§1 Preliminaries

We find it most convenient to treat general iteration theory in terms of complete Boolean algebras. We write $A \leq B$ to mean that $A \subseteq B$ are Boolean algebras, $A$ is complete, and $A$ is completely contained in $B$ (i.e., $\text{id}_A$ is a complete homomorphism). We write "IBA" as an abbreviation for "Boolean algebra". If $A \leq B$ and $b \in B$ we define the function $h = h_{IB}$ by:

$$h(b) = \bigcap \{a \in A \mid bca^3\}.$$  

Then $h(\bigcup b) = \bigcup h(b)$, $h(a \land b) = a \land h(b)$ if $a \in A$, and $b = 0 \iff h(b) = 0$.

(We denote the maximal and minimal elements of a BA by $1, 0$.)

For each complete BA we can form the canonical maximal IB-valued model $\mathcal{V}^IB$. The elements of $\mathcal{V}^IB$ are called names and there is a valuation function $\varphi \rightarrow [\varphi]_{IB}$ attaching to each statement $\varphi = \varphi' \in IB$ a truth value $[\varphi']_{IB} \in IB$. (Here $\varphi'$ is a ZFC formula and names are names.) If $A : \mathcal{V}^IB \rightarrow IB$, we may also add a predicate $A^*$. 


The language, setting:
\[ [t \in A] = \bigcup_{x \in V^B} [x = t] \wedge A(t) \]

We can enrich our language further by closing the \( \Sigma_0 \) formulas under infinite disjunction \( \bigvee_{i \in I} \varphi_i \) and conjunction \( \bigwedge_{i \in I} \varphi_i \) where \( I \) is any set. We shall generally take \( V^B \) as an identity model in the sense:
\[ [x = t] = 1 \iff x = t. \]

Thus e.g., \( \{ x \mid [x \in y] = 1 \} \) is a set, not a proper class. We also write:
\[ \vdash_0 \varphi \iff (a \in B \setminus \{0 \} \wedge a \in [\varphi]) , \]
\[ \vdash \varphi \iff \vdash_0 \varphi \because 1 \vdash \varphi. \]

\( \forall \alpha \in \mathbb{A} \subseteq V^B, \) there is a
\( \sigma \colon V^\mathbb{A} \to V^B \) s.t.
\[ [\varphi(t')] = [\varphi(\sigma(t'))] \]

where \( \mathbb{A} \) is a \( \Sigma_0 \) formula, \( t_1, \ldots, t_n \in V^B \)
and \( [x \in \sigma(t)] = \bigcap_{t' \in V^\mathbb{A}} [x = t'] \wedge [x' \in t]. \)

For \( x \in V^B, t \in V^\mathbb{A}, \)

\[ \therefore \text{we have defined } V^B \text{ without this property we can achieve it by a simple factorization.} \]
Since it is unimportant what the actual elements of $\langle V^A, V^B \rangle$ are, we can easily arrange that $\sigma = \text{id}$, when dealing with specific $\langle A, B \rangle$. We shall also follow this convention when dealing with a chain $\langle B_i \mid i < \alpha \rangle$ s.t. $B_{i+1} \subseteq B_i$, for $i \leq i' < \alpha$.

The simplicial $BA$ is $2 = \{0, 1\}$, $V^2$ is isomorphic to $V$ in an obvious sense.

Let $\kappa \mapsto \tilde{\kappa}$ be the isomorphism of $V$ onto $V^2$.

Following the convention that the above map $\sigma : V^2 \rightarrow V^B$ is the identity, we have $\{t \in \tilde{x} \mid t = \tilde{x}\} \in V^B$.

$\mathcal{W}$ is an inner model of $\text{ZFC}$ and $\mathcal{W} \subseteq W$ is complete in the sense of $W$.

We can of course form $W^B = (V^B)^W$ by relativizing the definition of $V^B$ to $W$. (However, we may not be able to define $W^B$ internally as an identity model if $W$ does not model full $\text{ZFC}$ or have a definable well ordering.) If $\mathcal{W} \subseteq W$ is complete in $W$, we say that $G \subseteq B$ is a $B$-generic set over $W$ iff $G$ is an ultrafilter on $B$ which is complete in $W$ in the sense;
\[ \bigcup b_i \in G \iff \forall i \in X \ b_i \in G \]

for \( X \in IB \), \( X \in W \). (Equivalently, let \( G \subseteq IB \setminus \emptyset \) un\( \exists \Delta \in W \) whenever \( \Delta \in W \).

For each \( b \in IB \setminus \emptyset \), \( \vdash IB \) is complete in \( V \).

It is often useful in analyzing \( VIB \) to work in a mythical larger universe in which:

(k) \( V \) is an inner model and \( \emptyset (IB) \ni V = \omega \).

This is harmless, since if \( G \) collapse \( 2^{\omega^V} \) to \( \omega \), then \((k)(V, IB)\) holds in \( V^G \).

Let \( \forall \mathcal{G} \in VIB \) \( \exists \mathcal{G} \subseteq IB \) and \( \| b \in \mathcal{G} \| = b \) for \( b \in IB \).

Thus, if \( \mathcal{G} \) is generic over \( V \), then \( \mathcal{G} = \mathcal{G}^V \).

We call \( \mathcal{G} \) the \underline{canonical IB-generic name}.

Now let \( \mathcal{A} \subseteq IB \). \( \vdash \mathcal{G} \in \mathcal{A} \)-generic over \( V \), then \( \mathcal{G}' = \{ b \in IB \mid \forall a \in \mathcal{G} \ a \subseteq b \} \) is a complete filter on \( IB \) and we can form the factor algebra \( IB / \mathcal{G}' \) (which we normally denote by \( IB / \mathcal{G} \)), \( \vdash IB \) is complete in \( V \), then \( IB / \mathcal{G} \) is complete in \( V [\mathcal{G}] \). There is a canonical homomorphism \( \sigma : IB \to IB / \mathcal{G} \) such that:

\[ \sigma (b) \subseteq (c) \iff \exists b \in \mathcal{G} \]

When the context permits, we write \( b / \mathcal{G} \) for \( \sigma (b) \).
Now let $1B_0 \leq 1B_1$ and let $G_0$ be $1B_0$-generic over $V$. Let $\tilde{G}$ be $1B_1/G_0$ - generic over $V[G_0]$. Set $G_1 = G_0 \times \tilde{G} = \{ b \in 1B_1 \mid b G_0 \in \tilde{G} \}$. Then $G_1$ is $1B_1$ - generic over $V$ and $V[G_1] = V[G_0][\tilde{G}]$.

Conversely, if $G_1$ is $1B_1$ - generic over $V$, and we set:

$$G_0 = 1B_0 \cap G_1, \quad \tilde{G} = \{ b \in G_1 \mid b \in 1B_0 \},$$

then $G_0$ is $1B_0$ - generic over $V$, $\tilde{G}$ is $1B_1/G_0$ - generic over $V[G_0]$ and $G_1 = G_0 \times \tilde{G}$.
Omitting most proofs, we mention some basic facts:

**Fact 1** Let $A \subseteq B$ and let $h = h_{A/B}$ be the function defined above. Then

$$h(b) = \left\{ \begin{array}{ll}
0 & \text{if } b \in G
\end{array} \right\}/A$$

where $G$ is the $\mathcal{M}$-generic name.

**Fact 2** $A \subseteq B \subseteq B'$, $G$ $\mathcal{M}$-generic, and $\sigma_h : B_h \rightarrow B_h / G$ is the canonical homomorphism ($h = 0, 1$), then there is a complete injective homomorphism $\delta : B_h / G \rightarrow B / G$ defined by

$$\delta(\sigma_h(b)) = \sigma_A(b)$$

**Note** Then $\delta : B_h / G \rightarrow B_h / G$. In such a context we shall often identify $B_h / G$ with $B_h / G$ to have $B_h / G \subseteq B / G$.

**Fact 3** $A \subseteq B$ and $\vdash b \in B / G$, then there is a unique $b \in B$ s.t.

$$\vdash b = b / G$$

**Proof** To see uniqueness, let $\vdash b / G = b' / G$. Then $\vdash b', b / G = 0$. Hence $h(b', b) = h(b / G, b') = 0$. Hence $b = b'$. Similarly $b' \in b$. 

Hence $b \leq b'$. Similarly $b' \leq b$. 


To see existence, note that $\Delta = \{a \in A \mid \forall b \in \Delta \exists c \in c_0 \text{ such that } b \not\leq c\}$. For any $\bar{X}$ be a maximal antichain in $\Delta$.

Let $a \in \Delta$ be a maximal antichain in $\Delta$. Set $b = \bigcup_{a \in X} a \cap b$. Then $\forall b \in \Delta \exists c \in c_0 \text{ such that } b \not\leq c$. Hence $b \not\leq c \iff b \not\leq c / c$, by genericity. Hence $b \not\leq c \iff ba / c = b / c$.

**QED (Fact 3)**

**Fact 4** Let $A \models IB$, then

$$\forall a \in A \forall b \in B \ (b \not\leq c / c) \implies a \cap b \cap c,$$

**Fact 5** Let $A \models IBCC$. Then $G$ be $A$-generic, let $h = h_{IB} \ (\text{in } V)$ and $h = h_{IB / \sigma} \ (\text{in } V[G])$. Then $h(c) / \sigma = h(c / \sigma)$ for all $c \in C$.

**Fact 6** Let $A \models IB \in V/A$ not.

Then $\exists a \in A$ such that $\forall b \in B \not\leq a$.

There is a complete $IB \geq A$ not, $\forall b / \sigma = IB$.

**Proof (sketch)** Form the complete $BA/A+IB$

consisting of the points $\exists b \in A \models b \in IB$ and with the operations.
\( b = \text{that } b' \text{ s.t. } \vdash b = b' \)

\( \cup A = \text{that } b' \text{ s.t. } \vdash b' = \bigcup A \in x \in \Sigma \) \ni \tilde{b} \)

\( \bigwedge A = \text{'' } (\vdash b' = \bigwedge A) \)

\( A \times \tilde{b} \) is then a complete BA and there is a complete homomorphic injection \( \sigma : IA \to A \times \tilde{b} \) defined by:

\[ \sigma(a) = \text{that } a' \text{ s.t. } \vdash a' = a \wedge a' \in \tilde{a} \text{ and } a' \in \tilde{a} \]

It hence we can find \( \tilde{b}, \tilde{b}' : I A \times \tilde{b} \mapsto IA \text{ s.t. } \sigma \circ \tilde{b} \circ \tilde{b}' = \text{id} \), QED (Fact 6)

**Note** In the following we shall often deal with sequences \( \tilde{b} = \langle \tilde{b}_i | i < \alpha \rangle \) s.t. \( \tilde{b}_i \subseteq \tilde{b}_j \) for \( i \leq j < \alpha \). We then set \( h_i = h_{\tilde{b}_i} \) and 

where \( \tilde{b} = \bigcup_{i < \alpha} \tilde{b}_i \). We let \( G_\tilde{b} \) be the \( \tilde{b} \)-generic name and write \( h_{\tilde{b}_i} \) for \( h_{\tilde{b}_i} \) and \( [G]_i \) for \( [G]_{\tilde{b}_i} \). We assume \( V | \tilde{b}_i \)

is defined that \( \text{id} \in V | \tilde{b}_i \) is the canonical map \( \sigma : V | \tilde{b}_i \to V | \tilde{b}_i \) mentioned above for \( i \leq j < \alpha \), \( \forall G_\tilde{b} \subseteq \tilde{b} \)-generic and \( \sigma_i : | \tilde{b}_i / G_\tilde{b} \to \tilde{b} / G_\tilde{b} \) is the injective homomorphism of Fact 2, we shall, whenever the context permits, identify \( \tilde{b}_i / G_\tilde{b} \) with \( \text{ring} (\sigma_i) \), so that \( \sigma_i = \text{id} \).
Aterations

**Def** Let $B_i = \{ \mathbf{b} \mid i < \alpha \}$ be an iteration of

- $B_i \subseteq B_j$ for $i \leq j < \alpha$ and $B_0 = \{ 0, 1 \}$
- $B_i$ is generated by $\cup B_i$ for limit $\lambda < \alpha$

We adopt the notation introduced above in dealing with iterations.

By a $\mathbf{B}$-sequence, we mean a $b = (b_i \mid i \in \mathbb{N})$

$s.t., \forall i \in \mathbb{N}, b_i \in B_{i+1}$ for $i \in \mathbb{N}$.

We like to pretend that $b_i$ is defined for all $i < \alpha$ and therefore set:

$$b_i = 1 \text{ if } i < \alpha \text{ and } \delta (b).$$

The support of $b$ is defined by:

$$\text{supp} (b) = \{ \delta \mid b_i \neq 1 \}.$$  

**Def** $\forall \delta \in \text{supp} (b)$ bounded in $\lambda$, set:

$$b^* = \bigcap_{i < \alpha} b_i \quad \text{(in } B_i \text{ where } \text{supp} (b) \text{ is)}.$$  

**Def** $b$ is a good sequence for $B$

($GS(b, B)$) if:

- $b$ is a $\mathbf{B}$-sequence
- $h_i((b^{i+1})^*) = (b_i)^*$ for $i \leq j < \alpha$
- $h_i(b_i) = 1$ for $i+1 < \alpha$

Note: Then $\forall i, (b_i)^* \neq 0$ for all $i < \alpha$, since

$$h_0 ((b_0^i)^*) = \emptyset^* = 1.$$
Note: The condition $h_i(b_i) = 1$ could be weakened to $h_i(b_i) > (b \upharpoonright i)^*$. However, we lose nothing by adopting the stronger requirement, since if $b$ satisfies the weaker condition and we let $\hat{b}_i = b_i \cup \neg h_i(b_i)$, then $\hat{b}$ is a good sequence and $(\hat{b} \upharpoonright i)^* = (b \upharpoonright i)^*$ for all $i < \omega$.

Def: Let $1B = \langle 1B_i, i < \omega \rangle$ be an iteration and let $G_i$ be $1B_i$-generic. Set:

$$1B / G_i = \langle 1B_i + j / G_i \mid j < \omega - i \rangle.$$ 
Without proof we mention:

Lemma 1.1 $1B / G_i$ is an iteration in $V[G_i]$.

Def: Let $1B_i, G_i$ be as above and let $GS(b, 1B_i)$. Set:

$$b / G_i = \langle b_{i+j} / G_i \mid j+1 < \omega - i \rangle.$$ 

Lemma 1.2 $GS(b / G_i, 1B / G_i)$ in $V[G_i]$.

Lemma 1.3 Let $i < \omega$. Then

$$GS(b_i, 1B_i) \iff \langle GS(b_i \upharpoonright i, 1B_i) \land$$

$$\land (b_i \upharpoonright i)^* \vert_{i < \omega} GS(\hat{b}_i \upharpoonright i, \hat{b} / G_i) \rangle$$

(where $\hat{b}_i$ is the $1B_i$-generic name).
Def: \( b \) has countable support in \( IB \) \( (CS(b, IB)) \)
\[ \Rightarrow \text{GS} (b, IB) \land \supp(b) \leq \omega. \]

Def: \( IB \) is a countable support iteration
\[ \Rightarrow \text{for all limit } \lambda < \alpha \text{ we have:} \]
(a) \( \forall \lambda \in \text{CS}(b, IB \upharpoonright \lambda), \text{ then } b^* = 0 \text{ in } IB \lambda \)
(b) \( \exists b^* \mid \text{CS}(b, IB \upharpoonright \lambda) \) is dense in \( IB \lambda \setminus \{0 \} \).

Then:

Lemma 2.4 Let \( IB = \langle IB_i, i < \alpha \rangle \) be a countable support iteration. Let \( \lambda < \alpha, \text{ Lim}(\lambda), \)
\( \forall \lambda \in \text{CS}(b, IB \upharpoonright \lambda), \text{ then } h_i(b^*) = (b_i^*) \) for \( i < \lambda \).

However, the analogue of Lemma 1.3 can fail for the property \( CS(b, IB) \). This happens if cofinalities of ordinals are changed by the iteration. For this reason we shall employ revised countable support iteration.

However, we use Donder's simplified notion in place of Shelah's definition.

Def: Let \( \text{GS}(b, IB), \lambda \leq \alpha, \text{ Lim}(\lambda). \)
\[ U_\lambda (b) = U_\lambda (b, IB) = \{ a \in IB, \\{0 \} \mid a \subset (b^*_i) \} \]
\[ S_\lambda (b) = S_\lambda (b, IB) = \{ a \subset U_\lambda (b) \mid \text{either } \forall i < \lambda \text{ all } c_f(\lambda) = \omega, \text{ or else} \]
\[ a \subset (b^*_i) \} \text{ for all } i < \lambda. \]
Def. \( b \) has revised countable support in \( IB \) 
\( (R(b, IB)) \upharpoonright \mathcal{H} \) for all limit \( \lambda < \alpha, \)
\[ S_\lambda(b) \in \text{close in } U_\lambda(b) \]

Def. \( IB \) is a revised countable support iteration (RCS iteration) \( i \upharpoonright \mathcal{H} \) for all limit \( \lambda < \alpha, \)
(a) \( A \upharpoonright (R(b, IB)) \lambda \), then \( b^* \neq 0 \) in \( IB \lambda \)
(b) \( \exists b^* \mid R(b, IB) \lambda \downarrow \) is close in \( IB \lambda \setminus \{0\} \)

Lemma 2.2
(a) \( CS(b, IB) \rightarrow R(b, IB) \)
(b) \( R(b, IB) \leftrightarrow R(b, IB^{c+1}) \)
\[ \text{if supp}(b) \subset c < d \]
(c) \( R(b, IB) \rightarrow R(b^{c'}, IB) \rightarrow R(b, IB^{c'+1}) \)
\[ \text{for } c < \alpha, \]

We get the analogue of Lemma 1.3:

Lemma 2.3 Let \( i < \alpha, \) then
\[ R(b, IB) \leftrightarrow (R(b, IB^{c'}) \uparrow \uparrow (b^{c'})^* \upharpoonright IB^{c'/c'} R(b^{c'}, IB^{c'/c'})) \]
\[ \text{where } G_{c'} \text{ is the } IB_{c'} \text{-generic name.} \]

Finally:

Lemma 2.4 Let \( IB \) be an RCS iteration, let \( j < \alpha, \) then
(a) \( R(b, IB^{j'}) \leftrightarrow R(b, IB) \)
(b) \( R(b, IB^{j'}) \rightarrow h_c(b^*) = (b^{c'})^* \) for \( c < j' \)
(c) \( \exists b^* \mid R(b, IB^{j'}) \downarrow \) is close in \( IB_j \)
Lemma 2.5 - Let \( IB \) be an RCS iteration.

Let \( \lambda < \alpha \) be a limit point,

\( R(b, IB \lambda), R(b', IB \lambda) \), then

\[ b^* \subseteq b'^* \iff \forall i < \lambda (b^*_i)^* \subseteq (b'^*_i)^*. \]

Note that \( \lambda < \alpha \) is a limit. Let \( R(b, IB \lambda) \).

Clearly there \( \exists c \subset b \) s.t. either

\[ (c^*_i)^* \nmid c^*_i = \omega \quad \text{for an } i < \lambda \]

or else \( \text{supp}(c) \subset c^*_i \quad \text{for an } i < \lambda \)

(hence \( (c^*_i)^* = c^*_i \)).

Hence the set of such \( c^*_i \) is dense in \( IB_\lambda \).

Donders original definition of "RCS-iteration" was that for each limit \( \lambda < \alpha \), the set of such \( c^*_i \) (with \( GS(c, IB \lambda) \)) be dense in \( IB \lambda \setminus \emptyset \). This is easily seen to be equivalent to the present definition.