§ 2. Attaining Subproper and Subcomplete Forcing

We essentially repeat our earlier definition.

Def. A complete BA $\mathcal{B}$ is **subproper** if

for sufficiently large cardinals $\Theta$;

Let $\mathcal{B} \in H_\Theta$ and $N = \langle L_{\mathcal{E}}[\mathcal{A}], A_{\mathcal{E}}[\mathcal{A}] \rangle$ where $\mathcal{E} > \Theta$ is

regular and $H_\Theta \in N$. Let $\sigma: \bar{\mathcal{B}} \subset N$ be

countable and full. Let $\sigma(\bar{\mathcal{E}}, \bar{\mathcal{B}}, \bar{\mathcal{A}}, \bar{\lambda}, \Xi, \bar{\lambda}) = \Theta, \mathcal{B}, A, \lambda, \lambda_i (1 \leq i \leq m)$. where $\lambda_i \in (\omega_1, \Theta)$ in

irregular and $\bar{\lambda} < \lambda$ for $i = 1, \ldots, m$.

Set $\bar{\lambda}_0 = 0 \cap \bar{\mathcal{B}}$. For any $a \in \bar{\mathcal{B}} \setminus \{0\}$ the

$\bar{\mathcal{B}} \in \bar{\mathcal{B}} - \text{generic}$, there is $\sigma_0 \in V[\bar{\mathcal{G}}]$ with:

(a) $\sigma_0: \bar{\mathcal{B}} \subset N$
(b) $\sigma_0(\bar{\mathcal{E}}, \bar{\mathcal{B}}, \bar{\mathcal{A}}, \bar{\lambda}, \bar{X}, \bar{\lambda}) = \Theta, \mathcal{B}, A, \lambda, \lambda_i (1 \leq i \leq m)$
(c) $\text{sup} \ \sigma_0^{\bar{\lambda}}(\bar{X}) = \text{sup} \ \sigma^{\bar{\lambda}}(\bar{X}) (i = 0, \ldots, m)$
(d) $\bar{G} = \sigma_0^{-1}(\bar{G}) \in \bar{\mathcal{B}} - \text{generic}$ over $\bar{\mathcal{N}}$.

It is trivial to see that a subproper forcing cannot collapse $\omega_1$. Clearly every proper

forcing is subproper, since properness is

just the case $\sigma = \sigma_0$.

Note $\sigma_0$ extends uniquely in $V[\bar{\mathcal{G}}]$ to a map

$\sigma_0^*: \bar{\mathcal{G}} \subset N[\bar{\mathcal{G}}] < N[\bar{\mathcal{G}}]$ s.t. $\sigma_0^*(\bar{G}) = G$. But

$\mathcal{G}[\bar{\mathcal{G}}]$ is then full in $V[\bar{\mathcal{G}}]$. Thus the

stage $\bar{\mathcal{G}}$ is set to handle a subproper

$\bar{\mathcal{B}} \in V[\bar{\mathcal{G}}]$. 
Def. We say that \( \alpha \) verifies the subproperness of IB if the above holds for all cardinals \( \theta \geq \alpha \).

Def. Let \( N = L_\alpha[A] \), \( \sigma : \bar{N} \prec N \) be as above. We then say that \( \sigma \) witnesses the subproperness of IB wrt. \( N \). We also say that \( X = \text{rng}(\sigma) \) witnesses the subproperness of IB.
An apparent weakening of the notion of subproperness is:

Define $IB$ is weakly subproper iff there is a set $\mathcal{X}$ such that for sufficiently large cardinals $\Theta$, if $N = L^\mathcal{X}[\mathcal{A}]$ as above and $X < N$ is countable and full set, $\mathcal{X}, \Theta, IB \in N$, then $X$ witnesses the subproperness of $IB$.

However:

**Lemma 1.** If $IB$ is weakly subproper, then it is subproper.

*Proof.*

Let $\Theta$ be least set, for some $z \in H_{\Theta}$, $\mathcal{X} \geq \Theta$ in a cardinal and $N$ as above and $X < N$ is countable and full set. If $\Theta, IB \in N$, then $X$ witnesses subproperness.

If $\mu > \Theta$ and $\mu = \mathcal{V}_{\mu_0}$, let $\Theta^\mu$ be defined as above in $\mathcal{V}_{\mu}$ rather than $\mathcal{V}$. Clearly $\Theta^\mu \leq \Theta^\mu \leq \Theta$ for $\mu \leq \mu'$.

But then there is $\mu_0$ such that $\Theta^\mu_0 = \Theta$. If $\mu \geq \mu_0$, then...
let $A_\mu$ be the set of $z \in H_{\theta_0}$ s.t.
if $\theta_0 \leq \theta < \mu$, $\theta$ is a cardinal, $N$ is an
above, $X \subseteq N$ is countable and full
s.t. $\theta, IB, z \subseteq X$, then $X$ witnesses
subproperness. Then $\mu_0 \leq \mu \leq \mu' \implies$
$A_\mu \supset A_{\mu'} \in H_{\theta_0}$. Hence there is
$\mu_1$ s.t. $A_\mu = A_{\mu_1}$ for $\mu_1 \leq \mu$. But
Then $A_{\mu_1} = A_\infty$ where $A_\infty$ is defined
as above with $\infty$ in place of $\mu$.
Now let $\theta > \mu_1$ be a cardinal +
let $N = L_{\infty} [A, I]$ be as above. Then
$< \theta, \infty > \in N$ is definable in $\theta, IB$. Hence
$\mu_0 \in N$ definable in $\theta, IB$. Hence
$\langle A, \in \rangle, \mu_0 \in NT \in H_{\theta} \in H_{\theta_0} \in H_{\theta_0}$
is $N$-definable in $\theta, IB$. Hence
$\mu_1, A_\mu = A_\infty$ are $N$-definable in
$\theta, IB$. Now let $X \subseteq N$ be full
and countable s.t. $\theta, IB \subseteq X$.
Then $A_\infty \cap X \neq \emptyset$. Hence $X$

witnesses subproperness. QED
Lemma 2 Let $B_0 \subseteq B_1$ where $B_0$ is subproper and $\mathcal{F}_{B_1} / \mathcal{G}_0$ is subproper. Then $B_1$ is subproper.

Proof: Choose $\delta$ large enough that $B_0 \in H_\delta$, $\delta$ verifies the subproperness of $B_0$ and $H_\delta \theta$ verifies the subproperness of $\mathcal{F}_{B_1} / \mathcal{G}_0$.

Let $\mathcal{V} = L_\omega[A]$ where $\omega > \theta$ is regular and $H_\omega < \mathcal{V}$. Let $X \subseteq \mathcal{V}$ be full and countable. Let $X \subseteq \omega$, $\mathcal{A} \in \mathcal{V}$ regular and $\omega < \lambda_i$, $\lambda_i \in (\omega_1, \theta)$ is regular and $\omega < \lambda_i$.

Let $\sigma : \mathcal{V} \rightarrow X$, $\sigma(\omega_1, B_0, B_1, \lambda_i, \theta) = \omega_1, B_0, B_1, \lambda_i, \theta$. Let $x_0 = \text{On} \cap \mathcal{V}$.

By the subproperness of $B_0$ there is $b_0 < c_0 = \mathcal{F}_{B_0} / \mathcal{G}_0$.

By $h_0(a)$ which forces the existence of $\sigma_0 \in V[G_0]$ s.t.

(a) $\sigma_0 : \mathcal{V} \rightarrow \mathcal{V}$

(b) $\sigma_0(\omega_1, \lambda_i, B_0, B_1, \theta) = \omega_1, \lambda_i, B_0, B_1, \theta$ ($i = 0, \ldots, m$)

(c) $\mu \sigma_0(\omega_1, \lambda_i, B_0, B_1, \theta) = \omega_1, \lambda_i, B_0, B_1, \theta$ ($i = 0, \ldots, m$)

(d) $\mathcal{V} = \sigma_0^{-1} \Pi \mathcal{G}_0$ in $\mathcal{F}_{B_0} - \text{generic over } \mathcal{V}$.

But then $\sigma_0$ extends uniquely to a $\sigma_0^* : \mathcal{V} [\mathcal{G}] \rightarrow \mathcal{V} [\mathcal{G}]$ s.t. $\sigma_0^*(\mathcal{G}) = \mathcal{G}$.
Hence $X_0 = \text{rng}(s_o^*) \leq N[G_o] \text{ in full in } V[G_o]$. Note that $b_o \in h_o(a) = \prod \bar{x}^G/\alpha_{1B'}$ \iff $\alpha' = \alpha^G \neq 0 \text{ in } IB' = IB'/G_o$ and $a' \in X_0$.

But then there is $b' \in a'$ in $IB'$ which forces the existence of $\sigma' \in V[G_o, G']$ (where $G' \in IB'$ - generic over $V[G_o]$) \iff

(a') $\tilde{\sigma}' : \bar{N}[\bar{G}_o] \leq N[G_o]$,

(b') $\sigma'(\bar{X}, \bar{X}_i, \bar{B}_o, \bar{B}_1, \bar{B}', \bar{G}_o, \bar{G}) = \bar{x}_i \lambda_{iB}$, $iB_0, iB_1, iB'/G_o$, $G$

where $i = 1, m$ and $\bar{B}' = \sigma^-1(\bar{B}') = \bar{B}_1/\bar{G}_o$.

(c') $\cup \sigma''(\bar{X}_i = \bar{x}_i : i = 0, \ldots, m)$

(d') $\bar{G}' = \sigma^-1(G'_o) \cup \bar{B}'$ - generic over $\bar{N}[\bar{G}_o]$.

We may assume $b' = b^G$, where $b' \in IB'/G_o$ and $b_o$ forces that $b'$ forces the above to hold. But then there is $b_o \in IB_1$ \iff

$ib_0/b_i/\bar{G}_o = b$. Since $b_o \upharpoonright b = b_i/\bar{G}_o \neq 0$, we have

$ib_0/b_i/\bar{G}_o = b$. Since $b_o \upharpoonright b_0/\bar{G}_o \subseteq \alpha^G/\bar{G}_o$, we have $b = b_o \land b_1 \subseteq \bar{a}$. Now let $G_o \ni b \in IB_1$ - generic, so that

$G_o = G_o \land IB_1$ \iff $G' = \{b/\bar{G}_o : b \in G_o\}$

Then $G_o \in IB_1$ - generic and $G' \in IB' = IB'/G_o$ - generic over $V[G_o]$. Since $b_o \in G_o$ there is $\sigma_o \in V[G_o]$ satisfying (a) - (d) above, thus gives $\bar{\sigma}_o^* : \bar{N}[\bar{G}_o] \leq N[G_o]$ \iff $\bar{\sigma}_o^*(\bar{G}_o) = G_o$, and $b' = b^G$, $G_o = b'/G_o$. Since $b' \in G'$, there is $\sigma' \in V[G_o, G'] = V[G_o]$.
satisfying (a') - (d') above. Set:
\[ \sigma_1 = \sigma \upharpoonright N. \]
Then
\[ (a) \sigma_1 : N \rightarrow N \]

\[ (b) \sigma_1 (\bar{\tau}, \bar{x} \bar{1}, \bar{B} \bar{1}, \bar{\theta}) = \Sigma \lambda \bar{1}, \bar{B} \bar{1}, \bar{\theta} \ (i = 1, \ldots, m) \]

\[ (c) \sup \sigma_1 (\bar{x}_i) = \bar{x}_i \ (i = 0, \ldots, m) \]

The remainder only to show:

**Claim** \( \overline{G}_1 = \sigma_1 \upharpoonright G_1 \) is \( \overline{B}_1 \)-generic over \( N \).

Clearly \( \overline{G}_1 = G_0 \star G' = \{ b \mid b/G_0 \in G' \} \).

Set \( \overline{G}_1 = \overline{G}_0 \star G' \). Then \( \overline{G}_1 \) is \( \overline{B}_1 \)-generic over \( N \), since \( \overline{G}_0 \) is \( \overline{B}_0 \)-generic over \( N \) and \( \overline{G} \) is \( \overline{B} \)-generic over \( N[\overline{G}_0] \). But

\[ b \in \overline{G}_1 \iff b/G_0 \in \overline{G}' \iff \sigma_1 (b/G_0) = \sigma_1 (b) \in \overline{G}_1, \]

**Q.E.D. (Lemma 2)**

We recall the definition of subcompleteness:

**Def.** \( \overline{B} \) is subcomplete iff for all \( \lambda \) large cardinals \( \theta \): \( A \upharpoonright \overline{B} \in \mathcal{H} \) and \( N, \Sigma, \lambda, \bar{x}, \bar{N} \):

\( \overline{B}, \bar{\theta}, \bar{x}, \bar{N} \) are as in the def. of "subprime" and \( \overline{G} \) is \( \overline{B} \)-generic over \( N \), then there is

\[ b \in \overline{B} \] which forces the existence of \( \sigma_0 \) satisfying (a) - (d).

[The difference is that \( \overline{G} \) was chosen in advance.]
We leave it to the reader to see that subcomplete forcing adds no reals, and it is also straightforward to define the notion "weakly subcomplete" and prove the analogue of Lemma 1.

By a slight modification of the proof just given we obtain:

**Lemma 3** Lemma 2 holds with "subcomplete" in place of "subproper".

**proof (sketch)**

Given at the outset and we set:

\[ G_0 = \overline{G}_1 \cap \overline{\bar{B}}_0, \quad \overline{G} = \{ b/\bar{e}_b \mid b \in \overline{G}_1 \}. \]

Then \( \overline{G}_1 = \overline{G}_1 \times \overline{G} \), \( \overline{G}_1 \) a \( \overline{\bar{B}}_0 \)-generic over \( N \) and \( \overline{G} \) a \( \overline{\bar{B}} \)-generic over \( N[G_0] \), where \( \overline{\bar{B}} \) is \( \overline{\bar{B}}_0 \)-generic over \( N[G_0] \), where \( \overline{\bar{B}} = \overline{\bar{B}}_0 / G_0 \). As before we are assuming \( \sigma : \overline{\bar{N}} \to N \) to be full and

\[ \sigma(\overline{\lambda}, \overline{\lambda}_i, \overline{\bar{B}}_0, \overline{\bar{B}}_1, \overline{\theta}) = \check{\lambda}, \check{\lambda}_i, \check{\bar{B}}_0, \check{\bar{B}}_1, \check{\theta}. \]

Let \( b, c \in \bar{B}_0 \setminus \bar{e}_0 \) force the existence of \( \sigma \in V[G_0] \) satisfying (a)–(d). Then \( \sigma \) extends uniquely to \( \overline{\sigma} : \overline{\bar{N}}[\bar{G}_0] \to \overline{\bar{N}}[\bar{G}_0] \) as before, where \( \overline{\bar{N}}[\bar{G}_0] \) is full in \( V[G_0] \).

Then \( \overline{\sigma}^*(\overline{\bar{B}}) = \overline{\bar{B}} = \overline{\bar{B}}_1 / G_0 \). Since \( \overline{G} \) is \( \overline{\bar{B}} \)-generic over \( \overline{\bar{N}}[\bar{G}_0] \) and \( \overline{\bar{B}} \) is subcomplete, there is \( b, c \in \bar{B}_0 \setminus \bar{e}_0 \)

forcing the existence of \( \sigma \) satisfying (a')–(d'). Let \( b, b' \) be as before.
Just as before, \( b = b_0 \cap b_1 \neq 0 \). We then let \( G \ni b \) be \( 1\beta \)-generic and finish the proof exactly as before. QED (Lemma 3)

Without proof we mention:

Lemma 4: Lemma 2 holds with “semi-subproper” in place of “subproper”.

The proof is again essentially the same, but the modification is somewhat more complex.

(At facilitates proofs of this sort if we replace “full” by “weakly full” (in the sense of the next section §3), in the definition of “semi-subproper”.

Thus is an inessential weakening.)
Theorem 5 Let $IB = \langle IB_i \mid 1 \leq i \leq d \rangle$ be an RCS iteration such that $IB_i + 1 / G_i \in \text{subproper}$ for $i < d$.

Suppose moreover that $\bar{\bar{z}} \leq IB_{\bar{\bar{z}}}$ and $IB_{\bar{\bar{z}}} \cap \bar{\bar{z}} + 1$ for $\bar{\bar{z}} < d$. Then each $IB_i$ is subproper.

Proof

By induction on $i$ we show:

(*) $IB_i / G_i \in \text{subproper}$ for $i \leq d$.

$h = 0$ then gives the desired result. We note that if $G_h \in IB_h - \text{generic}$ and $IB = IB / G_h$ then $\bar{\bar{IB}}_{i-h} = IB_i / G_h$ for $h \leq i \leq d$. Moreover $IB_{i-h+1} / G_{i-h} \in \text{subproper}$ for $h \leq i < d$.

$\bar{\bar{IB}}_{i-h}$ holds in $V[G_h]$. (To see this, let $\bar{\bar{G}}$ be $\bar{\bar{IB}}_{i-h} - \text{generic}$ over $V[G_h]$. Then $G_h \times \bar{\bar{G}} = G_i \in IB_i - \text{generic}$ over $V$.)

Moreover $\bar{\bar{IB}}_{i-h+1} / G = (IB_{i-h} / G_h) / \bar{\bar{G}} = IB_{i+1} / G_i \in \text{subproper}$ in $V[G_h][\bar{\bar{G}}]$.

This means in practice that once we have shown $IB_i$ to be subproper, we can simply repeat the proof in $V[G_h]$ to show that $IB_i / G_h = IB_{i-h} / G_h \in \text{subproper}$, where $G_h \in IB_h - \text{generic}$.
Case 1. \( i = 0 \). Trivial since \( B_0 = \emptyset, 1 \) is subproper.

Case 2. \( i = 1 + 1 \).
\( h = i \) is trivial.
\( h = 0 \). \( 1B_i \) is subproper by the ind. hyp. and \( 1B_i \not\subseteq \mathcal{G}_h \), thus subproper. Hence \( 1B_i \not\subseteq \mathcal{G}_h \) is subproper by Lemma 2. QED

To get the result for other \( h \geq 1 \), simply repeat the proof in \( V[\mathcal{G}_h] \).
QED (Case 2)

Case 3. \( i = \lambda \), \( \text{Lim}(\lambda) \).
\( h = \lambda \) is again trivial. For the usual reason, it will suffice to prove it for \( h = 0 \).

Case 3. 1. \( \text{cf}(\lambda) \leq \max(\omega_1, 1B_i) \) for an \( i < \lambda \).

Then there is \( \gamma < \lambda \) s.t. \( 1 \vdash \text{cf}(\lambda) \leq \omega_1 \).

Since we know that \( B_i \) is subproper, it suffices by Lemma 2 to show:
Claim \( 1 \neg \left( \exists \gamma (1B_i \not\subseteq \mathcal{G}_\gamma \text{ is subproper}) \right) \).

But this says that, if \( \mathcal{G}_\gamma \not\subseteq 1B_i \), then \( 1B_{\lambda - \gamma} \not\subseteq \mathcal{G}_\gamma \). Hence we may assume without loss of generality, that \( \text{cf}(\lambda) \leq \omega_1 \) in \( V \) (taking \( V \) as \( V[\mathcal{G}_\gamma] \), \( 1B \) as \( \bar{B} \) and \( \lambda \) as \( \lambda - \gamma \)).
Let $\theta$ be big enough that $1B \in H_\theta$ and $\theta$
verifies the subproperness of $1B_i$ for $i < \lambda$.
Let $\bar{\theta} > \theta$ be regular and set $N = \langle L_{\bar{\theta}[A]}, A, ... \rangle$
where $H_\theta \subset N$. Let $\sigma : \bar{\theta} \subset N$, $\sigma(\bar{\theta}, 1B, \bar{x})$
$= \theta, 1B, \lambda$, where $\bar{\theta}$ \(\bar{\text{is}}\) countable and full.
Claim $\sigma$ witnesses the subproperness of $1B_\lambda$ w.r.t. $N$.

An other words, if $\sigma(\bar{x}) = \xi$ and $\sigma(\bar{x}_i) = \lambda_i$
for $i = 1, \ldots, m$, where $\lambda_i$ \(\bar{\text{is}}\) regular
with $1B_{\lambda_i} < \xi_i$ ($i = 1, \ldots, m$) and given
$a \in 1B_{\lambda_i} \neg \{\xi_i \}$, there is $b \subset a$, $b \in 1B \neg \{\xi_i \}$,
\(\forall \tau \), whenever $G \models b \in 1B_{\lambda_i} - \text{generic}$
\(\forall \tau \), there is $\tilde{\sigma} \in V[G]$ s.t.
\(\forall \tau \),
(a) $\tilde{\sigma} : \bar{\theta} \subset N$
(b) $\tilde{\sigma}(\bar{\theta}, 1B, \bar{x}, \bar{x}_i, \lambda, \lambda_i)$
(c) $\sup \tilde{\sigma}(\bar{x}_i) = \bar{x}_i = \sup \sigma(\bar{x}_i)$ for
\(i = 1, \ldots, m\), where $\bar{x}_0 = \text{On} \cap \bar{N}$.
(d) $\sigma(\tilde{\sigma}) \models \forall \tau \\bar{\text{generic}}$ over $\bar{N}$.

Proof.
Let $\langle x_i | i < \omega \rangle$ enumerate $\bar{N}$ and let
$\langle y_i | i < \omega \rangle$ enumerate the $\Delta \in \bar{N}$ s.t.
$\langle y_i | i < \omega \rangle$ denote the $\Delta \in \bar{N}$ s.t.
$\Delta$ \(\bar{\text{is}}\) generic in $1B_{\lambda_i} \neg \{\xi_i \}$. Since $\bar{\theta}(\lambda) \leq \omega_1$
\(\forall \tau \), there is $\tilde{f} \in \bar{N}$ s.t. $\tilde{f} = \sigma(\tilde{f})$ maps $\omega_1$ to
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
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\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
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\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$
\(\forall \tau \), $\sup \tilde{f}(\omega_1) = \lambda$. Pick $\bar{y}_i$ s.t. $\tilde{y}_i$ \(\bar{\text{maps}}\) to $\lambda$.
We first define a sequence \( \langle b_n \ (n < \omega) \rangle \) in \( \mathbb{R} \), \( b_n^{x} < b_{n+1}^{x} \), and
\( b_{m+1}^{x} \star_{m+1}^{x} \star_{m+1} = b_{m}^{x} \star_{m+1}^{x} \) as follows:

Pick \( b_0 \) s.t. \( b_0^{x} \leq \bar{a} \) and \( b_0^{x} \in \Delta_0 \). Given
\( b_n \) we construct \( b_{m+1} \) s.t.
\[ \exists \alpha \in \mathbb{R}_{\star_{m+1}^{x}} \quad \alpha \land b_{m+1}^{x} \in \Delta_{m+1} \] is dense
below \( \bar{a} = h_{m+1}^{x} (b_{m}^{x}) = (b_{m}^{x} \star_{m+1}^{x})^{x} \).

We accomplish this as follows:
Set \( \Delta = \{ b \mid \beta \in \mathcal{R}(b, \overline{B}_n) \land b^* \in b^*_m \land b^* \in \Delta_{m+1} \} \).

Then \( \exists b^* \mid b^* \in \Delta_0 \subseteq \overline{B}_{n-1} \setminus \{0, 3\} \). Thus \( \Delta' = \{ h^*_m(b^*)_a \mid b^* \in \Delta_0 \subseteq \overline{B}_{n-1} \setminus \{0, 3\} \} \) is dense below \( \overline{\alpha} = \bigcup_{i} \beta^*_m(b^*_a) \subseteq \overline{B}_{n-1} \setminus \{0, 3\} \).

Let \( \mathcal{A} \) be a maximal antichain in \( \Delta' \). Then \( \mathcal{U} \mathcal{A} = \overline{\alpha} \). For each \( a \in \mathcal{A} \), choose \( b^*_a \in \Delta_0 \subseteq \overline{B}_{n-1} \setminus \{0, 3\} \) such that \( a = \bigcup_{i} \beta^*_m(b^*_a) \subseteq \overline{B}_{n-1} \setminus \{0, 3\} \).

Set:

\[
\begin{aligned}
b_{n+1}(i) &= \begin{cases} b_n(i) & \text{if } i \leq \overline{3}_{n+1} \\
 \bigcup_{a \in \mathcal{A} \setminus \{a \}} (a \cap b_n(i)) \cup \overline{\alpha} & \text{if } i > \overline{3}_{n+1}
\end{cases}
\end{aligned}
\]

We claim that \( b_{n+1} \) has the desired properties. We first show that it is a good sequence for \( \overline{B}_{n-1} \).

1. \( h^*_c(b_{n+1}(i)) = 1 \) for all \( i < \overline{3}_{n+1} \).

Trivial for \( i < \overline{3}_{n+1} \). Now let \( i > \overline{3}_{n+1} \). Then

\[
\begin{aligned}
h^*_c(b_{n+1}(i)) &= \bigcup_{a \in \mathcal{A} \setminus \{a \}} (a \cap b_n(i)) \cup \overline{\alpha} \\
&= a \cup \overline{\alpha} = a \cup \overline{\alpha} = 1, \\
\end{aligned}
\]

Thus \( h^*_c(b_{n+1}(i)) = 1 \). \( \square \) (PDL) Let \( b = \bigcup_{i \in I} b_i \), where \( b_i \cap b_j = 0 \) for \( i \neq j \).

Let \( a_i^c \subseteq b_i \) for \( i \in I \), \( i \in J \). Then

\[
\bigcap_{i \in J} a_i^c = \bigcap_{i \in J} a_i^c.
\]

Hence

\[
\begin{aligned}
\bigcap_{i \in J} a_i^c &= b \cap \bigcap_{i \in J} a_i^c \\
&= \bigcap_{i \in J} (b_i \cap a_i^c) = \bigcap_{i \in J} a_i^c.
\end{aligned}
\]

\( \square \) (PDL)
As a step toward proving (2) we first note:

(3) If \( j \geq \overline{\frac{s}{3}} m + 1 \), then \( (b_{m+n} \cap \bar{i})^* = \bigcup_{a \in A} (b_a \cap \bar{i})^* \).

With:

\[
(b_{m+n} \cap \bar{i})^* = \tilde{a} \cap \bigcap_{i \in [\overline{s_{m+n}}]} b_{m+n} (i) = \tilde{a} \cap \bigcap_{i \in [\overline{s_{m+n}}]} \left( a \cap b_a (i) \right) = \tilde{a} \cap \bigcap_{a \in A} \left( a \cap b_a (i) \right) \bigcap_{i \in [\overline{s_{m+n}}]} = \tilde{a} \cap \left( \bigcap_{a \in A} (b_a \cap \bar{i})^* \right) = \bigcup_{a \in A} (b_a \cap \bar{i})^* = (b_{m+n} \cap \bar{i})^* \]

we have \( a = (b_a \cap \overline{s_{m+n}})^* \) for \( a \in A \). QED (3)

We now prove (2). For \( j \in [\overline{s_{m+n}}] \), have:

\[
h_j (b_{m+n}^*) = h_j \left( \bigcup_{a \in A} b_a^* \right) = \bigcup_{a \in A} h_j (b_a^*) = \bigcup_{a \in A} (b_a \cap \bar{i})^* = (b_{m+n} \cap \bar{i})^* \]

If \( j < \overline{s_{m+n}} \), we have \( h_{\overline{s_{m+n}}} (b_{m+n}^*) = \bigcup_{a \in A} = \tilde{a} \).

By (3) and hence:

\[
h_j (b_{m+n}^*) = h_j \left( b_a^* \cap \overline{s_{m+n}} \right) = h_j (\tilde{a})^* = h_j ((b_{m+n}^*)^*) = (b_{m+n} \cap \bar{i})^* = (b_{m+n} \cap \bar{i})^* \]

QED (2)

By (1) + (2) \( b_{m+n} \) is a good sequence:

(4) GS \((b_{m+n}, \overline{1B \cap \bar{x}})\),

But then:
(5) \[ P(b_{m+1}, B' \cap \hat{\alpha}) \]

\textit{Mf. Let } c \in U_\hat{\alpha}(b_{m+1}). \text{ Then } c \in \overline{B_c} \text{ and } c \in (b_{m+1} \setminus c)^* \text{ for some } c. \text{ We may assume } c \geq \overline{S}_m. \text{ Hence } c \in \\
\in \bigcup_{a \in A} (b_a \setminus c)^*. \text{ Let } c' = c \cap (b_a \setminus c)^* \neq \emptyset. \text{ Then } c' \in U_\hat{\alpha}(b_a) \text{ and there is } d \in c' \cap \text{ d} \in S_\hat{\alpha}(b_a). \text{ It follows easily that } d \in S_\hat{\alpha}(b_{m+1}). \text{ QED (5)}

Finally, we note:

(6) \[ \{ a \in \overline{B_{m+1}} \setminus \overline{S}_{m+1} | a \cap b_{m+1}^* \in \Delta_{m+1} \} \]

\text{denote below } \hat{a} = h_\hat{\alpha}(b_{m+1}) \in \overline{B_{m+1}} \setminus \overline{S}_{m+1}.

\text{pf. It suffices to show that it } \hat{a} \text{ is dense. But } U\hat{a} = \hat{a} \text{ and }
\in \bigcup_{a \in A} b_a^* = b_a^* \in \Delta_{m+1} \text{ for } a \in A. \text{ QED (6)}

This completes the construction of \( \langle b_m \mid m < \omega \rangle \).
By induction on $n < \omega$ we construct:

- $\langle c_m \mid m < \omega \rangle \ni t \vdash R(c_m, B, \vec{z}_m)$

  (Hence $c_0 = \emptyset$, $c^*_0 = 1$, and $\omega \vec{z}_0 = 0$)

- $\langle \vec{s}_m \mid m < \omega \rangle, \langle \vec{u}_m \mid m < \omega \rangle, \langle \vec{q}_m \mid m < \omega \rangle$

  s.t. $\vec{s}_m, \vec{u}_m, \vec{q}_m \in VIB_{\vec{z}_m}$

We inductively verify:

(a) $c_{m+1} \upharpoonright \vec{z}_m = c_m$

(b) $\vec{s}_0 = \vec{\omega}$, $\vec{q}_0 = \{1\}$, $u_0 = \bar{u}_0$, where

  $u_0 = \langle \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m, \vec{f}, \vec{1B}, \vec{\theta}, \vec{\chi} \rangle$

(c) $c^*_m$ forces the following to hold in $VIB_{\vec{z}_m}$:

  - $\vec{q}_m \in \vec{1B}_{\vec{z}_m}$ - generic over $N$

  - $(b_m, m^*_{\vec{z}_m})^* \in \vec{q}_m$

  - $\vec{s}_m : \bar{N}[\vec{q}_m] \prec N[G_m] \land \vec{s}_m(\vec{q}_m) = G_m$

  (where $G_m$ is the canonical generic name in $VIB_{\vec{z}_m}$)

(d) $A \vdash m > 0$, then $c^*_m$ forces:

  - $\vec{s}_m(\vec{u}_{m-1}) = \vec{s}_{m-1}(\vec{u}_{m-1})$

  - $\vec{u}_m = \langle \vec{x}_m, \vec{z}_m, \ldots, \vec{z}_m, \vec{u}_{m-1}, \vec{q}_m, \vec{b}_m \rangle$

  where:

  - $\vec{z}_m$ = the least $z < \vec{x}_m$ s.t. $\vec{s}_m(z) \trianglelefteq \vec{z}_m$

  (where $\langle x_i \mid i < \omega \rangle$ enumerates $\bar{N}$, $\langle z^i_m \mid m < \omega \rangle$

  is a monotone, cofinal sequence in $\bar{X}$, and $X_i = \sup \vec{\sigma}^{\vec{z}_m} \vec{x}_i$ ($i = 0, \ldots, m$), and

  $\bar{X}_0 = \text{Om} \cap \bar{N}$.
Since $c_n = c_{n+1} \upharpoonright \bar{r}_n$ and $\bar{\lambda} = \sup_{n \in \omega} \bar{r}_n \in \omega$ - cofinal in $\tau$, we conclude that

$P(c, IB_{\bar{\lambda}}^{\bar{\lambda}})$, where $c = \cup_{n \in \omega} c_n$.

But then $c^* \in IB_{\bar{\lambda}}^{\bar{\lambda}} \setminus \{0^\bar{\lambda}\}$ and $h_{\bar{r}_n}(c^*) = c^*_n$.

Let $G \ni c^*$ be $IB_{\bar{\lambda}}^{\bar{\lambda}} -$ generic. Set:

$G_m = G \cap IB_{\bar{r}_m} \equiv \bar{G}_m, \quad g_m = \bar{g}_m, \quad \sigma_m = \bar{\sigma}_m G_m$.

Note that $\sigma_m(\bar{f}_m) = \bar{f}_m$ for all $m, h < \omega,$

and $\sigma_m(f) = f$ for all $m$. Then

$\sigma_m: \bar{N}[g_m] \subset \bar{N}[G_m] \cap \sigma_m(f) = G_m$ ($m < \omega$).

Since $\sigma_m(x_h) = \bar{\sigma}_h(x_h)$ for $h < m$, we can define a new map $\bar{\sigma}: \bar{N} \to \bar{N}$ by:

$\bar{\sigma}(x_1) = \bar{N}, \bar{\sigma}_m(x_1) = \sigma_m(x_1)$ for all $m, x \in \bar{N}$.

Since $\sigma_m(g_{n-1}) = \bar{g}_{n-1}(\bar{g}_{n-1}) = G_{n-1} = G \cap \bar{N}_{\bar{r}_{n-1}}$ for $n > 0$, it follows easily that:

$\sigma_m(g_h) = G_h = G \cap \bar{r}_h$ for $h \leq m < \omega$.

Thus $g = \sigma_m^{-1} \bar{G}_m = \bar{\sigma}_m^{-1} G_m$ ($m < \omega$).

Set: $g = \bar{\sigma}_m^{-1} \bar{G}_m$. Then $G_m \cap \bar{r}_{\bar{r}_m} \equiv \bar{g}_m$.

Claim 1: $g \ni \bar{r}_{\bar{r}_m} -$ generic over $\bar{N}$.

_pick_ $b_n \in \bar{r}_{\bar{r}_m}^*$, $g_{m+1} \subset \bar{g}_m$ for $m < \omega$.

where $(b_n \in \bar{r}_{\bar{r}_{m+1}}^*) = (b_{m+1} \in \bar{r}_{m+1}^{m+1})^*$.
Hence \((b_m \uparrow \bar{\alpha}_{m+i})^* \in \mathcal{G}\) for \(m, i < \omega\). Hence \(b_m^* = \bigcap_{i < \omega} (b_m \uparrow \bar{\alpha}_{m+i})^* \in \mathcal{G}\). But

\[
\Delta = \exists u \in (b_m \uparrow \bar{\alpha}_m)^* \exists v \in \Delta_m \bar{3} \text{ such that } \exists B \in \{0, \bar{3}\}^m \text{ in } \Delta_m^m \text{ with } u \cap v \in \Delta_m \text{ and } u \cap B = \emptyset, \text{ hence } q_m \cap \Delta \neq \emptyset.
\]

Then \(a \cap b_m^* \in \mathcal{G}\), where \(a \cap b_m^* \in \Delta_m\). Hence \(q \cap \Delta_m \neq \emptyset\) for all \(m\). QED (Claim 1)

\underline{Claim 2} \: \sup \bar{\alpha}_i = \bar{\lambda}_i = \sup \sigma^* \bar{\alpha}_i.

For \(i = 0, \ldots, m\)

\(\exists \bar{\sigma} \in (\bar{\alpha}_m^m) = \bar{\sigma}_m \in (\bar{\alpha}_m^m) \supseteq \bar{\alpha}_m^m \) where \(\sup \bar{\alpha}_m^m = \bar{\lambda}_m^m \)

\(\forall \bar{\sigma} \in (\bar{\alpha}_m^m) \supseteq \bar{\sigma}_m \in (\bar{\alpha}_m^m) \quad \bar{\alpha}_i \in \bar{\lambda}_i^m \) for some \(m\) if \(x \in \bar{\alpha}_i \)

QED (Claim 2)

(Note that by Claim 2, we have:

\(\bar{\sigma} : \bar{\mathcal{N}} \rightarrow \bar{\mathbb{N}}\) cofinally, since \(\lambda_0 = 0 \cap \bar{\mathcal{N}}\).

Trivially)

\underline{Claim 3} \: \bar{\sigma}(\bar{\theta}, \bar{\mathcal{B}}, \bar{\lambda}, \bar{\alpha}, \bar{\lambda}_i) = \bar{\theta}, \bar{\mathcal{B}}, \lambda, \lambda_i \bar{\lambda}_i

This proves that \(\sigma\) witnesses the subproperness of \(\mathcal{B}\).

It remains only to carry out the construction of \(c_m\) (\(m < \omega\)).
The construction of \( c_0 = \phi, \sigma_0 = \delta, \tau_0 = \eta_0, q_0 = \xi \) has already been given. Now let
\( c_m, \sigma_m, \tau_m, q_m \) be given s.t. (a)--(d) hold.
Let \( G_m \subset C_m \) be \( \beta \) generic. Set:
\[
\sigma_m = \sigma_m, \quad \tau_m = \tau_m, \quad q_m = q_m. \quad \text{Then} \quad \sigma_m : N [ q_m ] \rightarrow N [ G_m ] \quad \text{and} \quad \sigma_m ( q_m ) = G_m.
\]
Hence \( X_m = \sigma_m ( C_m ) \subset N [ G_m ] \) is full in \( \mathcal{U} [ G_m ] \). But \( \widetilde{\mathcal{B}} = \mathcal{B} \times G_m \) is subproper in \( \mathcal{U} [ G_m ] \). Clearly \( \sigma_m ( \mathcal{B} \times G_m ) \subset G_m \).
Moreover \( \sup \sigma_m " \lambda \ell = \lambda \ell = \mathcal{B} \times G_m \). For \( l = 0, \ldots, m \). Since:
\[
\left\llangle (b_m^r \times G_m^r) / G_m \neq 0 \right\rrangle \mathcal{B}_{\mathcal{S}_{m+n}} = h_{\mathcal{S}_{m+n}} (b_m^r \times G_m^r) = h_{\mathcal{S}_{m+n}} (b_m^r \times G_m^r) = (b_m^r \times G_m^r) / G_m \quad \text{we have} \quad (b_m^r \times G_m^r) / G_m \neq 0.
\]
Set:
\[
b = (b_m^r \times G_m^r) / G_m \quad \text{and} \quad \sigma_m ( b ) = (\sigma_m (b_m^r) \times G_m) / G_m.
\]
Then \( b \in \mathcal{B} \perp X \) and \( b \neq 0 \). Hence there is a condition \( \mathcal{C} \in \mathcal{B} \setminus \{ 0 \} \) s.t. \( \mathcal{C} \neq b \) and \( \mathcal{C} \) forces the following to hold:
(**) Let $G \in C$ be $\bar{B}$-generic over $U[G_m]$. Then
There is $\bar{\sigma} \in U[G_m][\bar{G}]$ s.t.
(a) $\bar{\sigma} : \bar{N}[q_m] \to N[G_m]$, s.t. $\bar{\sigma}(q_m) = G_m$
(b) $\bar{\sigma}(u_m) = \sigma_m(u_m)$
(c) $\forall i \in \bar{\sigma}^{-1} \bar{\lambda}_i = \lambda_i$ (i = 0, ..., m)
(d) $\bar{\sigma} = \bar{\sigma}^{-1} \bar{G} \in \bar{B}_{\bar{\sigma}}[q_m] - \text{generic over } N[G_m].$

Note that $\bar{\sigma}(b_m) = \sigma_m(b_m)$ by (b). Hence $\bar{\sigma}(\bar{b}) = \sigma_m(b) = b$. Since $\bar{\sigma} \subset \bar{b}$, we then have:

$$(b_m \upharpoonright \frac{\bar{\sigma}}{q_m}) \in \bar{G}.$$ 

Without loss of generality, we may assume $\bar{G} = c^*$. We may also assume $c = \bar{G}_m$, where $c = \bar{G}_m$. Hence $R(c_i, \bar{B}_{\bar{\sigma}}[q_m] / \bar{G}_m)$ and

$c_m$ forces the statements which ensure (**) for $c_m \in \bar{G}_m$

$$1 - \inf \frac{\bar{\sigma}}{q_m} \leq i < \frac{\bar{\sigma}}{q_m}, \quad \text{let} \quad c' / i = \text{that } b \in \bar{B}_{c_i}, \text{s.t. } |\frac{b}{c_i}| = c_i.$$ 

$|c' / i| = 1$ for $i < \frac{\bar{\sigma}}{q_m}$. Then

$$R(c', \bar{B}_{\bar{\sigma}}[q_m] / \bar{G}_m) = R(c', \bar{B}_{\bar{\sigma}}[q_m] / \bar{G}_m).$$

Moreover, $h_{\frac{\bar{\sigma}}{q_m}}(c' / i) = [c' / \bar{G}_m \neq 0]_{\bar{B}_{\bar{\sigma}}[q_m]} = [c' / \bar{G}_m \neq 0]_{\bar{B}_{\bar{\sigma}}[q_m]} = 1$, $\Delta$ we then act:

$$C_{m+1}(i') = \begin{cases} c_m (c_i) & i < \frac{\bar{\sigma}}{q_m} \\ c' / i & \frac{\bar{\sigma}}{q_m} \leq i < \frac{\bar{\sigma}}{q_m} \end{cases}$$

We have:
\(-21-\)

\[c_{m+1}^* = c_m^* \cap c^* \neq 0, \quad \text{and} \quad \text{h}_m^* (c_m^* \cap c^*) = c_m^* \cap c^* = c_m^* \neq 0. \]

Since \(R(c_{m+1}/G_m, B \cap S_m)\) and

\[(c_{m+1} \cap S_m)^* \cap R(c_{m+1}/G_m, B \cap S_{m+1}/G_m),\]

we conclude \(R(c_{m+1}, B \cap S_{m+1})\).

Hence \(c_{m+1}^* \neq 0 \in B \cap S_{m+1}\), and \(B \cap S_{m+1}\) is an RCS iteration. Now let \(G_{m+1} \supseteq C_{m+1}\) be \(B \cap S_{m+1}\) - generic. Then \(G_m = G_{m+1} \cap B \cap S_m \cup B \cap S_{m+1}\) - generic and \(c_m^* \in G_m\). Set:

\[
\tilde{B} = B \cap S_{m+1}/G_m, \quad c = c_{m+1}/G_m, \quad \tilde{G} = \{b/G_m \mid b \in E_{m+1}\}.
\]

Then \(\tilde{G} \cup \tilde{B}\) - generic over \(N[G_m]\) and \(\tilde{c} = c_{m+1} \in \tilde{G}\), Thus (1) holds. Set:

\[
\tilde{g} = \tilde{c}^{-1} \circ \tilde{g}, \quad \text{where } \tilde{g} \text{ is given by (1),}
\]

\(\tilde{c}\) extends uniquely to \(a\)

\[\sigma_{m+1} : N[q_m] [\tilde{g}] \rightarrow N[G_m] [\tilde{G}]\]

\(\sigma_{m+1}\) is ... 

\[g_{m+1} = g \cdot \tilde{g} = \{b \in B \cap S_{m+1} \mid b/g \in G\}\]

We know: \(G_m \cdot \tilde{g} = G_{m+1}\). Since \(\sigma_{m+1}(u_m) = \sigma_m(u_m)\), we have \(\sigma_{m+1}(q_m) = G_m\)
Hence $\sigma_{m+1} : \bar{N}[\sigma_{m+1}] \leq N[G_{m+1}]$ and $\sigma_{m+1}(\sigma_{m+1}) = G_{m+1}$. We also note that $(b_m \upharpoonright \bar{\sigma}_{m+1})/\bar{\sigma}_m \in \bar{q}'$.

By our construction:

- $q_{m+1} \upharpoonright \bar{\sigma}_{m+1} = q_m \upharpoonright \bar{\sigma}_{m+1}$,
- $\sigma_{m+1} : \bar{N}[\sigma_{m+1}] \leq N[G_{m+1}]$,
- $\sigma_{m+1}(\sigma_{m+1}) = G_{m+1}$,
- $\sigma_{m+1}(\underline{u}_m) = \sigma_m(\underline{u}_m)$

Now let $\underline{u}_{m+1} = \langle x_{m+1}, z^0, z^1, \ldots, z^m \rangle$, where $z^l$ is the least $z < \bar{\lambda}_l$ s.t. $\sigma_{m+1}(z) \geq \bar{\sigma}_{m+1}$ (for $l = 0, \ldots, m$).

All of this is forced by $c_{m+1}$. So there are terms $\tilde{\sigma}_{m+1}(\tilde{\sigma}_{m+1})\langle \underline{u}_{m+1} \rangle$, $\tilde{\sigma}_{m+1} = \tilde{\sigma}_{m+1} \circ G_{m+1}$, $\tilde{\sigma}_{m+1} = G_{m+1}$, $\underline{u}_{m+1} = \underline{u}_{m+1}$, and the above statements are forced.

This completes the construction.

QED (Case 3, 1)
Case 3.2 Case 3.1 fail
Then $\lambda > \omega_i$ is regular and $\overline{b}_i < \lambda$ for $i < \lambda$. Hence $\lambda$ is regular in $\bigcup IB_i$ for $i < \lambda$. Thus, by the definition of RCS-iteration, we have $IB_\lambda$ the minimal completion of $\bigcup IB_i$ (i.e., $\bigcup IB_i \setminus \{0\}$ is dense in $IB \setminus \{0\}$).

We again let $\Theta$ s.t. $IB \subseteq H_\Theta$ and $\Theta$ verifies the subproperness of $IB_i$ for $i < \lambda$. We claim that $\Theta$ verifies the subproperness of $IB_\lambda$. Let $\mu > 6$ be regular and let $N = \langle L[A], A, \ldots \rangle \thicksim H_\Theta \subseteq N$. Again, let $\sigma : N \subseteq N$ s.t. $(\overline{\Theta}, \overline{\lambda}, \overline{x}) = \Theta, IB, \lambda \setminus \overline{\sigma}$, where $\overline{\sigma}$ is countable and full.

Claim $\overline{\sigma}$ witnesses the subproperness of $IB_\lambda$ w.r.t. $N$.

We again let $\sigma(\overline{\sigma}) = \lambda$ and suppose that $\sigma(\overline{\lambda}_i) = \lambda_i$ for $i = 1, \ldots, m$, where $\lambda_i$ is regular and $\overline{b}_i < \lambda_i$ for $i = 1, \ldots, m-1$. We set $\lambda^m = \lambda$, We are given $a \in IB_\lambda \setminus \{0\}$ and claim that there is $b \in a \in IB_\lambda \setminus \{0\}$ s.t. whenever $G \ni b \in IB_\lambda$ generic, then there $\Theta \notin G \in [G]$ s.t. (a) $\Theta$ hold as before.

Note We are of course not constrained to prove $\sup \overline{\sigma}(\overline{\lambda}_m) = \sup \overline{\sigma}(\overline{\lambda}_m)$ since we do not have $\overline{b}_\lambda < \overline{\lambda}_m$. However, this will come out of the proof, and
including $\lambda = \lambda^m$ in our list of regular cardinals facilitates our proof. We shall, of course, exploit the fact that $\overline{1_B} \subseteq \lambda$ for $i \leq \lambda$.

To prove this we again pick a cofinal monotone sequence $\langle \overline{\delta}_i | i \leq \omega \rangle$ in $\lambda$ with $\overline{\delta}_0 = 0$. However, we are no longer able to enforce that $\varphi_h (\overline{\delta}_m) = \varphi_h (\overline{\delta}_n)$, where $\langle \varphi_h | h \leq \omega \rangle$ is the sequence of maps we intend to add, converging to $\overline{\sigma}$. This will make our construction more complex. We will be able to enforce $\sup \varphi_h (\overline{\chi}) = \sup \varphi_h (\overline{\sigma}) \lambda$.

We again let $\langle x_i | i \leq \omega \rangle$ enumerate $\overline{N}$ and $\langle d_i | i \leq \omega \rangle$ enumerate the strongly closed subsets of $\overline{B}_x = \varphi^{-1} (\overline{B}_x)$ in $\overline{N}$. We define the sequence $\langle b_i | i \leq \omega \rangle$ exactly as before. Set $\overline{\chi} = \overline{\chi}_m = \sup \varphi^{-1} (\overline{\chi}_m)$. Then $\overline{\chi} \subseteq \lambda$.

$\overline{B}_x$ will now in large part play the role that $\overline{B}_\lambda$ played in Case 3.1.

We again choose a monotone cofinal sequence $\langle \overline{\delta}_i | i \leq \omega \rangle$ in $\overline{\chi}_m = \sup \varphi^{-1} (\overline{\chi}_m)$ for $i = 0, \ldots, m$. We take $\overline{\delta}_i = \overline{\delta}_i \uparrow$ where $\langle \overline{\delta}_i | i \leq \omega \rangle$ is defined as above.
We wish to construct $c_m$ ($m < \omega$) which will play the same role as in Case 3.1. In particular $c_m^\ast$ should force the existence of $\bar{c}_m$ r. t.:

1. $q_m \in \bar{B}_{\bar{3}_m}^\ast$ - generic over $\bar{N}$
   - $b_m \cap \bar{3}_m \subseteq q_m$
   - $\bar{c}_m : N[q_m] \nrightarrow N[G_m]$ r. t. $c(q_m) = G_m$, however, $c_m^\ast$ cannot itself fix the value of $\bar{c}_m$, so it makes no sense to require $R(c_m, IB_{\bar{3}_m}(\bar{3}_m))$. The best we can require is that $R(c_m, IB_{\bar{3}_m})$ and that if $G_\lambda \exists C_m^\ast \in IB_{\bar{3}_m}$ then (1) holds with $G_m = G_\lambda \cap IB_{\bar{3}_m}(\bar{3}_m)$.

If $s = <y_0, \ldots, y_{m-1}>$ is any monotone sequence in $\bar{X}$, we simultaneously construct

- $e_s \in IB_{\bar{3}_m}$ (if $m = 0$)
- $e_s \cap C_m^\ast$ fixes the value of $\bar{c}_m(\bar{3}_m)$ as $y_h$ for $h < m$. We will then have $e_s \cap C_m^\ast \in IB_{\bar{3}_m}$. 


Moreover, if \( m > 0 \), then \( e_\lambda \subseteq C_{\lambda - 1}^* \).

(We can, of course, have \( e_\lambda = 0 \), but not for all \( \lambda \).)

**Def.** For \( m < \omega \) let \( S_m = \{ \text{the set of monotone} \}
\lambda \mapsto \lambda \). Set \( S = \bigcup_m S_m \). For \( \lambda \in S \)
let \( \lambda = \text{dom}(\lambda) = m \) if \( \lambda \in S_m \). We also
set \( \max(\lambda) = \sup\{\nu \in \lambda \} \) (hence \( \max(\emptyset) = 0 \)).

By induction on \( m \) we construct:

- \( c_m \ni \text{t. } P(c_m, IB^*_\lambda) \)
- \( \{ e_{\lambda} \mid \lambda \in S_m \} \ni \text{t. } e_{\lambda} \in IB_{\max(\lambda)} \)
- \( \{ \hat{\epsilon}_{\lambda} \mid \epsilon_{\lambda} \neq 0 \}, \{ \hat{\mu}_{\lambda} \mid \epsilon_{\lambda} \neq 0 \}, \{ \hat{\eta}_{\lambda} \mid \epsilon_{\lambda} \neq 0 \} \)

\( \ni \text{t. } \hat{\epsilon}_{\lambda}, \hat{\mu}_{\lambda}, \hat{\eta}_{\lambda} \in \bigvee \text{IB}_{\max(\lambda)} \).

We inductively verify:

(a). \( \emptyset = 1 \in IB_c \), \( e_{i\nu} \in IB_{\max(\lambda)} \)

(\where \( \lambda = 2^{<\nu} \))
- \( e_{i\nu} \neq e_{i\nu'} \) if \( \nu \neq \nu' \).
- \( \bigvee_{\nu} e_{i\nu} = e_{i\nu} \cap C_{1_{\lambda \nu}}^* \) (hence \( e_{i\nu} \subseteq C_{1_{\lambda \nu}}^* \)).
- \( e_{i\nu} \subseteq C_{1_{\lambda \nu}}^* (i') \) for \( i \geq \max(\lambda) \)

(hence \( e_{i\nu} \subseteq C_{1_{\lambda \nu}}^* = e_{i\nu}(C_{1_{\lambda \nu}} \cap \max(\lambda)) \subseteq IB_{\max(\lambda)} \))
- \( c_m (i') \subseteq c_m (i') \) for \( m < m \)

(hence \( C_m^* \subseteq C_m^* \))
- \( e_{i\nu} \cap C_{1_{\lambda \nu}}^* (i') = e_{i\nu} \cap C_{1_{\lambda \nu}}^* (i') \) for \( m \geq \lambda, i' \leq \max(\lambda) \)
(b) $\hat{\phi} = \phi$, $g'_\phi = \{1 \hat{\phi} \}$, $u_\phi = \hat{u}_\phi$, where
\[ u_\phi = \langle x_0, \hat{x}, \bar{x}_1, \ldots, \bar{x}_m, b, b_0 \rangle, \text{ where} \]
\[ \langle \chi, i < \omega \rangle \text{ enumerates } \bar{N}. \]

(c) $\mathcal{E}_\lambda \models c_{i, 1}^* \Rightarrow$ forces the following in $V^{|B|, \text{max}(a)}$:
\[ \tilde{g}' \models \tilde{B} \models \tilde{g}' \models \tilde{B}_1 \]
\[ (b_{\lambda, 1} \models \tilde{B}_{\lambda, 1})^* \leq g' \]
\[ \tilde{g}' : \bar{N} [g'] \prec \bar{N} [G], \text{ and } \tilde{g}' (g') = G, \]
where $G$ is the canonical generic name.

(d) If $\lambda = \bar{\nu}$, then $\mathcal{E}_\lambda \models c_{i, 1}^*$ forces:
\[ \tilde{g}' (u_{\lambda, 1}) = \tilde{v}' (u_{\lambda, 1}) \]
\[ u_{\lambda, 1} = \langle x_{\lambda, 1}, \bar{x}_1, \ldots, \bar{x}_m, g_0, b_{\lambda, 1} \models \tilde{B}_{\lambda, 1} \rangle, \text{ where} \]
\[ \bar{x}_l = \text{ the least } e \prec \bar{x}_l, \text{ s.t. } \tilde{g}' (\bar{v}_l) \models \tilde{g}' (\bar{x}_l) \]
\[ (l = 0, \ldots, m) \]
\[ e \models \mathcal{E}_{\lambda, 1} \models \text{max}(a), \tilde{g}' (\tilde{f}_{\lambda, 1}) = \nu \]

Note. By (a) we easily have: $n(i) \neq n'(i) \Rightarrow \mathcal{E}_{n, i} = 0$ for $n, n' \in S$.

We shall delay the construction of $c_{i, 1} \models \tilde{u}_1 \models \tilde{f}_1$ and the verification of (a) - (e) until later.
We note:

(1) Let $n > 0$. Then $\bigcup_{1 \leq l = n} c^*_{m-1}$


Proof: And, on $m$

$m = 1$: $\bigcup_{1 \leq l = m} e = e_0 = e_0 \cap c^* = 1 = c^*_m$

$m = m + 1$: $\bigcup_{1 \leq l = m} e = \bigcup_{1 \leq l = m} e_{l^*} = \bigcup_{1 \leq l = m} (e_{l^*} \cap c^*_m) = c^*_{m-1} \cap c^*_m = c^*_m$ Q.E.D

Our intention is to fuse the $c^*_m \ (m < \omega)$ into a $c \ i.t. \ R(c, IB^I \widetilde{X})$ just as in Case 3.1, but it will be somewhat trickier in the present care. We set:

$\overline{c}(i) = \bigcap_{m < \omega} c^*_m(i), \ \ c(i) = \overline{c}(i) \cup \overline{c}(i)$.

Claim ca) $b_i(c_\overline{c}, i^*) = \overline{c}(i)^*$ for $l \leq i \leq \widehat{i}$

(Hence $\overline{c}(i)^* = c(i)^*$)

(b) $R(c, c, i, IB^I \widetilde{X})$ for $i \leq \widehat{i}$.

The proof will be by induction on $i$.

In the following suppose that $j < i$ and that (a), (b) hold for $i < j$. Suppose moreover that $a \in U_i(c_\overline{c}, i)$ - i.e. there is $h < i$ s.t. $a \in c_\overline{c}(i)^*$ and $a \in IB^I_h$.

(2) $\text{anc}^*_m \neq 0$ for all $n < \omega$.

Proof. Clearly $a \in U_i(c_m^*)$. Let $a \overline{c}(i)^*, a \in IB^I_h, i < j$.

Then $a \overline{c}(i)^* \neq 0$. Also $\overline{h}(a \overline{c}(i)^*) = an \overline{c}_m^* = a \neq 0$ Q.E.D

(3) For $n < \omega$ there is $u \in S^m, u^*_m, \text{anc}^*_m \neq 0$.

Proof. $\bigcup_{1 \leq l = m + 1} c^*_m$. Hence $\text{anc}^*_{m+1}$ for an

$N \ni \bigcup_{1 \leq l = m + 1} c^*_m$. Hence $\text{anc}^*_{m+1} \text{anc}^*_m$, Q.E.D.
(4) Let \( a_n \cap c^*_m \neq 0 \) s.t. \( \max (a) < i \).

Then \( a_n \cap c^*_m \in U_i (\overline{e_i} i) \) (Moreover, 
\[ a_n \cap c^*_m \in I_B h, a_n \cap c^*_m \in (\overline{e_i} h)^* \] for an \( h < i \) s.t. \( h \geq \max (a) \).

**Proof**

Let \( m = \max (a) \). Then \( e_n \cap c^*_m = e_n \cap (c^*_m \cap \mu)^* = e_n \cap (c^*_m \cap \mu)^* \) for all \( m \geq n \) by (a). Hence \( e_n \cap c^*_m = e_n \cap (c^*_m \cap \mu)^* \) by the induction.

Let \( a \in I_B h, a \in (c^*_h)^* \), \( h < i \). Then \( a_n \cap c^*_m \in (c^*_h)^* \cap (c^*_h \cap \mu)^* \) where \( i = \max (c^*_h) \). Obviously \( a_n \cap c^*_m \in I_B i \).

**Q.E.D.**

(5) There \( i \) s.t. \( \max (a) \geq i \) and

\[ a_n \cap c^*_i = 0. \]

**Proof**

Let \( n \) be big enough that \( \frac{i}{m} = \frac{5}{m} \geq \gamma \)
(hence \( m > 0 \)). Set \( a' = a_n \cap c^*_m \neq 0 \) where \( 1 \leq m = \max (a) \), we are done.

At not, let \( m = \max (a) < i \). By (4) we have:

\[ a' \in U_i (\overline{e_i} i), a' \in I_B i, a' \in (c^*_i)^* \]

where \( \mu \leq i < i \). Note that

\[ e_n \cap c^*_m \cap (\overline{e_i} i) \geq \sum_{5 = 5}^{\frac{5}{m}} \gamma. \]

For each \( 5 < \lambda \) set \( d_5 = e_n \cap c^*_m \cap (\overline{e_i} i) \).

Then \( d_5 \in I_B m \) and \( e_n \cap c^*_m = \sum_{5} d_5 \).

Hence \( \alpha = a' \cap d_5 \neq 0 \) for some \( 5 < \lambda \).

Hence \( \alpha \in I_B i, \alpha \in (c^*_i)^* \).

Hence \( \alpha \in U_i (\overline{e_i} i) \).
Now let $5 < \frac{\gamma}{3p}$, where $\gamma \leq p$. Let 
$\tilde{a} \wedge c_{p+1} \neq 0$, where $|a'| = p+1$.

**Claim** \( \max(a') \geq \gamma \)

**Proof.**
Let $\tilde{a} \wedge c_{p+1} \in G$, where $G \subseteq B_l$ - generic.

Then $e_{\tilde{a}} \wedge c_{p+1} \in G$. Hence $\varepsilon_{p+1} = 5$, where $\varepsilon_{p} = \varepsilon_{p+1} G$, and $\sigma_{\tilde{a}}(5) \geq \varepsilon_{p+1} \geq \gamma$, where $\sigma_{\tilde{a}} = \tilde{a} G$. But $e_{\tilde{a}} \wedge e_{\tilde{a}} \neq 0$; hence $\varepsilon_{\tilde{a}} = 2^{|\tilde{a}|}$.

Hence $\delta_{\tilde{a}}(\varepsilon_{p+1}) = \sigma_{\tilde{a}}(\varepsilon_{p+1})$ and

$\gamma \leq \delta_{\tilde{a}}(5) \leq \delta_{\tilde{a}}(\varepsilon_{p+1}) = \max(a')$. QED (5)

We now prove the Claim by induction on $i \leq \tilde{a}$.

**Case 1** $i = 0$. Trivial since $\varepsilon_{\tilde{a}}^1 = \emptyset$.

**Case 2** $i = i^1 + 1$ (hence $i \leq \tilde{a}$).

Then $(c_{i^1})^* = (\varepsilon_{i^1})^*$ and $R(c_{i^1} \mid B_1^1 \tilde{a})$.

We must show;

**Claim** $h_i(c_{i^1} \setminus B_1^1 \tilde{a})^*$.

(Hence $(c_{i^1})^* = (\varepsilon_{i^1})^*$, $h_i(c_{i^1}) = 1$, and hence $R(c_{i^1} \mid B_1^1 \tilde{a})$)

Suppose not. Let $a = (c_{i^1})^* \setminus h_i(c_{i^1})$.

Then $a$ is in $(c_{i^1})$, since $a \in B_{i^1}^1$, ac$(\varepsilon_{i^1})^*$.

By (5) we can find $m < a$, $x \in S_m$

such that $\max(a) \geq m$ and $e_{\varepsilon_{m} \cap a} \neq 0$.

Choose $a$ minimal for this property,
Then \( \lambda = \lambda' \), where \( \max(\lambda') \leq \gamma \). Hence \( \lambda' \in \text{IB}_j \). By (a1) we have
\[
e^{-\lambda} c_m(h) = e^{-\lambda} c_m(h) \quad \text{for } m \geq n, h \leq 1.
\]
Hence \( e^{-\lambda} \alpha \neq (\lambda') \). But
\[
\lambda'\{e^{-\lambda} \alpha\} \neq (\lambda')\{e^{-\lambda} \alpha\} = 0
\]
Contradiction! \( \text{QED} \) (Case 2)

**Case 3** \( i' = \gamma < \lambda \), \( \text{Lin}(\gamma) \).

By the incl. hypothesis (a) holds below \( \gamma \).
Hence \( \text{GS}(c_{\gamma} \gamma, \text{IB}_j \gamma) \).

**Claim** \( \text{R}(c_{\gamma} \gamma, \text{IB}_j \gamma) \)

(Hence \( \text{R}(c_{\gamma} \gamma, \text{IB}_j) \) since \( \text{IB}_j \) is an RCS iteration.)

Let \( \alpha \in U_{\gamma}(c_{\gamma} \gamma), \alpha \in \text{IB}_j \setminus \gamma, \alpha \in (c_{\gamma} \gamma)^* \)
where \( i' \leq \gamma \). We must find \( \alpha' \in \alpha \)
\text{r.t. } \alpha' \in S_{\gamma}(c_{\gamma} \gamma).

**Case 3.1** There \( \alpha' \in \alpha \) r.t. \( \alpha' \in U_{\gamma}(c_{\gamma} \gamma) \)
and \( V_i \leq \gamma \) \( \alpha' \{f_i(V_i)\} = \theta \).

Then trivially \( \alpha' \in S_{\gamma}(c_{\gamma} \gamma) \).

**Case 3.2** Case 3.1 fails.

Let \( \alpha \in S_{\gamma} \) r.t. \( \max(\alpha) \geq \gamma \) and \( e^{-\lambda} c_m \alpha \neq 0 \)
with \( m \) chosen minimally. Then \( \lambda = \lambda' \),
\( V \geq \gamma \) and \( \max(\lambda') < \gamma \). Set:
\[
\alpha' = \alpha e^{-\lambda} c_{\gamma} c_{m-1}^{-1} \cdot \text{Then } \alpha \in U_{\gamma}(c_{\gamma} \gamma) \text{ with } \alpha' \in (c_{\gamma} \gamma)^*, \alpha' \in \text{IB}_j \text{ for } \alpha' \geq \max(\lambda') \text{ by (6).}
But \( an_c \in a \), and \( an_c \in IB_{\gamma} \), since \( \epsilon_1 \in IB_{\max(i)} \). Hence \( an_c \in U_{\gamma}(c \epsilon \gamma) \).

But then \( an_c \in U_{\gamma}(cm \epsilon \gamma) \). Hence there is \( \tilde{a} \subseteq a \) s.t. \( \tilde{a} \in S_{\gamma}(cm \epsilon \gamma) \).

Hence \( \tilde{a} \subseteq (cm \epsilon i)^* \) for all \( i < \gamma \) since Case 3.1 faith. More over \( \tilde{a} \subseteq IB_i \) for any \( i < \gamma \).

By (a), \( e_{\epsilon_1 \gamma}(cm \epsilon i)^* = e_{\epsilon_1 \gamma}(cm \epsilon i)^* \) for all \( m \geq 1, i < \gamma \), since \( \max(a) \gamma \).

Hence \( \tilde{a} \subseteq (c \epsilon i)^* \) for \( i < \gamma \), since a e \( c \).

Hence \( \tilde{a} \subseteq S_{\gamma}(c \epsilon \gamma) \). QED (Case 3, 2)

Case 4. \( i = \tilde{x} \).

Then \( R(c \epsilon\gamma, IB_1 \tilde{x}) \) for \( i < \tilde{x} \) and \( cf(\tilde{x}) = \omega \) in \( \mathcal{V} \). The conclusion is trivial.

QED (Claim)

Since \( R(c, IB_1 \tilde{x}) \) and \( IB_1 \) an RCS iteration, we conclude: \( R(c, IB) \). (In particular, \( R(c, IB_1 \tilde{x}) \)).
We now show that \( c \) has the desired properties. Let \( G \) be \( 1B_\alpha \) - generic with \( c^* \in G \). We claim that there is \( \vec{a} \in V[G] \) such that 

(a) \( \sigma_0 \colon \vec{N} \prec \vec{N} \)

(b) \( \sigma_0 (c_1, \vec{B}, \vec{x}) = c_1 B_1, \ldots, c_m \) (\( i = 1, \ldots, m \))

(c) \( \sup \sigma_0 (\vec{x}_i) = \vec{x}_i \) (\( i = 0, \ldots, m \))

(d) \( \vec{a} \in \vec{q} = \sigma_0^{-1} \vec{c} \) and \( \vec{q} \in \vec{B} \) - generic over \( \vec{N} \).

We know that \( c^*_m \in G \) for \( m < \omega \). Since \( c^*_m = \omega \cup \omega \), we know that for each \( m \) there is exactly one \( \vec{a} \) such that \( \vec{a}_m = \omega \cup \omega \) and \( \vec{a}_m \in G \).

Set \( \vec{a} = \lim_{n \to \omega} \vec{a}_n = \langle c^n_i \mid i < \omega \rangle \). Since \( c^*_m \cap \vec{a}_m \downarrow \vec{a}_m \), \( \vec{N} \models \int_{\vec{a}_m} \vec{N} [\vec{q}_m] \prec \vec{N} [G_m] \), we have that \( \vec{a}_m : \vec{N} [\vec{q}_m] \prec \vec{N} [G_m] \), where 

\( \vec{q}_m = \vec{q}_m \Downarrow \vec{a}_m = \lim_{n \to \omega} c^n_m \), \( G_m = \vec{N} \cap G_m \), \( G_m = \vec{N} \cap G_m \), \( G_m = \vec{B} \vec{x}_m \), and \( \vec{q}_m (\vec{x}_m) = G_m \). Clearly \( \vec{q}_m (\vec{x}_i) = \vec{x}_i \) for \( i < m \).

Hence we can define \( \vec{a} : \vec{N} \prec \vec{N} \) exactly as before and \( \vec{q}_m \) too.

Finally, note that exactly as before, \( \lim_{n \to \omega} \vec{x}_i = \vec{x}_i \) (\( i = 0, \ldots, m \)) and hence \( b_m \in \vec{a}_m \). Moreover, \( b_m \in \vec{B}_\alpha \) - generic over \( \vec{N} \), exactly as before.

Thus \( \vec{q} \in \vec{B}_\alpha \) - generic over \( \vec{N} \). But \( b_m \in \vec{a}_m \), hence \( \vec{a} \in \vec{q} \). Q.E.D.
We have thus shown that $1 \mathcal{B}_x$ is not proper. All that remains is to define $e_1, e_2, e_3, \ldots$, and verify (a) – (e). By recursion on $m$ we define

$$\Gamma_m = \langle c_m, \langle e_2 | 12! = n \rangle, \langle \hat{e}_2 | 12! = m \wedge e_2 \neq 0 \rangle, \langle \hat{e}_3 | 12! = m \wedge e_3 \neq 0 \rangle, \langle \hat{e}_4 | 12! = m \wedge e_4 \neq 0 \rangle \rangle$$

and verify (a) – (e) (e.g., (e) will then be verified for $12! = n$), $\Gamma_m$ is defined by (a), (b) for $n = 0$. The verifications are trivial. Now let $\Gamma_m$ be given s.t. (a) – (e) hold. Before proceeding further, we note that by the disjoint distributive law used in the construction of $\langle b_m | m < \omega \rangle$, the following holds in all complete BA's:

(1) Let $b = \bigcup_{i \in I} b_i \wedge b_i \neq 0$ for $i \neq 1$, and $a_i \subseteq b_i$. Then $\bigcap_i (a_i \cup b_i) = \bigcup_i a_i$.

Set $a_i^0 = \{ a_i \mid i \neq 1 \}$. Then

$$\bigcap_i (a_i \cup b_i) = \bigcup_i a_i^0 = \bigcup_i a_i = \bigcup_i a_i^0 = \bigcup_i a_i.

\text{QED (1)}$$

We now define $e_1$ for $n = m + 1$:

(2) $e_{1/2} = c_m \wedge e_2 \wedge \left[ y = \frac{1}{2} \left( \frac{1}{2} \right)_{m+1} \right] B_{m+1} (x)$

for $12! = n$. 

We immediately get:

$$e_{\alpha'} = c^* \wedge e_{\alpha}$$

- \(e_{\alpha'} \wedge e_{\alpha'} = 0\) for \(\alpha \neq \alpha'\)
- \(e_{\alpha'} \in 1B_{\text{max}(n)}\)

for \(1 \leq n \leq m\). From this we can prove:

$$e_{\alpha} = c^*$$

as before.

Now let \(G \in e_{\alpha} \), be \(B_{\text{max}(n)}\) - generic, where

\(1 \leq n \leq m\). Set \(\alpha = \frac{x}{a}, u_\alpha = \frac{u}{a}, q_\alpha = \frac{q}{a}\). Then

\(\alpha \in N[q_\alpha] \subseteq N[G], \quad \alpha(q_\alpha) = G, \quad \text{and} \)

\(N[q_\alpha] \cup \text{full in } V[G]\). Since

\(\nu = \alpha(\frac{x}{a}, \frac{n}{a}) \in G\), we have \(\alpha(\frac{n}{a}) = \nu\).

Set \(\tilde{B} = \{B_x / G\}. Then \tilde{B} \cup \text{full proper in } V[G]\). Clearly

$$\alpha(\tilde{B}_{\frac{x}{a}} / q_\alpha) = B_x / G = \tilde{B}.$$ 

Moreover, \(\sup \tilde{\varepsilon}^* x_{\alpha} = X_{\alpha}^*\) for \(1 \leq \alpha \leq m\),

Recall that \(h_{\frac{x}{a}} (b^*)_m = h_{\frac{x}{a}} (b^*_{m+1}) = (b^*_{m+1})^* = (b_m^* | \frac{b_{m+1}}{a})^*\). Since

\(\nu (b_m^* | \frac{b_{m+1}}{a})^*/q_\alpha \in q_\alpha\), we have

\((b_m^* | \frac{b_{m+1}}{a})^*/q_\alpha \neq 0\). Hence,
\[ b \mid a \cdot (b \cdot \bar{b}) \]
(\$G$ being the canonical generic name in \(\mathcal{U}^{\text{IB}_1(\text{max}(\text{i}))}\).

We may also assume:

\begin{align}
(7) & \vdash (\forall \mu \in \mathcal{C}_1 \rightarrow c_\mu) \text{ for all } \mu \\
(8) & \vdash (\forall \mu \in \mathcal{C}_1 \rightarrow \mathcal{C}_0) = 1 \text{ for all } \mu
\end{align}

But then there is a unique \(d = d_\mu = \gamma \in \text{IB}_1(\text{max}(\text{i}))\) such that \(d(i) \in \text{IB}_1(\text{i} + 1)\) and

\begin{align}
(9) & \vdash d(i)^T = \mathcal{C}_0(i - \text{max}(\text{i})) \text{ for } i \geq \text{max}(\text{i}) \\
\end{align}

and \(d(i) = 1 \text{ for } i < \text{max}(\text{i})\).

Hence:

\begin{align}
(10) & \vdash \text{R}(d, \text{IB}_1(\text{max})) \text{ since } \text{R}(d, \text{IB}_1(\text{max})\text{)} \text{ and} \\
\end{align}

\begin{align}
(11) & \vdash \text{R}(d, \gamma^T, \text{IB}_1(\text{max}))^T, \text{ where } \mu = \text{max}(\text{i})
\end{align}

Define \(d_\mu = \langle d_{\mu}(i) \mid i < \text{max}(\text{i}) \rangle\) by:

\[d_{\mu}(i) = \bigcup_{\mu_1 = \mu} d_{\mu_1}(i),\]

We now analyze \(d_{\mu}\).

Let \(\langle \tilde{e}_n \mid 1 \leq n \leq g \rangle\) be a 1-1 enumeration of \(\{e_\mu \mid \mu_1 = \mu, e_\mu \neq 0\}\).

Let \(\tilde{e}_0 = 1\) and \(\tilde{e}_n = \tilde{e}_0 \cap \mathcal{C}_0^* \text{ (by (11))}\)

Let \(h < 2\) and:

\[d_{\mu}(i) = \begin{cases} 
\tilde{e}_0 & \text{if } h = 0 \\
\bigcup d_{\mu_1}(i) \cap \tilde{e}_n & \text{if } \tilde{e}_n = e_\mu
\end{cases}\]

Then:
\[(11) \quad 1 = d^0(i) \cup \bigcup_{i \neq 0} \tilde{E}_i \]

- \[d_{\nu'}(i) = d^\nu(i) \cup \bigcup_{i \neq \nu} \tilde{E}_i \quad \text{for} \quad \tilde{E}_h = e_{1,1} \]

\[M \overline{B} \]

The first equation is clear. The second is trivial for \( i < \max(a) \) since then \( d^\nu(i) = 1 \).

Let \( i \geq \max(a) \). (C) is trivial, since \( d_{\nu'}(i) = d^\nu(i) \cup \bigcup_{i \neq \nu} \tilde{E}_i \).

\[\tilde{E}_h \]

Let \( l \neq h \). Then \( \tilde{E}_l \cap \bigcup_{i = \max(a)}^{\nu} \tilde{E}_i \neq \emptyset \). Hence \( \tilde{E}_l \cap \frac{d^\nu(i)}{\nu} = \left( \nu - \max(a) \right) = 1 \). Hence \( \tilde{E}_l \cap d_{\nu'}(i) \), since otherwise, letting \( \alpha = \tilde{E}_l \setminus d_{\nu'}(i) \), we have all \( d_{\nu'}(i) \cap \alpha = 0 \).

\[\square \text{ED (11)}\]

Hence:

\[(12) \quad d_m(l') = \bigcap_{h \geq 0} \left( d^h(i) \cup \bigcup_{\nu \neq h} \tilde{E}_i \right) = \bigcup_{h \geq 0} d^h(i) \]

by (11). (Hence \( d_m(i) \neq \emptyset \)).

Since \( d_{\nu'}(i) \in B_{i+\alpha} \), for all \( i \), the definition of \( d_m \) gives:

\[(13) \quad d_m(i) \in B_{i+\alpha}.\]

Since \( h_1(d_{\nu'}(i)) = 1 \), we have:

\[(14) \quad h_1(d_m(i)) = \bigcup_{i \in \tilde{E}_l} \left( h_1(d^\nu(i)) \cap \tilde{E}_l \right) = 1 \]
By (12):

\[ (15) \quad \tilde{e}_0 \cap d_m(i) = d^0(i) = \tilde{e}_0 \]
\[ \tilde{e}_n \cap d_m(i) = \tilde{d}_n(i) = \tilde{e}_n \cap d_m(i) \quad \text{where} \quad \tilde{e}_n = e_n. \]

\[ (16) \quad \tilde{e}_0 \subset d_m^* \]
\[ \tilde{e}_n \cap (d_m(i))^* = \tilde{e}_n \cap (d_n(i))^* \quad \text{for} \quad \tilde{e}_n = e_n. \]

We now define:

\[ \tilde{C}(i') = C_m(i') \cap d_m(i') \quad \tilde{C}(i') = C_{m+n} \]
\[ \tilde{C}(i') = \tilde{C}(i') \cup \tilde{C}(i) \text{ for } i' < \tilde{x}, \]

\[ (17) \quad R(c_{i'}, \overline{B} \tilde{x}) \quad \text{for } i \leq \tilde{x} \]

(Hence R(c_{1}, B \tilde{x}) \text{ since } \overline{B} \text{ is an RCS iteration})

By induction on \( i \leq \tilde{x} \) we show:

(a) \( h_{d'}((\tilde{c}_{i'}))^* = (\tilde{c}_{i'})^* \quad \text{for } d \leq d' \)

(Hence \( h_{d'}((\tilde{c}_{i'}))^* = (\tilde{c}_{i'})^* \) and \( GS(\tilde{c}_{i'}, B) \).

(b) \( R(c_{i'}, \overline{B} \tilde{x}) \).

**Case 1** \( i = 0 \). Trivial since \( c_{i'} = \tilde{c}_{i'} = \emptyset \).

**Case 2** \( i = i' + 1 \)

It suffices to show:

\[ (\tilde{c}_{i'})^* \subset h_{i'}(\tilde{c}_{i'}) \]

(Hence \( (\tilde{c}_{i'})^* = (c_{i'})^* \) and \( h_{i'}(\tilde{c}_{i'})^* = (\tilde{c}_{i'})^* = (c_{i'})^* \).
Suppose not. Let $a = (e_1)^* \setminus h'_1(c_1')$

**Case 2:1** Let $e_1 = 0$ where $i_1 = n$, $\max(n_1) = 1$.
Set $a' = a e_{1\nu}$. Then $a' \subseteq c_m^* \subset c_m(1')$. Hence

$$a' \cap e_1 = a' \cap c_m(1') \cap d_m(1') = a' \cap d_m(1')$$

But $a' \cap d_m(1') = 0$, since $h'_1(a' \cap d_m(1')) = a' \cap h'_1(c_1') = 0$. (Since $a' \subseteq B_1$). Hence

$$a = a \cap h'_1(c_m(1')) = h'_1(a \cap d_m(1')) = h'_1(a \cap d_m(1')) = 0, \text{ Contr! QED}$$

**Case 2:2** Case 2:1 fails.
Set $I = \{i : e_i \cap a = 0 \}$. Then if $l \notin I$ we have either $l = 0$ and $e_l \subseteq c_l^*$ or $e_l = e_{1\nu}$, $\max(n) = 1$. Hence

$$\bar{e}_l \cap d_m(1') = e_l \cap (c_l(1')^*) = e_l$$

since $(c_l(1')^*) = 1$, Hence $\bar{e}_l \subseteq (d_m(1')^*)$. Hence

$$a = \bigcap_{l \in I} e_l \subseteq c_m(1')^* \cap d_m(1')^*$$

As before, $a \cap e_l = 0$. Hence

$$a \cap c_m(1') = a \cap c_m(1') \cap d_m(1') = a \cap d_m(1') = 0$$

Hence $a = a \cap h'_1(c_m(1')) = h'_1(a \cap c_m(1')) = 0, \text{ Contr! QED(Case 2)}$
Case 3 \( i = \gamma \), \( \text{Lim}(\gamma) \).

Since \( R(c^1_\gamma, IB_\gamma^\times) \) for \( i < \gamma \), we have \( GS(c^1_\gamma, IB_\gamma^\times) \).

Claim \( R(c^1_\gamma, IB_\gamma^\times) \).

Let \( a \in U_\gamma(c^1_\gamma) \). We must find \( a \notin \text{Lim} \), \( a \notin \Sigma(c^1_\gamma) \). Suppose not. Then there \( \exists \mu \neq a \leq a \), \( a \in U_\gamma(c^1_\gamma) \) and \( \forall j < \gamma \), \( a^j \text{Lim}(\gamma) = \omega \), since then \( a \notin \Sigma(c^1_\gamma) \).

Let \( a \in IB_\gamma \setminus \{ \emptyset \} \), \( a \in (c^1_\gamma)^* \), where \( i < \gamma \).

Since \( (a) \) holds below \( \gamma \), we know \( (c^1_\gamma)^* = (c^0_\gamma)^* \) for \( i \leq \gamma \).

Case 3. \( \exists \mu \neq a \) where \( \| \mu \| = \mu \) and \( \text{max}(a, \mu) < \gamma \).

Let \( \text{max}(a, \mu) = a \). Then \( a \in \Sigma(a) \) and \( a \in \Sigma_\mu \). Set \( a' = a \in \Sigma(a) \). Then \( a' \in \Sigma(a) \), \( (c^1_\mu)^* = c^1_\mu \cap (d^1_\mu)^* = c^1_\mu \cap (d^1_\mu)^* = 1 \). Hence \( a' \in (d^1_\mu)^* \) and \( a' \in \Sigma(c^1_\mu) \) and \( a' \in IB_\mu \), where \( j = \text{max}(c^1_\mu, \mu) \). Hence \( a' \in U_\mu (d^1_\mu)^* \).

Hence there \( \exists \tilde{a} \in S_\gamma (d^1_\mu)^* \) set \( \tilde{a} \leq a' \). Then \( \tilde{a} \in (d^1_\mu)^* \) and \( \tilde{a} \in IB_\mu \) for \( \text{all } \mu < \gamma \).
Hence $\tilde{c} \subseteq \xi_{v} \cap (d_{1} \gamma)^{*} = \xi_{v} \cap (d_{m} \gamma)^{*} = \xi_{v} \cap (d_{m} \gamma)^{*} \cap (c_{m} \gamma)^{*} \cap (c_{1} \gamma)^{*} = (c_{1} \gamma)^{*}$. Hence $\tilde{c} \subseteq \alpha$ and $\alpha \in S_{\gamma}(c_{1} \gamma)$. Contradiction! QED (Case 3.1)

Case 3.2 Case 3.1 fails.

Set $I = \{ h | a \tilde{e}_{h} \neq 0 \}$. If $h \in I$, then either $h = 0$ and $\tilde{c} = e_{1}$, or else $\tilde{e}_{h} = e_{1}$ where $\max(h) \geq \gamma$. Hence $\tilde{e}_{h} \subseteq (d_{1} \gamma)^{*} = 1$. Hence $\tilde{e}_{h} \subseteq \xi_{v} \cap (d_{1} \gamma)^{*} = \xi_{v} \cap (d_{m} \gamma)^{*}$. Then $\alpha \subseteq \cup_{h \in I} \tilde{e}_{h} \subseteq (d_{m} \gamma)^{*}$. Thus $\alpha \subseteq \cup_{h \in I} \tilde{e}_{h} \subseteq (d_{m} \gamma)^{*}$ and $\alpha \in B_{1}$, hence $\alpha \in U_{\gamma}(c_{m} \gamma)$. Set $a \subseteq a \cap t$, $a \in S_{\gamma}(c_{m} \gamma)$. Then $a \subseteq (c_{m} \gamma)^{*} \cap (d_{m} \gamma)^{*}$, where $a \subseteq B_{1}$, for $a \subseteq B_{1}$. Hence $a \subseteq (c_{m} \gamma)^{*} \cap (d_{m} \gamma)^{*} = (c_{m} \gamma)^{*} = (c_{1} \gamma)^{*}$. Hence $\alpha \in S_{\gamma}(c_{1} \gamma)$, Contradiction! QED (17)
This gives us $c_{n+1}$ not $P(c_{n+1} \cup B)$. We verify the remaining cases of (a).

- $e_{r_2} \subset c_{n+1}$ (c) for $|s| = n$, $i \geq r = \max(s)$,
  since $e_{r_2} \subset c_n$ and $e_{r_2} \cap d_n (i) = e_{r_2} \cap d_n(i) = e_{r_2}$,
  since $d_n(i) = 1$ for $i \geq r$.

- $\left( \frac{|d_n(i)|}{c_{\max(s)}} = c_{\max(s)}^{-1} \right) = 1$ for $i \geq r$,
  since $|d_n| = \max(s)$.

- $c_{n+1}(i) = c_n(i) \cap d_n(i) \subset c_n(i)$ is total.

- $e_{r_2} \cap c_m(i) = e_{r_2} \cap c_{n+1}(i)$ for $|s| = n$, $i < \max(s)$,
  since $e_{r_2} \cap c_{n+1}(i) = e_{r_2} \cap c_m(i) \cap d_m(i) = c_m(i) \cap e_{r_2} \cap d_m(i) = c_m(i) \cap e_{r_2}$, since $d_m(i) = 1$ for $i < \max(s)$.

Thus uses $e_{r_2} \cap d_n(i) = e_{r_2} \cap e_{r_2} = 0$ if

$\tilde{e}_n = e_{r_2}$, not $d(i)$, and

$c_{n+1}(i) \cap d_0(i) = c_n(i) \cap (d(i) \cup c_n) = 0$.

This completes the verification of (a),

(b) needs no further verification.

We now define $\tilde{g}_2$, $\tilde{g}_2$ and verify (c) for $|s| = n+1$. 
Let \( e_{t, v} \cap G^{n+1} \in G \), where \( m = n, e_{t, v} \neq 0 \), and \( G : IB_{v} \)-generic. Let \( \alpha = \max(\varepsilon) \). Set \( G_{\mu} = G \cap IB_{\mu} \). Then \( e_{\nu} \cap G_{\mu} \in G_{\mu} \) and \( G_{\mu} : IB_{\mu} \)-generic. Set \( \hat{B} = IB_{v} / G_{\mu} \), \( \hat{G} = \{ b G_{\mu} \mid b \in G \} \). Then \( \hat{G} \) is \( \hat{B} \)-generic over \( V[G_{\mu}] \) and \( V[G_{\mu}] [\hat{G}] = V[G] \).

Moreover \( G = G \times \hat{G} = \{ b \} b G_{\mu} \in \hat{G} \} \).

Choose by \( \nu = \frac{\nu}{\mu} (\bar{x}_{m}) \in G_{\mu} \), so \( \nu = \bar{x}_{m} (\bar{y}_{m}) \), where \( \bar{x}_{m} = \frac{\nu}{\mu} e_{\nu} \). Set \( \nu_{m} = \frac{\nu}{\mu} \).

Then \( \hat{G} : N[\nu_{m}] \leq N[G_{\mu}] \), \( \mathcal{U}(\nu_{m}) = G_{\mu} \), and \( \nu_{m} : IB_{\mu} \)-generic over \( N \). Since \( e_{\nu} \cap G_{\mu} \in G_{\mu} \) and \( d_{v} \cap G_{\mu} \in d_{v} \in G \), \( d_{v} / G_{\mu} = \hat{G} \), where \( \hat{G} \) is \( G \)- and \( (5) \).

Then in a \( \hat{G} \in V[G] = V[G_{\mu}] [\hat{G}] \) satisfying (a)-(d) of (5).

Then, in particular, \( \hat{G} = \hat{G} - \frac{\nu}{\mu} e_{\nu} \) is \( \hat{G} : IB_{\mu} / \mu_{\hat{G}} \)-generic over \( N[\nu_{m}] \). \( \hat{G} \)

expands uniquely to a \( \sigma^{*} \) s.t.

\( \hat{G} : N[\nu_{m}] [\hat{G}] \leq N[G_{\mu}] [\hat{G}] \) and

\( \hat{G} = \bar{G} \). As remarked before, we know that \( (b_{n} \bar{x}_{n+1}) \bar{y} / \eta \in \hat{G} \).
We can wisely take \( \sigma^* = \frac{\partial}{\partial x} \), where \( \Sigma_{m+1} \) forces \( \tau_{uv} \) to have the above properties.

If we set: \( \bar{q}_{uv} = \{ b \in \mathbb{R}^2 \mid b/q \in \bar{q}_{m+1} \} \), then \( \tau_{uv} (\bar{q}_{uv}) = G \) and \( \bar{q}_{uv} \in \mathbb{R}^2 - \text{generic over } N \).

It follows easily that

\[ \sigma^* : \mathbb{C} \mathbb{N} \mathbb{L} [\bar{q}_{uv}] \rightarrow N \mathbb{C} \mathbb{L} [G] \quad \text{and} \quad \sigma^* (\bar{q}_{uv}) = G. \]

Moreover, \( b_{m+1} \bar{q}_{m+1} \in \bar{q}_{uv} \). But we know that \( b_{m+1} \bar{q}_{m+1} = b_{m+1} \bar{q}_{m+1} \). We can arrange \( \bar{q}_{uv} = \bar{q}_{m+1} \) where \( \Sigma_{m+1} \) forces the above to hold. The verification of (e1) is then immediate. It remains only to define \( \bar{q}_{uv} \) and to verify (e1), but this is straightforward. The verification of (e1) follows from the def. of \( \Sigma_{uv} \).

QED (Theorem 5)
Lemma 6 Thm 5 holds with "subcomplete" in place of "subproper",

Proof (sketch for $i = \lambda, \lim(\lambda)$)

Let $\sigma: \mathbb{N} \to \mathbb{N}$ be as before. We claim that $X = \sigma$ is a witness to the subcompleteness of $\mathcal{B}_\lambda$.

Let $\mathcal{B}_\lambda$ be $\mathcal{B}_\lambda$-generic over $\mathbb{N}$. Let $\mathcal{B}_i$ (for $i < \omega$) be as before and set: $q_i = q \cap \mathcal{B}_i$.

Consider the Case 3.1. The construction is the same except that $q_i$ plays the role of $q_m$. The sequence $\langle b_m, m \in \omega \rangle$ is omitted. In constructing $c_{m+1}$, $\sigma_{m+1}$ etc. from $c_m, \sigma_m$ etc., we let $G_m \supseteq c_m$ be $\mathcal{B}_m$-generic, define $\mathcal{B}$ as before, and note that by the subcompleteness of $\mathcal{B}_i$, there is $\bar{c} \in \mathcal{V}[\mathcal{D}_m]$.

Choose $\bar{c} \in \mathcal{B}$ which forces $(*)$ (a) (d) as before, except that now $\bar{c} = \langle b_q, q \in \mathcal{B}_m \rangle$.

Having constructed $c_m, \sigma_m$ etc. ($m < \omega$), we again set $c = \lim c_m$ (hence $c = \lim c_m$), and let $G \supseteq c$ be $\mathcal{B}_{\lambda}$-generic.

We define $\bar{c}$ as before and again observe that $\bar{c} \in \mathcal{B}_m$ (m < \omega).
The verification are as before except that we must now verify \( y = \overline{\sigma}^{-1} G \). Let be a

Claim: \( \overline{\sigma}(b) \subseteq G \).

Recall that by careful prior factoring we know that either of \( (x) = \omega \) in \( V \)

or of \( (x) = \omega_1 \) in \( V \). In the first case,

\( \langle \overline{\sigma} \rangle_{m < \omega} \in \overline{\mathbb{N}} \) and \( \overline{B}_x \) is the inverse

limit of \( \langle \overline{\sigma} \rangle_{m < \omega} \). By the genericity

of \( \overline{\sigma} \) there exist \( \langle b_m \rangle_{m < \omega} \subseteq \overline{\mathbb{N}} \) s.t.

\( b_m \in \overline{\mathbb{N}} \) and \( \bigcap b_m \subseteq b \). Hence \( \langle \overline{\sigma}(b_m) \rangle_{m < \omega} \in \overline{\mathbb{N}} \) and \( \overline{\sigma}(\bigcap b_m) = \bigcap \overline{\sigma}(b_m) \in \overline{\sigma} \) by

the genericity of \( \overline{\sigma} \), but \( \overline{\sigma}(\bigcap b_m) \subseteq \overline{\sigma}(b) \).

An the second case, \( \overline{B}_x \) is the dual limit and hence there is \( b' \subseteq y \) s.t.

\( b' \subseteq b \) and \( b' \subseteq \overline{\mathbb{N}} \) for some \( m \), hence

\( \overline{\sigma}(b') \subseteq \overline{\mathbb{N}} \subseteq G \) and \( \overline{\sigma}(b') \subseteq \overline{\sigma}(b) \).

QED (Claim)

An Case 3: 2 the construction of \( \overline{G} \) is as before, but \( \overline{G} \) takes

the place of \( \overline{\sigma} \) for \( \overline{G} \subseteq \overline{G} \). The verification

of \( y = \overline{\sigma}^{-1} G \) is straightforward, since

\( \overline{B}_x \) is a direct limit. QED (Lemma 6)
Without proof we mention:

Lemma 7 Thm 5 holds with "semi-subproper" in place of "subproper".

We can prove a slightly less restrictive version of Theorem 5. We first modify the definition of subproper to:

Define a complete BA in subproper above \( \alpha \) for sufficiently large \( \theta \) we have:

Let \( B \in H_\theta, N = \langle L_\xi[A], \ldots \rangle \) be as before with \( \mu < \theta \). Let \( \sigma : \bar{N} \rightarrow N \) be as before with \( \sigma(\bar{\mu}) = \mu \). For any \( a \in \bar{B} \setminus \{0\} \) there is \( b \in \sigma(\bar{a}) \) which forces that if \( G \models B \) -
gen 1: There is \( \bar{v} \in V[G] \) satisfying

\[(a)-1d) \text{ as before and } \sigma^* H(\bar{N}) = \sigma^* H(\bar{N}).\]

(Note Even subproper forcing is subproper above \( \omega + 1 \).)

Theorem 8 Let \( B = \langle B_\xi \ | \xi \leq \alpha \rangle \) be an RCS iteration

\[\mu \in H(\bar{B}_\xi) \text{ is subproper above } \mu_i,\]

where \( \mu_i \leq \mu \) for \( i < \xi \). Suppose moreover, that \( B_\alpha := \text{coll} \bar{B}_\alpha \rightarrow \mu_i \). Then each \( B_\xi \in \mu_0 \)-subproper.

(Note \( B_{\xi+1} \) will certainly collapse \( \bar{B}_\xi \) to \( \mu_i \) if \( \mu_i \geq \bar{B}_\xi \).)
The proof is virtually the same. We first redo the proof of Lemma 2 to get:

**Lemma 9.** Let \( \mathcal{B}_0 \subseteq \mathcal{B}_1 \), where \( \mathcal{B}_0 \) is subproper above \( \mu_0 \) and \( \mathcal{B}_0 \) is subproper above \( \mu_1 \),

where \( \mu_1 \geq \mu_0 \). Then \( \mathcal{B}_1 \) is subproper above \( \mu_0 \).

The proof is virtually unchanged. The proof of Lemma 8 involves only a slight modification of the original proof. In Case 2 we use Lemma 9. In Case 3.1 we now have: \( \forall \gamma \forall \gamma \leq \mu \) for some \( \gamma < \lambda \). By Lemma 9 it suffices to show \( \forall \gamma \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \).

Hence we may assume \( \forall \gamma \forall \gamma \leq \mu \) that \( \gamma = 0 \) and \( \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \).

Letting \( X \) be as before (with \( \mu \in X \)), there is \( \forall \gamma \leq \mu \) \( \forall \gamma \leq \mu \) such that \( \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \).

Hence we may assume \( \forall \gamma \forall \gamma \leq \mu \) that \( \gamma = 0 \) and \( \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \).

Defining \( f \) as before, we are guaranteed that \( \forall \gamma \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \).

If \( \forall \gamma \forall \gamma \leq \mu \) \( \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \). We run the proof of Case 3.1 exactly as before, ensuring that \( \forall \gamma \forall \gamma \leq \mu \) \( \mathcal{B}_1 \) is subproper above \( \mu_1 \) for all \( \gamma \).
The proof in Case 3.2 is unchanged except that we assume:
\[ c_{\infty} \leq e_1 \leq 2 \mu_0 = c_{\max(\mu)} \]
for \( e_1 \neq 0 \).

The definition of subcomplete can of course also be altered to give: subcomplete above \( \mu \). Clearly we get:

Lemma 10 Theorem 8 holds with "subcomplete" in place of "subproper".

This could conceivably be of interest in connection with Prikry forcing, since the forcing for adding a Prikry sequence to a measurable \( \kappa \) is subproper above \( \kappa \) for all \( \mu > \kappa \).