ in $N$ such that $\beta > \alpha + 1$ and $L_{\beta_\alpha} [S_{\alpha \beta_\alpha + 1}, c_{\alpha \beta_\alpha + 1}] = ZFC$.

Define $S^* = \langle S^*_\alpha | \alpha < \omega_1 \rangle$ by $S^*_\alpha = \{ c_{\alpha \xi} | M_\xi \}$.

Set $M = \langle M_\alpha | \alpha < \omega_1 \rangle$.

We shall generically extend $N$ to an $N[G]$ s.t. $S^*$ is a $\Diamond^+$-sequence in $N[G]$.

Lemma 1 Let $\mathcal{M} = \langle A_i, \varepsilon, \ldots \rangle \in H_{\omega_2}$ be transitive. Then in $N$ there is $X < \mathcal{M}$ s.t. $X \subseteq \omega_1$, $A = \omega_1 \setminus X$ is transitive, and $\mathcal{M} \subseteq M_\alpha$, where $\mathcal{M}$ is the transitive closure of $X$.

$\forall N$ plays the role of $N' = N[G]$ in §1.
proof of Lemma 1.

Assume without loss of generality that $\omega_1 < \omega_2$ and let $f : \omega_1 \rightarrow \omega_2$. Let $T = \omega_1 \cdot f = \langle \omega_1, f, \nu (\nu < \omega_1) \rangle$ the complete theory of $\langle \omega_1, f, \nu (\nu < \omega_1) \rangle$ (with a constant $x$ for $\nu < \omega_1$). We suppose this theory to be coded in such a way that $T \subseteq \omega_1$. Let

$$C = \{ \xi \mid < \omega_1, f^\xi, x > < < \omega_1, f > \}$$

Then $C$ is a $T$-consistent in $\omega_1$. Let $\delta \in C$ and $T \cap \delta = \Delta$. Then, letting $\sigma : < \omega_1, f > \rightarrow < \omega_1, f^\delta, x >$, we have $\sigma \cdot f = f^\delta$, $\sigma \cdot r = id$, $\sigma \cdot \omega_1 = \omega_1$, and $T \cap \delta = the complete theory of $\langle \omega_1, f, \nu (\nu < \delta) \rangle$. Since $T \cap \delta \subseteq M_\delta$, we conclude $< \omega_1, f^\delta > \subseteq M_\delta$; since $< \omega_1, f >$ is uniquely recoverable from $T \cap \delta = theory (\langle \omega_1, f^\delta, x > < < \omega_1, f > \rangle)$ in any $\mathsf{ZFC}^-$ model containing $\omega_1$, $\omega_1 \cdot f$. Hence the lemma holds with $X = f^\delta$. QED (Lemma 1)
Def. Let $A \subset \omega_2$ be $\nabla$-stable. $L_{\omega_2}[A] = H_{\omega_2}$.

Define $\langle \rho | \tau < \omega_2 \rangle$ by:

$\rho = \text{the least } \rho > \omega_1 \text{ s.t. } \rho > \sup_{\tau < \omega_3} \rho_{\tau} \text{ and }$ $\nu \in L_{\rho}, \text{ cf}(\rho) = \omega_1 \text{ and }$ $L_\rho[A] = \{ \exists \xi \in \omega_2 \forall x \quad x \leq \omega_1 \}$

Set: $\tilde{\rho} = \omega_1 \cup \sup_{\tau < \omega_3} \rho_{\tau} \quad (\text{Heuristically } \tilde{\rho} = \rho_{\nu})$

$\mathcal{M}_\nu = \langle L_\rho[A], \mathcal{G}, \mathcal{A} \rho_{\nu}, \mathcal{M} \rangle$

For $\nu > 0$ set:

$\hat{\mathcal{M}}_\nu = \bigcup_{\nu < \tilde{\nu}} \mathcal{M}_{\tilde{\nu}} = \langle L_\rho[A], \mathcal{G}, \mathcal{A} \rho_{\tilde{\nu}}, \mathcal{M} \rangle$

Then:

**Lemma 2**

(\begin{enumerate}
\item $\langle \rho | \tau < \nu \rangle$ is uniformly $\mathcal{M}_\nu$-definable.
\item $[x]^{\omega} \in \mathcal{M}_\nu$ for $x \in \mathcal{M}_\nu$
\item $[\mathcal{M}_\nu]^{\omega} \subset \mathcal{M}_\nu$
\end{enumerate})

Proof:

(\begin{enumerate}
\item Trivial.
\item Follows by $\mathcal{M}_\nu \models \forall x \; x \leq \omega_1$ and $[\omega_1]^{\omega} \in \mathcal{M}_\nu$ (since $[\omega_1]^{\omega} \subset \mathcal{M}_\nu$).
\item Follows by (b) and $\text{cf}(\rho_{\nu}) = \omega_1$.
\end{enumerate})
Set: \( f_\nu = \text{the } \mathcal{M}_\nu \text{-least } f: \omega_\nu \rightarrow \vec{p}_\nu \).

\( \vec{a}_\nu = \text{the } 3 \text{-th } \vec{a} \in \omega_\nu \text{ in } L_\omega_2[\vec{a}] \).

\( \vec{a}_\nu = \{ \exists \lambda, \mu \in \omega_\nu \exists \langle f_\nu, (\mu) \rangle^{\vec{a}_\nu} \} \).

Then:

**Lemma 2** (d) \( \langle f_\nu, 1^\nu \rangle \) is uniformly \( \mathcal{M}_\nu \text{-definable} \) (and \( \mathcal{M}_\nu \text{-definable} \).

(e) \( f_\nu \in \mathcal{M}_\nu \) is uniformly \( \mathcal{M}_\nu \text{-definable} \).

(f) \( \alpha_\nu \) is uniformly \( \mathcal{M}_\nu \text{-definable} \).

(g) \( \vec{a}_\nu \in \mathcal{M}_\nu \) is uniformly \( \mathcal{M}_\nu \text{-definable}. \)

We now define forcing conditions \( \text{IP} = \text{IP}^\mathcal{A} \).
Def. \( IP_x = \mathbb{P}^A_x \) = The set of maps \( p \) such that \( p \) is closed, bounded in \( \omega_1 \), and \( x \leq p \rightarrow \exists y \in M_x \),

Set: \( m_p = \max(p) \) for \( p \in IP_x \),

\( p \leq q \iff q = p \wedge (m_q + 1) \) for \( p, q \in IP_x \).

Def. \( IP = \mathbb{P}^A = \mathbb{P}^A_1 \) the set of maps \( p \) such that \( \text{dom}(p) \subset \omega_2 \) is countable, \( p(v) \in IP_x \) for \( v \in \text{dom}(p) \), and whenever \( v \in \text{dom}(p) \), then:

\[
\begin{align*}
& (a) \quad \forall v m(p(v)) \\
& (b) \quad m(p(z)) \geq m(p(v)) \quad \text{for } z \in p(v) \\
& (c) \quad d \in p(v) \rightarrow p(v) \subseteq M_d \\
& \quad \text{where } \hat{p}(v) = \{ \langle \mu, z \rangle \leq m(p(v)) \mid \mu \in p(f(v)) \},
\end{align*}
\]

\( p \leq q \iff \text{dom}(q) \subseteq \text{dom}(p) \wedge \forall v \in \text{dom}(q) \ p_v \leq q_v \in IP_x \).

Def. For \( p \in IP \) set:

\( m_p = \min\{ m(p(v)) \mid v \in \text{dom}(p) \} \)

\( l_p = \text{lub}(\text{dom}(p)) \)
At is immediate from the definition that
\[ IP \subseteq L_{\omega_1}[A] \subseteq L_{\omega_1}[A] \text{ definable}. \]
At is also clear that, if \( W \) is an inner model of \( N \) with \( [W]^{\omega_1} \subseteq W \) in \( N \),
then \( IP \subseteq W \). Hence \((A), (I)\) of \( \S 1 \)
are proven. \((B)\) is also trivial from
the definition of \( IP \). For \( \nu < \omega_1 \) not \( 1 \).
\[ \tilde{IP}_\nu = \{ p \in IP \mid \text{dom}(p) < \nu \} \text{.} \]
At it is apparent that
\[ \tilde{IP}_\nu = IP^{-}\nu = \{ p \in IP \mid \exists \mu < \nu \} \text{.} \]
(\( \text{Hence} IP \subseteq IP^{-}\nu \).
(\( \text{In this section we generally write} \tilde{IP}_\nu \)
Instead of \( IP^{-}\nu \). The reason is that \( \S 2 \)
was largely written before \( \S 1 \).) At it is apparent that, if \( p \in IP \) and
\( q \in p \cap \nu \), then \( pq \subseteq IP \). Hence
\((D), (E)\) are proven. Before making
the further verification we prove:
Lemma 3 Let \( p \in \Pi^A \) \( \delta \leq \omega_1 \) \( \delta \leq \omega_2 \). There is \( p' \leq p \) s.t. \( \delta \leq \text{mp}' \) \( \delta \leq l(p') \).

Proof.
Assume w.l.o.g. \( l(p) < \delta \), \( mp < \delta \).
Let \( \mathcal{M} = \langle L_\beta [\mathcal{A}], \mathcal{A}, M, p, \delta \rangle \), where \( \beta \) is least s.t.
\[ \langle L_\beta [\mathcal{A}], \mathcal{A}, M, p, \delta \rangle \leq \langle L_{\omega_2} [\mathcal{A}], \mathcal{A}, M, p, \delta \rangle. \]
Then \( \mathcal{M} = \bar{\mathcal{M}}_{\beta} \), \( \beta = \beta \).
Let \( X \subset \mathcal{M} \) be countable s.t.
\( \delta, \delta' \in X \) and \( \delta = \delta' \cap \omega_1 \) is transitive
and \( \bar{\mathcal{M}} \in M_{\mathcal{M}} \), where \( \bar{\mathcal{M}} \in \text{the transitive closure of } \mathcal{M} \mid X \).
Let \( \bar{\mathcal{M}} \leq \mathcal{M} \mid X \).
Let \( \mathcal{M} = \langle L_\beta [\mathcal{A}], \mathcal{A}, \bar{M}, \bar{p} \rangle \). Define \( \bar{p}' \) by:
\[ \text{dom} (p') = \omega_2 \cap X, \]
\[ p'(\nu) = \begin{cases} p(\nu) \cup \{ 3 \} & \text{if } \nu \in \text{dom} (p) \\ 3 & \text{otherwise} \end{cases} \]
Claim \( p' \in \Pi^A \) (hence \( p' \leq p \)).
(a) \( p'(\nu) \in \Pi^A \) (trivial)
(b) Let \( \nu \in \text{dom}(p') \). Then \( f_\nu \in X \).
Let \( \bar{f} = f_\nu \).
Then
\[ \bar{f} : 2 \leftrightarrow \bar{p}, \text{ where } \sigma (\bar{f}) = \bar{p}, \]

But $\overline{\alpha}_v \in X$. Let $\sigma(\overline{\alpha}) = \alpha_v$. Then
\[
\overline{\alpha} = \{ \langle \mu, z \rangle \in d | \mu \in f_\sigma(\overline{\alpha}) \} \overline{\alpha} = \{ \langle \mu, z \rangle \in d | \mu \in f_\sigma(\overline{\alpha}) \} \overline{\alpha} = \overline{\alpha}_v \land d. \quad \text{Hence}
\]
\[
\{ \langle \mu, z \rangle \in d | \mu \in f_\sigma(\overline{\alpha}) \} \overline{\alpha} \land d \in M_d.
\]
(d) $\overline{C}_{p', \mu} \in M_d$, when
\[
\overline{C}_{p', \mu} = \{ \langle \mu, z \rangle \in d | \nu \in p'(f_\sigma(\overline{\alpha})) \} \overline{C}_{p', \mu} = \{ \langle \mu, z \rangle \in d | f_\sigma(\overline{\alpha}) \in \text{dom}(p') \land \nu \in p'(f_\sigma(\overline{\alpha})) \} \overline{C}_{p', \mu} = \{ \langle \mu, z \rangle \in d | f_\sigma(\overline{\alpha}) \in \text{dom}(p') \land \nu \in \overline{p'}(f_\sigma(\overline{\alpha})) \} \overline{C}_{p', \mu} \in M_d.
\]
Let $\eta \in p'(\nu) \land d$, then
\[
p'(\nu) \land d = p'(\nu) + \text{hence}
\]
\[
\overline{C}_{p', \mu} \land \eta = \overline{C}_{p', \mu} \land \eta \in M_d.
\]
QED (Lemma 3)

We now verify (C) of §1!
Lemma 4 IP is $\omega_1$-distributive.

Proof.
Let $p \in \text{IP}$. Let $\mathcal{A}_i$ be strongly dense in IP for $i < \omega$. Claim $\forall p' \leq p$ $p' \subseteq \bigcap \mathcal{A}_i$.

Fix $\langle x_i, m_i \rangle$ | $i < \omega_1$. Let $x_i < j$ for $0 < i < \omega_1$.

(a) $m_i < \omega$; (b) $x_i < j$ for $0 < i < \omega_1$.

(c) $\forall i < \omega_1 \exists \langle y, n \rangle \in \omega_1 \times \omega_1$. Then

$\exists i < \omega_1 \exists \langle x_i, m_i \rangle = \langle y, n \rangle$ is unbounded in $\omega_1$.

Pick $\langle p_0, i_0 < \omega_1 \rangle$ s.t. $p_0 = p$ and $p_0 \leq p_i$, $p_0 \in \Delta m_i$. $m_i \geq i$ for $0 < i < \omega_1$.

Let

$\mathcal{M} = \langle L_{\beta}[A], A, \langle p_0, i_0 < \omega_1 \rangle, M >$

$< \langle L_{\omega_2}[A], A, \ldots >$

s.t. $\beta < \omega_2$, $cf(\beta) = \omega_1$.

Let $X \subseteq \mathcal{M}$ be countable s.t.,

$\Delta = \omega_1 \cap X$ is transitive and

$\overline{\Delta} \subseteq M_{\Delta}$, where $\overline{\Delta}$ is the

transcendence of $X$. Then

$\overline{\Delta} = \langle L_{\beta}[\overline{A}], \overline{A}, \langle p_0, i_0 < \omega \rangle, M_{\Delta} \rangle$.

Let $\Delta = \sup \Delta_i$, where $\Delta_i$ is

$\subseteq M_{\Delta}$ is monotone. An

$M_{\Delta}$ select $\langle p_i, i < \omega \rangle$.  


\[ a - \]

\[ \forall i. \: x^i < y^i, \: y^i = y^i, \: \sigma^i = i. \]

Define \( p' \) by:
\[
\text{dom}(p') = \bigcup_{i < \omega} \text{dom}(p^i) \]
\[
p'(v) = \bigcup_{i < \omega} \bigcup_{v \in \text{dom}(p^i)} \text{dom}(p^i) \quad \text{for } v \in \text{dom}(p'),
\]

At sufficient to show:

Claim \( p' \in P \) (Hence \( p' \leq p \in A \) for \( i < \omega \).)

\[ \text{Proof:} \]
\[ \text{dom}(p') \cap \omega_1 \text{ is trivially countable.} \]

Moreover:

(a) \( p'(v) \) is closed, bounded in \( \omega_1 \) for \( v \in \text{dom}(p') \), since \( p'(v) \cap \omega \) is countable in \( \omega = \max(p'(v)) \).

(b) \( \tilde{a}_v \land \forall \gamma \in M_p \) for \( \forall \gamma \in p'(v) \)

\[ \text{Proof:} \]

Case 1 \( v < \omega \). Then \( \forall \gamma \in \text{dom}(p'(v)) \) and hence \( \tilde{a}_v \land \gamma \in M_p \).

Case 2 \( v = \omega \).

Let \( v \in \text{dom}(p') \). Then \( v \in X \) since \( P \subseteq X \). Hence \( \tilde{a}_v \in X \). Let \( \sigma(v) = \tilde{a}_v \). Then \( \bar{a} = \tilde{a}_v \land \bar{a} \in M_p \).

QED(b)
(c) \( f'_{\nu} \uparrow m_{p'(\nu)} = f''_{\alpha} \cap \text{dom}(p') \)
for \( \nu \in \text{dom}(p') \).

Let \( \zeta = f'_{\nu} \uparrow m_{\nu} \), \( \mu < \alpha \), let \( \nu \in \text{dom}(p') \),
where \( \gamma_i > \mu \) (hence \( m_{p'} \geq \gamma_i > \mu \)).
Then \( m_{p'(\nu)} > \mu \) and \( \nu \in \text{dom}(p) \cap \text{dom}(p') \).

(d) \( m_{p'(\nu)} = m_{p(\nu)} = \alpha \) for \( \nu \in f''_{\alpha} \).

(e) \( \gamma \in p'(\nu) \rightarrow \tilde{c}_{p',\nu} \cap \gamma \in M_{p'} \).

Case 1 \( \gamma < \alpha \).

Trivial since \( \gamma \in p'(\nu) \) for an \( i < \omega \) and \( \gamma : \nu \cap \gamma = \tilde{c}_{p',\nu} \cap \gamma \).

Case 2 \( \gamma = \alpha \).

\( \tilde{c}_{p',\nu} = \bigcup_{i < \omega} \tilde{c}_{p',\nu} \).

But \( \nu \in \text{dom}(p') \).

The function \( \langle \tilde{c}_{p',\nu} \mid \nu \in \omega \cap \text{dom}(p') \rangle \)
is \( \mathcal{L} \)-definable. Let \( \tilde{\mathcal{L}} \).

\( \langle \tilde{c}_{p',\nu} \mid \nu \in \omega \cap \text{dom}(p') \rangle \) have
the same def. in \( \tilde{\mathcal{L}} \).
Then \( \bar{\varepsilon}_{\bar{p}_3, \nu} = \bar{\varepsilon}_{\bar{p}_3, \sigma(\nu)} \); hence
\[
\bar{\varepsilon}_{\bar{p}_3, \nu} = \bar{\varepsilon}_{\bar{p}_3, \sigma(\nu)} \land d. \text{ Since } \langle \bar{p}_i, 1; < \omega \rangle \leq M \rangle,
\]
we have: \( \bar{\varepsilon}_{\bar{p}, \nu} = \bigcup \bar{\varepsilon}_{\bar{p}_i, \nu} \), where \( \sigma(\nu) = \nu \).

\[\text{QED (Lemma 4.1)}\]

**Note:** A modification of this proof shows:

**Cor. 4.1** Let \( \nu < \omega_1 \). Set \( Q = \) the set of \( \langle p, q \rangle \in IP \times IP \) s.t. \( p \upharpoonright \nu = q \upharpoonright \nu \), with the ordering: \( \langle p, q \rangle \leq \langle p', q' \rangle \iff (p \leq p', q \upharpoonright \nu \leq q' \upharpoonright \nu) \).

Then \( Q \) in \( \omega_1 \) - distributive.

(Note \( Q = IP \times IP \) for \( \nu = 0 \))

As in § 1, a standard proof using \( (D), (E) \) gives:

**Lemma 5.2** \( IP \) satisfies the \( \omega_2 - \text{CC} \) (i.e. every antichain has cardinality \( \leq \omega_1 \)).

Hence \( IP \) preserves cardinals.
Lemma 6. Let $G$ be $IP_r$-generic over $N$. Set $C = U G$. Then
(a) $C \in \text{cub} \in \omega_1$ and $C \cap d \in M_d$ for all $d \subseteq C$.
(b) Let $a \in \omega_1$, $a \in L_p[A]$. Then there is $a_0 \subseteq C$ not and $a_0 \in M_d$ for all $d \subseteq C \setminus a_0$.

Proof. Trivial.

Definition. Let $G$ be $IP$-generic over $N$.
$G_{\nu} = \{ p(\nu) \mid p \in G \wedge \nu \in \text{dom}(p) \}$
$C_{\nu} = C_{\nu} = U G_{\nu}$
$\tilde{C}_{\nu} = C_{\nu} = \{ \mu, \nu  \mid \mu < \omega_1 \wedge \mu \in C_{\nu} \}$

Definition. $\tilde{IP}_r = \tilde{IP}_r = \{ p \in IP \mid \text{dom}(p) \subseteq \nu \}$
(Hence, $\tilde{IP}_r \subseteq IP_r$, by Lemma 2)

Lemma 7. Let $G$ be $IP_r$-generic over $N$.

(a) $G_{\nu}$ is $IP_r$-generic over $N$

(b) $d \subseteq C_{\nu} \rightarrow d \cap \tilde{C}_{\nu}, d \cap C_{\nu} \subseteq M_d$

(c) $N[G] = N[\langle C_{\nu} \mid \nu < \omega_1 \rangle]$

(d) $N[G \cap \tilde{IP}_r] = N[\tilde{C}_{\nu}]$

Proof

(a) Let $p \in IP_r$, $q \leq p(\nu) \in IP_r$.

Claim. There is $p' \leq p$ in $IP$ such that $p'(\nu) \leq q$ in $IP_r$.

(b) $d \subseteq C_{\nu} \rightarrow d \cap \tilde{C}_{\nu}, d \cap C_{\nu} \subseteq M_d$

(c) $N[G] = N[\langle C_{\nu} \mid \nu < \omega_1 \rangle]$

(d) $N[G \cap \tilde{IP}_r] = N[\tilde{C}_{\nu}]$
Amplifying the proof of Lemma 3 (applied to \( p' \)), we find an \( x > m_g \) and \( a \in \pi \not\ni p'' \) not \( p'' \leq prv \in \pi' \) and \( mp''(z) = a \) for all \( z \in \text{dom}(p'') \) and \( p''(z) \cap d = \{ p(z) \) if \( z \in \text{dom}(p) \}
\phi \) if not \( z \in \text{dom}(p'') \). Set \( p' = p'' \cup \{ \langle q, v \rangle \} \).

It is easily verified that \( p' \in \pi' \).

Hence \( p' \leq p \) and \( p'_v = q_v \), \( Q.E.D \) (a)

(b) trivial

(c) (\( \Rightarrow \)) trivial, (c) follows by i.

For \( p \in \pi \) we have:

\[ p \leq \pi \iff \forall v \in \text{dom}(p) \, (p(v) = c_v \cap mp(v)) \]

(d) \( N[\pi \cap \pi] = N[\pi] \)

follows as in (c), but

\[ N[\pi] = N[\pi] \), \( Q.E.D \) (Lemma7)

Using this, we verify (G) of \$1.'
Lemma 8. Assume $L_{\omega_2}[A] = H_{\omega_2}$ in $N$. Then $S^* = \langle S^*_3, 13 < \omega_1 \rangle$ in a $\square^+_L$ sequence in $N[G]$.

Proof. Let $B \subseteq \omega_1$ in $N[G]$. Claim: There is a club $C \subseteq \omega_1$ in $N[G]$ s.t.

\[ \forall \alpha \in C \quad B \cap \alpha \in M_\alpha, \quad C \cap \delta \in M_\delta, \]

Let $B = \vec{B}$. For each $\nu \in \omega_1$ choose in $N$ a maximal antichain $X_\nu$ in \{ $p \mid p \Vdash \nu \in \vec{B}$ \},

Then $\vec{X}_\nu \in \omega_1$, where $X_\nu \in H_{\omega_2}$. Hence $\langle X_\nu, \nu < \omega_1 \rangle \in L_{\omega_2}[A]$. We know:

(1) $\nu \in B \quad \Rightarrow \quad \forall \alpha X_\alpha \neq \emptyset$

Pick $\beta < \omega_2$ s.t.

\[ \mathcal{M} = \langle \mathcal{L}_\beta[A], A \cap \beta, M, X \rangle \in \langle L_{\omega_2}, X \rangle, \]

where $X = \langle X_\nu, \nu < \omega_1 \rangle$, and $\text{cf}(\beta) = \omega_1$.

Clearly $\beta = \sup \beta$, $\mathcal{M}^* = \langle \vec{M}_\beta, X \rangle$.

Since $\text{cf}(\beta) = \omega_1$, we have

(2) $[\mathcal{M}]^{\omega} \subseteq \mathcal{M}$; $\forall \tau \in \mathcal{M} \quad [x]^{\omega} \subseteq \mathcal{M}$

Hence:

(3) $\vec{B}_\beta \subseteq \mathcal{M}$ in $\mathcal{M}$ - definable.

Clearly (1) can be improved to

(4) $\nu \in B \quad \Rightarrow \quad (G \upharpoonright \vec{B}) \cap X_\nu \neq \emptyset$.

Note that $\rho > \beta$ and $\vec{f}_\beta \in L_{\rho}[A]$, where $\vec{f}_\beta : \omega_1 \rightarrow \beta$. 
Let \( D \subseteq \omega_1 \) code the complete theory of \( \langle \omega_1, f, \nu (r < \omega_1) \rangle \). Then \( D \) is uniquely recoverable from \( D \) in any transitive \( \mathcal{ZFC} \)-model containing \( D \) as a set.

Then \( D \in L_\beta [A] \). Hence there is \( d_0 \in C_\beta \) s.t. \( D \cap \Delta \subseteq GM_d \) for all \( d \in C_\beta \setminus d_0 \).

Set: \( \Omega^* = \langle \Omega \cap D, G \cap M_\beta \rangle \). Set: \( \Omega_d = \{ Y < \Omega^* \mid d \subseteq Y \} \) (\( d \subseteq \omega_1 \)).

Then \( \Omega^* = \Omega \cap \omega_1 \) and \( \Omega_d \) is countable for \( d < \omega_1 \). Set:

\[ C = \{ d \in C_\beta \mid d_0 \subseteq d = \omega_1 \cap \Omega_d \} \]

Then \( C \) is cub in \( \omega_1 \).

Claim: \( \exists d \in C. \) Then \( \forall d, C \cap d \subseteq GM_d \).

Proof:

Let \( \sigma : \Omega^* \rightarrow \Omega_d \), where \( \Omega^* = \langle \Omega, G, \bar{D} \rangle \) and \( \Omega = \langle L_\beta [A], \bar{A}, \bar{M}, \bar{X} \rangle \). Then:

\[ d = \sup \{ \sigma ( \delta ) \mid \delta \subseteq \omega_1 \}, \sigma ( \bar{A} ) = A \cap \beta, \]
\[ \sigma ( \bar{M} ) = \bar{M} \) (hence \( \bar{M} = M \cap A \)), \)
\[ \sigma ( \bar{X} ) = X \) (hence \( \bar{X} = \langle \bar{X}_r \mid r < d \rangle \), where \( \bar{X}_r \) is a maximal antichain in \( \bar{P} \), and \( \bar{P} \) is defined in \( \Omega \) as \( \bar{P}_\beta \) in \( \Omega \).
Finally, we note that \( D = D \cap d \in M_d \)
and \( \sigma(\bar{D}) = D \) and \( d \in C \beta \setminus d_0 \). Hence
\( (5) \quad \bar{\sigma} \in M \).

Since \( \bar{\sigma} \) is recoverable from \( D \) as \( \bar{\sigma} \) was recoverable from \( D \).
\( (6) \quad \bar{G} \in M_d \)

\( \mu \).

By the proof of Lemma 7 (d), \( G \cap \bar{P}_\beta \quad \langle \bar{\sigma}, \bar{C} \rangle \) - definable, where \( \bar{C} \) in turn, \( \bar{\sigma} \) - definable. Let \( \bar{C} \) have the same definition in \( \bar{\sigma} \). Then
\( \bar{C} = \bar{\sigma}^{-1} \bar{\sigma} \bar{C} = \bar{C} \cap d \in M_d \), since \( d \in C \beta \). But then \( \bar{G} \cap \langle \bar{\sigma}, \bar{C} \rangle \) - definable at \( G \cap \bar{P}_\beta \) was defined in \( \langle \bar{\sigma}, \bar{C} \beta \rangle \), since \( \sigma(\bar{G}) = G \cap \bar{P}_\beta \) and
\( \sigma : \langle \bar{\sigma}, \bar{C}, \bar{G} \rangle \rightarrow \langle \bar{\sigma}, \bar{C} \beta, G \cap \bar{P}_\beta \rangle \).

Hence \( G \in M_d \), since \( \langle \bar{\sigma}, \bar{C} \rangle \in M_d \).
QED (6)

By the proof of Lemma 7 (c):
\( \nu \in B \leftrightarrow (G \cap \bar{P}_\beta) \cap X_\nu \neq \emptyset \)
\( \leftrightarrow \bar{G} \cap X_\nu \neq \emptyset \) for \( \nu < \beta \).

Hence:
\( (7) \quad B \cap d \in M_d \), \( \nu \in \bar{G}, \langle X_\nu \mid \nu < \beta \rangle \in M_d \).
Finally we note that:

(17) $C \cap x \subseteq M_x$,

since $\bar{M}^* \subseteq M_x$ by (16), $C \cap x \subseteq M_x$, and $C \cap x$ is definable from $\bar{M}^*$, $C \cap x \subseteq x_0$ as $C$ was defined from $\bar{M}^*$, $C \cap x \subseteq x_0$. Q.E.D. (Lemma 8)

It remains only to verify (H), which will follow by Corollary 4.1:

**Lemma 9** Let $\widetilde{G}$ be $\bar{P}_0$-generic over $N$. Set $\mathbf{IP}_0^G = \{ p \in \mathbf{IP} \mid p \text{ non-generic} \}$. Then $\mathbf{IP}_0^G \times \mathbf{IP}_0^G$ is $\omega_1$-distributive.

**Proof.** Let $G_0 \times G_1$ be $\mathbf{IP}_0^G \times \mathbf{IP}_0^G$-generic over $N[\widetilde{G}]$. We must show that $N[\widetilde{G}] [G_0 \times G_1] = N[\widetilde{G}]$ contains no new countable subsets of $N[\widetilde{G}]$. Let $Q = \{ \langle p, q \rangle \in \mathbf{IP}_0 \times \mathbf{IP}_0^G \mid p \text{ non-generic} \}$ be as in Lemma 4.1. Set $Q^G = Q \cap (\mathbf{IP}_0^G \times \mathbf{IP}_0^G)$. It is easily seen that $Q^G$ is dense in $\mathbf{IP}_0^G \times \mathbf{IP}_0^G$. Hence $N[G_0 \times G_1] = N[\widetilde{G}]$, where $\widetilde{G} = G_0 \times G_1 \cap Q$, and $\widetilde{G}$ is $Q^G$-generic over $N[\widetilde{G}]$. 


At is apparent from the def. of $Q$ that:

(E') Let $<p, p', r> \in Q$. Let $q \leq p$ in $\bar{p}_r$. Then $<p, q, p', r, q> \in Q$.

Using this we can repeat the proof of (2) (following (G1)) in $\bar{Q}$ to get:

$Q \cap (G_0 \times G_1) \cap Q_+ - generic$ over $N$ if

$\bar{G} \models G_0 \cap \bar{p}_r \cap \bar{p}_1 - generic$ over $N$

and $Q \cap (G_0 \times G_1) \cup \bar{Q}_+ - generic$ over $N[\bar{G}]$. Hence $\bar{G} = Q \cap (G_0 \times G_1) \cup \bar{Q}_+ - generic$ over $N$, where $Q \cup \omega_1$ - distributive in $N$. Hence $N[G_0 \times G_1] = N[\bar{G}]$ contains no more countable sets of ordinals.

QED