§ 2 Iterating Subproper and Subcomplete Forcing

We essentially repeat our earlier definition.

<u>Def</u> A complete BA $\mathbb{B}$ is <u>subproper</u> iff for sufficiently large cardinals θ: Let $\mathbb{B} \in H_\theta$ and $N = \langle L_\tau[A], A, \in \rangle$ where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ be countable and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(1 \leq i \leq m < \omega)$ where $\lambda_i \in (\omega_1, \theta)$ is regular and $\overline{\overline{\mathbb{B}}} < \lambda_i$ for $i = 1, \ldots, m$. Set: $\bar{\lambda}_0 = On \cap \bar{N}$. For any $a \in \mathbb{B} \setminus \{0\}$ there is $b \subset \sigma(a)$ which forces that if $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, m)$

(c) $\sup \sigma_0 `` \bar{\lambda}_i = \sup \sigma `` \bar{\lambda}_i$ $(i = 0, \ldots, m)$

(d) $\bar{G} = \sigma_0^{-1} `` G$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$.

It is trivial to see that a subproper forcing cannot collapse ω_1. Clearly every proper forcing is subproper, since properness is just the case $\sigma_0 = \sigma$.

<u>Note</u> σ_0 extends uniquely in $V[G]$ to a map $\sigma_0^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma_0^*(\bar{G}) = G$. But $\bar{N}[\bar{G}]$ is then full in $V[G]$. Thus the stage is set to handle a subproper $\tilde{\mathbb{B}} \in V[G]$.

Def We say that μ verifies the subproperness of $\mathbb{B}$ if the above holds for all cardinals $\theta \geq \mu$.

Def Let $N = L_\tau[A]$, $\sigma : \bar{N} \prec N$ be as above. We then say that σ witnesses the subproperness of $\mathbb{B}$ w.r.t. N. We also say that $X = \mathrm{rng}(\sigma)$ witnesses the subproperness of $\mathbb{B}$.

An apparent weakening of the notion of subproperness is:

<u>Def</u> $\mathbb{B}$ is <u>weakly subproper</u> iff there is z s.t. for sufficiently large cardinals θ, if $N = L_\tau[A]$ is as above and $X \prec N$ is countable and full s.t. $z, \theta, \mathbb{B} \in N$, then X witnesses the subproperness of $\mathbb{B}$.

However:

<u>Lemma 1</u> If $\mathbb{B}$ is weakly subproper, then it is subproper.

prf.

Let θ_0 be least s.t. for some $z \in H_{\theta_0}$ if $\theta \geq \theta_0$ is a cardinal and N is as above and $X \prec N$ is countable and full s.t. $\theta, \mathbb{B}, z \in X$, then X witnesses subproperness.

If $\mu > \theta_0$ s.t. $\mu = \overline{\overline{V_\mu}}$, let θ_0^μ be defined as above in V_μ rather than V. Clearly $\theta_0^\mu \leq \theta_0^{\mu'} \leq \theta_0$ for $\mu \leq \mu'$. But then there is μ_0 s.t. $\theta_0^{\mu_0} = \theta_0$. If $\mu \geq \mu_0$,

let A_μ = the set of $z \in H_{\theta_0}$ s.t. if $\theta_0 \leq \theta < \mu$, θ is a cardinal, N is as above, $X \prec N$ is countable and full s.t. $\theta, \mathbb{B}, z \in X$, then X witnesses subproperness. Then $\mu_0 \leq \mu \leq \mu' \rightarrow$
$\rightarrow A_\mu \supset A_{\mu'} \subset H_{\theta_0}$. Hence there is μ_1 s.t. $A_\mu = A_{\mu_1}$ for $\mu_1 \leq \mu$. But then $A_{\mu_1} = A_\infty$, where A_∞ is defined as above with ∞ in place of μ.
Now let $\theta > \mu_1$ be a cardinal & let $N = L_\tau[A]$ be as above. Then $\langle \theta_0^{\mu} \mid \mu = \bar{\bar{V}}_\mu < \theta \rangle$ is N-definable in $\theta, \mathbb{B}$. Hence μ_0 is N-definable in $\theta, \mathbb{B}$. Hence $\langle A_\mu \mid \mu_0 \leq \mu \wedge V_\mu \in H_\theta \wedge \bar{\bar{V}}_\mu = \mu \rangle$ is N-definable in $\theta, \mathbb{B}$. Hence $\mu_1, A_{\mu_1} = A_\infty$ are N-definable in $\theta, \mathbb{B}$. Now let $X \prec N$ be full and countable s.t. $\theta, \mathbb{B} \in X$. Then $A_\infty \cap X \neq \emptyset$. Hence X witnesses subproperness. QED

<u>Lemma 2</u> Let $\mathbb{B}_0 \subseteq \mathbb{B}_1$ where $\mathbb{B}_0$ is subproper and $\Vdash_0 \check{\mathbb{B}}_1 / \mathring{G}_0$ is subproper. Then $\mathbb{B}_1$ is subproper.

prf.

Choose θ large enough that $\mathbb{B}_1 \in H_\theta$, θ verifies the subproperness of $\mathbb{B}_0$ and the subproperness of $\check{\mathbb{B}}_1 / \mathring{G}_0$.

$\Vdash_0 \check{\theta}$ verifies the subproperness of

Let $N = L_\tau[A]$ where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $X \prec N$ be full and countable s.t. $\theta, \mathbb{B}_0, \mathbb{B}_1 \in X$. By Lemma 1 it suffices to show that X witnesses the subproperness of $\mathbb{B}_1$. Let $s, \lambda_1, \ldots, \lambda_m \in X$ s.t. Let $a \in X \cap (\mathbb{B} \setminus \{0\})$. $\lambda_i \in (\omega_1, \theta)$ is regular and $< \lambda_2$.

Let $\sigma : \bar{N} \overset{\sim}{\longleftrightarrow} X$, $\sigma(\bar{s}, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\tau}, \bar{\lambda}_i, \bar{\theta}) = s, \mathbb{B}_0, \mathbb{B}_1, \tau, \lambda_i, \theta$. Let $\bar{\lambda}_0 = On \cap \bar{N}$,

By the subproperness of $\mathbb{B}_0$ There is $b_0 \subset a_0 =_{Df} h_0(a)$ which forces the existence of $\sigma_0 \in V[G_0]$ s.t.

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \theta$ $(i = 1, \ldots, m)$

(c) $\sup \sigma_0 {}'' \bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma {}'' \bar{\lambda}_i$ $\quad (i = 0, \ldots, m)$

(d) $\bar{G}_0 = \sigma_0^{-1} {}'' G_0$ is $\bar{\mathbb{B}}_0$ - generic over $\bar{N}$,

where G_0 is $\mathbb{B}_0$ - generic over V,

But then σ_0 extends uniquely to a $\sigma_0^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma_0^*(\bar{G}) = G$, Hence $X_0 = rng(\sigma_0^*) \prec N[G_0]$ is full in $V[G_0]$.

Note that $b_0 \subset h_0(a) = [\![\check{a}/\dot{G}_0 \neq 0]\!]_{\mathbb{B}_0}$. Hence

$a' = a/_{G_0} \neq 0$ in $\mathbb{B}' = \mathbb{B}_1/G_0$ and $a' \in X_0$.

But then there is $b' \subset a'$ in $\mathbb{B}'$ which forces the existence of $\sigma' \in V[G_0][G']$ (where G' is $\mathbb{B}'$-generic ~~over~~ $V[G_0]$) s.t.

(a') $\sigma' : \bar{N}[\bar{G}_0] \prec N[G_0]$

(b') $\sigma'(\bar{\lambda}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\mathbb{B}}', \bar{G}_0, \bar{\theta}) = \lambda, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \mathbb{B}', G_0, \theta$

where $i = 1, ..., m$ and $\bar{\mathbb{B}}' = \sigma'^{-1}(\mathbb{B}') = \bar{\mathbb{B}}_1/\bar{G}_0$.

(c') $\sup \sigma'{}''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, ..., m)$

(d') $\bar{G}' = \sigma'^{-1}{}''G'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$.

We may assume $b' = \mathring{b}^{G_0}$, where $\Vdash_0 \mathring{b} \in \check{\mathbb{B}}_1/\dot{G}_0$ and b_0 forces that $\mathring{b}$ forces the above to hold. But then there is $b_1 \in \mathbb{B}_1$ s.t. $\Vdash_0 \check{b}_1/\dot{G}_0 = \mathring{b}$. Since $b_0 \Vdash \dot{b} = \check{b}_1/\dot{G}_0 \neq 0$, we have $b \neq 0$, where $b = b_0 \cap b_1$. Since $b_0 \subset h_0(b_1)$. Hence $b \neq 0$, where $b = b_0 \cap b_1$. Since $b_0 \Vdash \check{b}_1/\dot{G}_0 \subset \check{a}/\dot{G}_0$, we have $b = b_0 \cap b_1 \subset a$. Now let $G_1 \ni b$ be $\mathbb{B}_1$-generic. Set:

$$G_0 = G_1 \cap \mathbb{B}_0 \; ; \; G' = \{ b/_{G_0} \mid b \in G_1 \}$$

Then G_0 is $\mathbb{B}_0$-generic and G' is $\mathbb{B}' = \mathbb{B}_1/G_0$-generic over $V[G_0]$. Since $b_0 \in G_0$, there is $\sigma_0 \in V[G_0]$ satisfying (a)-(d) above. This gives $\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G_0]$ s.t. $\sigma_0^*(\bar{G}_0) = G_0$, and $b' = \mathring{b}^{G_0} = b_1/G_0$. Since $b' \in G'$, there is $\sigma' \in V[G_0][G'] = V[G_1]$

satisfying (a') – (d') above. Set:
$\sigma_1 = \sigma' \upharpoonright \bar{N}$. Then

(a) $\sigma_1 : \bar{N} \prec N$

(b) $\sigma_1(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_1, \theta \quad (i = 1, \ldots, n)$

(c) $\sup \sigma_1 `` \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, m)$

It remains only to show:

<u>Claim</u> $\bar{G}_1 = \sigma_1^{-1} `` G_1$ is $\bar{\mathbb{B}}_1$-generic over $\bar{N}$.

pf.

Clearly $G_1 = G_0 * G' = \{ b \mid b/G_0 \in G' \}$.
Set $\bar{G}_1 = \bar{G}_0 * \bar{G}'$. Then $\bar{G}_1$ is $\bar{\mathbb{B}}_1$-generic over $\bar{N}$, since $\bar{G}_0$ is $\bar{\mathbb{B}}_0$-generic over $\bar{N}$ and $\bar{G}'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$. But

$$b \in \bar{G}_1 \longleftrightarrow b/\bar{G}_0 \in \bar{G}' \longleftrightarrow \sigma'(b/\bar{G}_0) = \sigma_1(b)/G_0 \in G'$$
$$\longleftrightarrow \sigma_1(b) \in G_1.$$

QED (Lemma 2)

We recall the definition of <u>subcompleteness</u>:

<u>Def</u> $\mathbb{B}$ is <u>subcomplete</u> iff for suff. large cardinals θ: If $\mathbb{B} \in H_\theta$ and $N, s, \sigma, \lambda_i, \bar{N}, \bar{\mathbb{B}}, \bar{\theta}, \bar{s}, \bar{\lambda}_i$ are as in the def of "subproper", and $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$, then there is $b \in \mathbb{B}$ which forces the existence of σ_0 satisfying (a) – (d).
[The difference is that $\bar{G}$ was chosen in advance.]

We leave it to the reader to see that subcomplete forcing adds no reals. It is also straightforward to define the notion "weakly subcomplete" and prove the analogue of Lemma 1.

By a slight modification of the proof just given we obtain:

<u>Lemma 3</u> Lemma 2 holds with "subcomplete" in place of "subproper".

proof (sketch)

$\bar{G}_1$ is given at the outset and we set:

$\bar{G}_0 = \bar{G}_1 \cap \bar{\mathbb{B}}_0$; $\bar{G}' = \{ b/\bar{G}_0 \mid b \in \bar{G}_1 \}$.

Then $\bar{G}_1 = \bar{G}_0 * \bar{G}'$, $\bar{G}_0$ is $\bar{\mathbb{B}}_0$-generic over $\bar{N}$ and $\bar{G}'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$, where $\bar{\mathbb{B}}' = \bar{\mathbb{B}}_1/\bar{G}_0$. As before we are assuming $\sigma : \bar{N} \prec N$ to be full and $\sigma(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \theta$.

Let $b_0 \in \mathbb{B}_0 \setminus \{0\}$ force the existence of $\sigma_0 \in V[G_0]$ satisfying (a)–(d). Then σ_0 extends uniquely to $\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G_0]$ as before, where $\bar{N}[\bar{G}_0]$ is full in $V[G_0]$. Thus $\sigma_0^*(\bar{\mathbb{B}}') = \mathbb{B}' = \mathbb{B}_1/G_0$. Since $\bar{G}'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$ and $\mathbb{B}'$ is subcomplete, there is $b' \in \mathbb{B}' \setminus \{0\}$ forcing the existence of σ' satisfying (a')–(d'). Let b, b_1 be as before.

Just as before, $b = b_0 \cap b_1 \neq 0$. We then let $G_1 \ni b$ be $\mathbb{B}_1$ - generic and finish the proof exactly as before. QED (Lemma 3)

Without proof we mention;

<u>Lemma 4</u> Lemma 2 holds with "semisubproper" in place of "subproper".

The proof is again essentially the same, but the modification is somewhat more complex.

(It facilitates proofs of this sort if we replace "full" by "weakly full" (in the sense of the next section §3), in the definition of "semi subproper". This is an inessential weakening.)

Theorem 5 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i \leq \alpha \rangle$ be an RCS iteration s.t. $\Vdash_i (\check{\mathbb{B}}_{i+1}/\dot{G}_i$ is subproper$)$ for $i < \alpha$.
Suppose moreover that $\bar{\bar{\xi}} \leq \bar{\bar{\mathbb{B}}}_\xi$ and $\mathbb{B}_{\xi+1}$ collapses $\bar{\bar{\mathbb{B}}}_\xi$ to ω_1 for $\xi < \alpha$. Then each $\mathbb{B}_\xi$ is subproper.

proof

By induction on i we show:

(*) $\Vdash_h (\check{\mathbb{B}}_i/\dot{G}_h$ is subproper$)$ for $h \leq i$.

$h = 0$ then gives the desired result. We note that if G_h is $\mathbb{B}_h$ - generic and $\tilde{\mathbb{B}} = \mathbb{B}/G_h$, then $\tilde{\mathbb{B}}_{i-h} = \mathbb{B}_i/G_h$ for $h \leq i \leq \alpha$. Moreover
$\Vdash_{\tilde{\mathbb{B}}_{i-h}} (\tilde{\mathbb{B}}_{(i-h)+1}/\dot{G}_{i-h}$ is subproper$)$ for $h \leq i < \alpha$
holds in $V[G_h]$. (To see this, let $\tilde{G}$ be $\tilde{\mathbb{B}}_{i-h}$ - generic over $V[G_h]$. Then $G_h * \tilde{G} = G_i$ is $\mathbb{B}_i$ - generic over V.
Moreover $\tilde{\mathbb{B}}_{i-h+1}/\tilde{G} = (\mathbb{B}_{i+1}/G_h)/\tilde{G} =$
$= \mathbb{B}_{i+1}/G_i$ is subproper in $V[G_i] =$
$= V[G_h][\tilde{G}]$.)

This means in practice that once we have shown $\mathbb{B}_i$ to be subproper, we can simply repeat the proof in $V[G_h]$ to show that $\mathbb{B}_i/G_h = \tilde{\mathbb{B}}_{i-h}$ is subproper, where G_h is $\mathbb{B}_h$ - generic.

Case 1 $i=0$. Trivial since $\mathbb{B}_0 = \{0,1\}$ is subproper.

Case 2 $i=j+1$.

$h=i$ is trivial.

$h=0$: $\mathbb{B}_j$ is subproper by the ind. hyp and $\Vdash_j \mathbb{B}_{j+1}/\dot{G}_j$ is subproper. Hence $\mathbb{B}_{j+1}$ is subproper by Lemma 2. QED

To get the result for other $h \leq j$, simply repeat the proof in $V[G_h]$.

QED (Case 2)

Case 3 $i=\lambda$, Lim(λ). For the usual reasons $h=\lambda$ is again trivial. It will suffice to prove it for $h=0$, it will suffice to prove $\mathbb{B}_i$ subproper for an $i<\lambda$.

Case 3.1 $cf(\lambda) \leq \max(\omega_1, \overline{\overline{\mathbb{B}_i}})$ for an $i<\lambda$.

Then there is $\eta<\lambda$ s.t. $\Vdash_\eta cf(\check{\lambda}) \leq \omega_1$. Since we know that $\mathbb{B}_\eta$ is subproper, it suffices by Lemma 2 to show:

Claim $\Vdash_\eta (\mathbb{B}_\lambda/\dot{G}_\eta$ is subproper)

But this says that, if G_η is $\mathbb{B}_\eta$-generic, then $\tilde{\mathbb{B}}_{\lambda-\eta}$ is subproper, where $\tilde{\mathbb{B}} = \mathbb{B}/G_\eta$. Hence we may assume w.l.o.g. that $cf(\lambda) \leq \omega_1$ in V (taking V as $V[G_\eta]$, $\mathbb{B}$ as $\tilde{\mathbb{B}}$ and λ as $\lambda-\eta$.)

Let θ be big enough that $\mathbb{B} \in H_\theta$ and θ verifies the subproperness of $\mathbb{B}_i$ for $i < \lambda$. Let $\tau > \theta$ be regular and set: $N = \langle L_\tau[A], A, \dots \rangle$ where $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$, $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}) = \theta, \mathbb{B}, \lambda$, where $\bar{N}$ is countable and full.

Claim σ witnesses the subproperness of $\mathbb{B}_\lambda$ w.r.t. N.

In other words, if $\sigma(\bar{s}) = s$ and $\sigma(\bar{\lambda}_i) = \lambda_i$ where λ_i is regular for $i = 1, \dots, n$, where $\bar{\bar{\mathbb{B}}}_\lambda < \lambda_i$ $(i = 1, \dots, m)$, and given $a \in \mathbb{B}_\lambda \setminus \{0\}$, there is $b \subset a$, $b \in \mathbb{B}_\lambda \setminus \{0\}$, s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$-generic, there is $\tilde{\sigma} \in V[G]$ s.t.

(a) $\tilde{\sigma} : \bar{N} \prec N$

(b) $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, \lambda, s, \lambda_i$ $(i = 1, \dots, m)$

(c) $\sup \tilde{\sigma}''\bar{\lambda}_i = \tilde{\lambda}_i =_{Df} \sup \sigma''\lambda_i$ for $i = 0, \dots, m$, where $\bar{\lambda}_0 = \mathrm{On} \cap \bar{N}$.

(d) $g = \tilde{\sigma}^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$.

proof.

Let $\langle x_i \mid i < \omega \rangle$ enumerate $\bar{N}$ and let $\langle \Delta_i \mid i < \omega \rangle$ enumerate the $\Delta \in \bar{N}$ s.t. Δ is strongly dense in $\bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$. Since $\mathrm{cf}(\lambda) \leq \omega_1$ there is $\bar{f} \in \bar{N}$ s.t. $f = \sigma(\bar{f})$ maps ω_1 to λ s.t. $\sup f''\omega_1 = \lambda$. Pick $\bar{\xi}_i \in \mathrm{rng}(\bar{f})$ s.t. s.t. $\langle \bar{\xi}_i \mid i < \omega \rangle$ is monotone and cofinal in $\bar{\lambda}$ with $\bar{\xi}_0 = 0$. Set $\xi_i = \sigma(\bar{\xi}_i)$.

We first define a sequence $\langle b_n \mid n < \omega \rangle$ s.t. $R(b_n, \bar{\mathbb{B}} \restriction \bar{\lambda})$ in $\bar{N}$, $b^*_{n+1} \subset b^*_n$, and $b_{n+1} \restriction \bar{\xi}_{n+1} = b_n \restriction \bar{\xi}_{n+1}$ as follows:

Pick b_0 s.t. $b^*_0 \subset \bar{a}$ and $b^*_0 \in \Delta_0$. Given b_n we construct b_{n+1} s.t.

$\{a \in \bar{\mathbb{B}}_{\bar{\xi}_{n+1}} \mid a \cap b^*_{n+1} \in \Delta_{n+1}\}$ is dense

below $\tilde{a} = h_{\bar{\xi}_{n+1}}(b^*_n) = (b_n \restriction \bar{\xi}_{n+1})^*$.

We accomplish this as follows:

Set $\Delta = \{b \mid \mathbb{P}(b, \overline{\mathbb{B}} \restriction \bar{\lambda}) \wedge b^* \subset b_n^* \wedge b^* \in \Delta_{n+1}\}$. Then $\{b^* \mid b \in \Delta\}$ is dense below b_n^* in $\overline{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$. Thus $\Delta' = \{h_{\bar{\bar{\xi}}_{n+1}}(b^*) \mid b^* \in \Delta\}$ is dense below $\tilde{a} =_{\text{df}} h_{\bar{\bar{\xi}}_{n+1}}(b_n^*)$ in $\overline{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$.

Let A be a maximal antichain in Δ'. Then $\bigcup A = \tilde{a}$. For each $a \in A$ choose a $b_a \in \Delta$ s.t. $a = h_{\bar{\bar{\xi}}_{n+1}}(b_a^*) = (b_a \restriction \bar{\bar{\xi}}_{n+1})^*$ Set:

$$b_{n+1}(i) = \begin{cases} b_n(i) & \text{if } i < \bar{\bar{\xi}}_{n+1} \\ \bigcup_{a \in A} (a \cap b_a(i)) \cup \neg \tilde{a} & \text{if } i \geq \bar{\bar{\xi}}_{n+1} \end{cases}$$

We claim that b_{n+1} has the desired properties. We first show that it is a good sequence for $\overline{\mathbb{B}} \restriction \bar{\lambda}$.

(1) $h_i(b_{n+1}(i)) = 1$

Trivial for $i < \bar{\bar{\xi}}_{n+1}$. Now let $i \geq \bar{\bar{\xi}}_{n+1}$. Then

$h_i(b_{n+1}(i)) = \bigcup_{a \in A} a \cup \neg \tilde{a} = \tilde{a} \cup \neg \tilde{a} = 1$,

since $h_i(b_a(i)) = 1$. QED(1)

(2) $h_i(b_{n+1}^*) = (b_{n+1} \restriction i)^*$.

We recall the <u>disjoint distributive law</u> which holds in every complete BA:

(DDL) Let $b = \bigcup_{i \in I} b_i$, where $b_i \cap b_j = 0$ for $i \neq j$. Let $a_i^j \subset b_i$ for $i \in I$, $j \in J$. Then

$$\bigcap_j \bigcup_i a_i^j = \bigcup_i \bigcap_j a_i^j.$$

pf.

$\bigcap_j \bigcup_i a_i^j = b \cap \bigcap_j \bigcup_i a_i^j = \bigcup_i (b_i \cap \bigcap_j \bigcup_h a_h^j) =$

$= \bigcup_i \bigcap_j (b_i \cap \bigcup_h a_h^j) = \bigcup_i \bigcap_j a_i^j$. QED(DDL)

As a step toward proving (2) we first note:

(3) If $j \geq \bar{\xi}_{n+1}$, then $(b_{n+1} \upharpoonright j)^* = \bigcup_{a \in A} (b_a \upharpoonright j)^*$.

prf.

$$(b_{n+1} \upharpoonright j)^* = \tilde{a} \cap \bigcap_{i \in [\bar{\xi}_{n+1}, j)} b_{n+1}(i)$$

$$= \tilde{a} \cap \bigcap_{i \in [\bar{\xi}_{n+1}, j)} \bigcup_{a \in A} (a \cap b_a(i))$$

$$= \tilde{a} \cap \bigcup_{a \in A} \bigcap_{i \in [\bar{\xi}_{n+1}, j)} (a \cap b_a(i))$$

$$= \tilde{a} \cap \bigcup_{a \in A} (b_a \upharpoonright j)^* = \bigcup_{a \in A} (b_a \upharpoonright j)^*$$

since $a = (b_a \upharpoonright \bar{\xi}_{n+1})^*$ for $a \in A$. QED(3)

We now prove (2). For $j \in [\bar{\xi}_{n+1}, \lambda)$ have:

$$h_j(b^*_{n+1}) = h_j(\bigcup_{a \in A} b^*_a) = \bigcup_{a \in A} h_j(b^*_a) =$$

$$= \bigcup_{a \in A} (b_a \upharpoonright j)^* = (b_{n+1} \upharpoonright j)^*.$$

For $j < \bar{\xi}_{n+1}$ we have $h_{\bar{\xi}_{n+1}}(b^*_{n+1}) = \bigcup A = \tilde{a}$

by (3) and hence:

$$h_j(b^*_{n+1}) = h_j h_{\bar{\xi}_{n+1}}(b^*_{n+1}) = h_j(\tilde{a}) =$$

$$= h_j((b_n \upharpoonright \bar{\xi}_{n+1})^*) = (b_n \upharpoonright j)^* = (b_{n+1} \upharpoonright j)^*.$$

QED(2)

By (1) + (2) b_{n+1} is a good sequence:

(4) $GS(b_{n+1}, \mathbb{B} \upharpoonright \lambda)$.

But then:

(5) $R(b_{n+1}, \bar{\mathbb{B}} \restriction \bar{\lambda})$

pf. Let $c \in U_{\bar{\lambda}}(b_{n+1})$. Then $c \in \mathbb{B}_i$ and $c \subset (b_{n+1} \restriction i)^*$ for some i. We may assume $i \geq \bar{\xi}_n$. Hence $c \subset \bigcup_{a \in A} (b_a \restriction i)^*$. Let $c' = c \cap (b_a \restriction i)^* \neq 0$. Then $c' \in U_{\bar{\lambda}}(b_a)$ and there is $d \subset c'$ s.t. $d \in S_{\bar{\lambda}}(b_a)$. It follows easily that $d \in S_{\bar{\lambda}}(b_{n+1})$. QED(5)

Finally we note:

(6) $\{a \in \mathbb{B}_{\bar{\xi}_{n+1}} \setminus \{0\} \mid a \cap b^*_{n+1} \in \Delta_{n+1}\}$ is dense below $\tilde{a} = h_{\bar{\xi}_{n+1}}(b_n)$ in $\mathbb{B}_{\bar{\xi}_{n+1}} \setminus \{0\}$.

pf. It suffices to show that it is predense. But $\bigcup A = \tilde{a}$ and $a \cap b^*_{n+1} = a \cap \bigcup_{a' \in A} b^*_{a'} = b^*_a \in \Delta_{n+1}$ for $a \in A$. QED(6)

This completes the construction of $\langle b_n \mid n < \omega \rangle$.

By induction on $n<\omega$ we construct:

- $\langle c_n \mid n<\omega \rangle$ s.t. $R(c_n, \mathbb{B}\restriction\xi_n)$
  (Hence $c_0=\emptyset$, $c_0^*=1$, since $\xi_0=0$)
- $\langle \dot\sigma_n \mid n<\omega\rangle, \langle \dot u_n \mid n<\omega\rangle, \langle \dot g_n \mid n<\omega\rangle$
  s.t. $\dot\sigma_n, \dot u_n, \dot g_n \in V^{\mathbb{B}_{\xi_n}}$

We inductively verify:

(a) $c_{n+1}\restriction\xi_n = c_n$

(b) $\dot\sigma_0 = \check\sigma$, $g_0 = \{\check 1\}$, $u_0 = \check u_0$, where
$u_0 = \langle \bar s, \bar\lambda_1, \ldots, \bar\lambda_m, \bar f, \bar{\mathbb{B}}, \bar\theta, \bar\lambda \rangle$

(c) c_n^* forces the following to hold in $V^{\mathbb{B}_{\xi_n}}$:

- $\dot g_n$ is $\check{\bar{\mathbb{B}}}_{\bar\xi_n}$ - generic over $\check{\bar N}$
- $(\check b_n\restriction\bar\xi_n)^\vee \in \dot g_n$
- $\sigma_n : \bar N[\dot g_n] \prec N[\dot G_n] \wedge \dot\sigma_n(\dot g_n) = \dot G_n$

(where $\dot G_n$ is the canonical generic name in $V^{\mathbb{B}_{\xi_n}}$)

(d) At $n>0$, then c_n^* forces:

- $\dot\sigma_n(\dot u_{n-1}) = \dot\sigma_{n-1}(\dot u_{n-1})$
- $\dot u_n = \langle \check x_n, \dot z_n^0, \ldots, \dot z_n^m, \dot u_{n-1}, \dot g_n, \check b_n \rangle$,

where:

- z_n^ℓ = the least $z<\check{\tilde\lambda}_\ell$ s.t. $\dot\sigma_n(z) \geq \check\xi_n^\ell$

(where $\langle x_i \mid i<\omega\rangle$ enumerates $\bar N$, $\langle \xi_n^\ell \mid n<\omega\rangle$ is a monotone, cofinal sequence in $\tilde\lambda_i = \sup\sigma''\bar\lambda_i$ $(i=0,\ldots,m)$, and $\bar\lambda_0 = On\cap\bar N$.