§3 Examples

§3.1 The transfer lemma for embeddings of ZFC⁻ models

We recapitulate and expand upon some facts developed in [J] §5. *)

Def Let $M = \langle |M|, \in, \ldots \rangle$ be a transitive ZFC⁻ model. Let $\pi : M \prec M'$, where M' is transitive. π is cofinal in M' iff $M' = \bigcup_{u \in M} \pi(u)$.

In the following, we suppose $M, N, \ldots$ to be transitive ZFC⁻ models unless otherwise stated.

Fact 1 Let $\pi : M \prec M'$ and set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec M'$ and $\pi : M \prec \tilde{M}$ cofinally.

(The proof uses: Let $x_1, \ldots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$. Then $\tilde{M} \models \varphi(x_1, \ldots, x_n) \leftrightarrow \langle x_1, \ldots, x_n \rangle \in X$, where $X = \{ \langle \vec{z} \rangle \in u_1 \times \ldots \times u_n \mid M \models \varphi(\vec{z}) \}$.)

Hence:

Fact 2 Let $\tau > \omega$ be regular in M, where $\pi : M \prec M'$. Set $\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$. Then $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally.

Def Let $\pi : M \prec M'$. Let τ be regular in M. π is τ-cofinal iff
$$M' = \bigcup \{ \pi(u) \mid u \in M \wedge \bar{\bar{u}} < \tau \text{ in } M \}$$
(Hence τ-cofinality implies cofinality.)

*) In ZFC⁻ "the axiom of choice reads": Every set is enumerable by an ordinal.

Def Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$. Let $\bar{\pi} : \bar{H} \prec H$ cofinally. By a *liftup* of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. M' is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ τ-cofinally.

(We also say: "$\langle M', \pi \rangle$ is a liftup of M by $\bar{\pi}$".)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. There is at most one liftup $\langle M', \pi \rangle$.

proof:

Clearly, every element of M' has the form $\pi(f)(x)$, where $f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$. But

$M' \models \varphi(\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)) \longleftrightarrow$

$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X)$, where

$X = \{ \langle z_1, \ldots, z_n \rangle \mid \varphi(f_1(z_1), \ldots, f_n(z_n)) \}$

(hence $X \in \bar{H}$).

This means that if $\langle M'', \pi' \rangle$ is a second liftup, we can define $\sigma : M' \overset{\sim}{\longleftrightarrow} M''$ by $\sigma(\pi(f)(x)) = \pi'(f)(x)$. Hence $\sigma = \mathrm{id}$, $M' = M''$. But $\pi(z) = \pi(\mathrm{const}_z(0)) =$ $\sigma(\pi(\mathrm{const}_z(0))) = \pi'(\mathrm{const}_z(0)) = \pi'(z)$ for $z \in M$; where $\mathrm{const}_z = \{\langle z, 0 \rangle\}$. QED (Fact 3)

Note By this analysis it follows easily that, if $\langle M', \pi \rangle$ is the liftup of M by $\pi : \bar{H} \prec H$, where $\bar{H} = H_{\bar{\tau}}^{M}$, and $\tau' = On_{M'} \cap H$, then $\pi(\bar{\tau}) = \tau'$ and $H = H_{\tau'}^{M'}$. $\bar{H}$ need not be an element of M, but if it is, it follows that $\pi(\bar{H}) = H$.

The proof of Fact 3 suggests a general method of constructing the liftup:

Def Let M be a transitive ZFC$^-$ model with predicates $A_1, \ldots, A_n$. Let $\bar{H} = H_{\bar{\tau}}^{M}$, where $\bar{\tau}$ is regular in M, and let $\pi : \bar{H} \prec H$ cofinally.

$$\mathbb{D} = \mathbb{D}_{M, \pi} = \langle D, E, I, \tilde{A}_1, \ldots, \tilde{A}_n \rangle$$

is defined by:

$$D = \{ \langle x, f \rangle \mid f \in M \wedge f : u \to M \text{ for a } u \in \bar{H} \wedge x \in \pi(u) \}$$

$$\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \langle x, y \rangle \in \pi(\{\langle z, w \rangle \mid f(z) \in g(w)\})$$

$$\langle x, f \rangle I \langle y, g \rangle \longleftrightarrow \langle x, y \rangle \in \pi(\{\langle z, w \rangle \mid f(z) = g(w)\})$$

$$\tilde{A}_i(\langle x, f \rangle) \longleftrightarrow x \in \pi\{z \mid A_i f(z)\}.$$

We then get Łoś Theorem in the form

$$\mathbb{D} \models \varphi(\langle x_1, f_1 \rangle, \ldots, \langle x_n, f_n \rangle) \longleftrightarrow$$

$$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \pi(\{\langle z_1, \ldots, z_n \rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\})$$

The proof is by induction on φ and is just like the proof of Łoś theorem for ultrapowers. Then $\mathbb{D} \models ZFC^-$ and $\mathbb{D}$ is an equality model with equality relation I.

This gives:

<u>Fact 4</u> The liftup of $\langle \bar{M}, \bar{\pi} \rangle$ exists iff iff E is well founded.

prf. (sketch)

$(\rightarrow)$ If $\langle M', \pi \rangle$ is the liftup, then $a E b \longleftrightarrow k(a) \in k(b)$ for $a, b \in \mathbb{D}$, where k is defined by $k(\langle x, f \rangle) = \pi(f)(x)$.

$(\leftarrow)$ Factor $\mathbb{D}$ by I to get $\mathbb{D}^* = \mathbb{D}/I$. Let $[u]$ be the equivalence class of u for $u \in \mathbb{D}$. Then $\mathbb{D}^*$ satisfies extensionality and has a well founded $\in$-relation. Hence there is $\sigma^* : \mathbb{D}^* \overset{\sim}{\longleftrightarrow} M'$, where M' is transitive by Mostowski's isomorphism theorem. Set: $\sigma(u) = \sigma^*([u])$ for $u \in \mathbb{D}$. We can define $\pi : M \prec M'$ by $\pi(x) = \sigma(\langle 0, \mathrm{const}_x \rangle)$, where $\mathrm{const}_x = \{\langle x, 0 \rangle\}$ = the constant function x on $\{0\}$. Set: $\tilde{\mathbb{D}} = \{\langle x, f \rangle \in \mathbb{D} \mid f \in \bar{H}\}$; $H' = \{\sigma(u) \mid u \in \tilde{\mathbb{D}}\}$.

H' is easily seen to be transitive.

But $H = \{\bar{\pi}(f)(x) \mid \langle x, f \rangle \in \tilde{\mathbb{D}}\}$

Moreover:

$$\bar\pi(f)(x) \in \bar\pi(g)(y) \leftrightarrow \langle x,y\rangle \in \bar\pi(\{\langle z,w\rangle \mid f(x)=g(y)\})$$
$$\leftrightarrow \sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle)$$

for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. Hence there is an isomorphism $i' : H \overset{\sim}{\leftrightarrow} H'$ defined by $i'(\bar\pi(f)(x)) = \sigma(\langle x,f\rangle)$. Hence $i = id$, $H = H'$ and $\sigma(\langle x,f\rangle) = \bar\pi(f)(x)$ for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. In particular,

$$\pi(z) = \sigma(\langle 0, const_z\rangle) = \bar\pi(const_z)(0) =$$
$$= const_{\bar\pi(z)}(0) = \bar\pi(z) \text{ for } z \in \bar H$$

Hence $\pi \restriction \bar H = \bar\pi$. But then for $\langle x,f\rangle, \langle y,g\rangle \in \mathbb{D}$, we have:

$$\sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle) \leftrightarrow \langle x,y\rangle \in \bar\pi(\{\langle z,w\rangle \mid f(x) \in g(w)\})$$
$$\leftrightarrow \pi(f)(x) \in \pi(g)(y).$$

Hence there is $i : M' \overset{\sim}{\leftrightarrow} M'' \subset M''$ defined by $i(\sigma(\langle x,f\rangle)) = \pi(f)(x)$. M'' is easily seen to be transitive, however, so $i' = id$ and each $z \in M'$ has the form $\pi(f)(x)$, where $\langle x,f\rangle \in \mathbb{D}$. It follows easily that $\langle M', \pi\rangle$ is the liftup of $\langle M, \bar\pi\rangle$. QED (Fact 4)

This gives us the <u>interpolation</u> <u>lemma</u>:

<u>Fact 5</u> Let $\pi' : \bar{M} \prec M'$. Let $\bar{\tau} \in \bar{M}$ be regular in $\bar{M}$ and set $\bar{H} = H^{\bar{M}}_{\bar{\tau}}$. Let $\pi : \bar{H} \prec H$ cofinally. Then:

(a) The liftup $\langle M, \pi \rangle$ of $\langle \bar{M}, \bar{\pi} \rangle$ exists

(b) There is a unique $\sigma : M \prec M'$ s.t. $\sigma\pi = \pi'$ and $\sigma \upharpoonright H = \mathrm{id}$.

prf.

To prove (a) we note that E is well founded, since $\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi'(f)(x) \in \pi'(g)(y)$. But for $\langle x_1, f_1 \rangle, \dots, \langle x_n, f_n \rangle \in \mathbb{D}$ we have:

$$M \models \varphi(\pi(f_1)(x_1), \dots, \pi(f_n)(x_n)) \longleftrightarrow$$
$$\longleftrightarrow M' \models \varphi(\pi'(f_1)(x_1), \dots, \pi'(f_n)(x_n))$$
$$\longleftrightarrow \langle x_1, \dots, x_n \rangle \in \bar{\pi}(\{\vec{z} \mid \bar{M} \models \varphi(f_1(z_1), \dots, f_n(z_n))\}).$$

Hence there is $\sigma : M \prec M'$ defined by $\sigma(\pi(f)(x)) = \pi'(f)(x)$ for $\langle x, f \rangle \in \mathbb{D}$. But this σ is characterized by the above conditions. QED (Fact 5)

The structure $\mathbb{D}^*$ will be of interest to us, however, even if it is ill founded. An embedding $\tilde{\pi} : M \prec \mathbb{D}^*$ is definable by $\tilde{\pi}(x) = [\langle 0, \mathrm{const}_x \rangle]$. This embedding is cofinal in the sense that for every $z \in \mathbb{D}^*$ there is $u \in M$ s.t. $\mathbb{D}^* \models z \in \tilde{\pi}(u)$.

In dealing with ill founded models of set theory it is useful to work with <u>solid</u> structures in the following sense:

<u>Def</u> Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, \dots \rangle$ model the extensionality axiom. $\mathfrak{A}$ is <u>solid</u> iff the well founded core $\mathrm{wfc}(\mathfrak{A})$ is transitive and $\in_{\mathfrak{A}} \cap \mathrm{wfc}(\mathfrak{A})^2 = \in \cap \mathrm{wfc}(\mathfrak{A})^2$.

($\mathrm{wfc}(\mathfrak{A})$ is the set of $x \in \mathfrak{A}$ s.t. $\in_{\mathfrak{A}} \cap X^2$ is well founded, where X is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$).

Clearly, every model is isomorphic to a solid model.

We note the following facts about solid models of ZFC^-:

<u>Fact 6</u> Let $\mathfrak{A}$ be a solid model of ZFC^-. Let $H = \mathrm{wfc}(\mathfrak{A})$. Then

(a) $\omega \subset H$; $\alpha \in H \to \alpha + 1 \in H$

(b) If $\alpha \in H$, $x \in \mathfrak{A}$, and $\mathfrak{A} \models \mathrm{rn}(x) \leq \alpha$, then $x \in H$

(c) H is admissible

prf.

(a), (b) are trivial. We prove (c).

(*Note* We take the replacement axiom of ZFC$^-$ as reading:

$$\bigwedge x \bigvee y\, \varphi(x,y,\vec{z}) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x,y,\vec{z})$$

for arbitrary formulae φ. The theory KP ("Kripke-Platek set theory") is obtained by restricting the fmla φ in this schema - and in the separation schema - to Σ_0 formulae. A transitive structure is called *admissible* iff it satisfies KP.)

By (b), H is easily seen to satisfy Σ_0-separation, as well as the trivial existence axioms: "$\emptyset$ is a set", "$\{x,y\}$ is a set", "$\bigcup x$ is a set". We prove Σ_0 replacement. Let $u \in H$. Let $H \models \bigwedge x \bigvee y\, \varphi(x,y,\vec{z})$. Let $R(x,y)$ mean: "$\varphi(x,y,\vec{z})$ and y is of minimal rank." Then there is $v \in \mathfrak{M}$ s.t. $\mathfrak{M} \models \bigwedge x \in u \bigvee y \in v\, R(x,y)$. But if we take v as being of minimal rank in $\mathfrak{M}$, it must have rank $\in H$. Hence $v \in H$. QED (Fact 6)

Note It follows that if $u \in H$ is transitive and $\delta = On \cap H$, then $L_\delta(u)$ is admissible.

We now extend some of our definitions to solid models of ZFC^-.

Def Let $\mathfrak{A}$ be a solid model of ZFC^-; let $\tau \in \mathrm{wfc}(\mathfrak{A})$ be regular in $\mathfrak{A}$ and let $\bar{H} = H_\tau^{\mathfrak{A}}$ (hence $\bar{H} \subset \mathrm{wfc}(\mathfrak{A})$). Let $\bar{\pi} : \bar{H} \prec H$ cofinally, where H is transitive. $\langle \mathfrak{A}', \pi \rangle$ is a <u>liftup</u> of $\langle \mathfrak{A}, \bar{\pi} \rangle$ iff $\mathfrak{A}'$ is solid, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : \mathfrak{A} \prec \mathfrak{A}'$ is τ-cofinal (i.e. for each $x \in \mathfrak{A}'$ there is $u \in \mathfrak{A}$ s.t. $\bar{\bar{u}} < \tau$ in $\mathfrak{A}$ and $\mathfrak{A}' \models x \in \pi(u)$).

A virtual repetition of the proof of Fact 3 gives:

<u>Fact 7</u> Let $\langle \mathfrak{A}, \bar{\pi} \rangle$ be as above. Up to isomorphism there is at most one liftup $\langle \mathfrak{A}', \pi \rangle$.

<u>Note</u> As before, $\pi(\tau) = \sup\{\bar{\pi}(\nu) \mid \nu < \tau\}$ in $\mathfrak{A}'$; hence $\pi(\tau) \in \mathrm{wfc}(\mathfrak{A}')$ and $H = H_{\pi(\tau)}^{\mathfrak{A}'} \subset \mathrm{wfc}(\mathfrak{A}')$. If $\bar{H} \in \mathrm{wfc}(\mathfrak{A})$, then $H = \pi(\bar{H}) \in \mathrm{wfc}(\mathfrak{A}')$. But we can then form $\mathbb{D}$ as before [taking $\langle x, f \rangle \in \mathbb{D}$ iff ($\mathfrak{A} \models f : u \to V$) for a $u \in \bar{H}$ and $x \in \bar{\pi}(u)$)].

Repeating the proof of Fact 4 we get:

Fact 8 Let $\langle \mathfrak{A}, \bar{\pi} \rangle$ be as above. Then the liftup exists.

(Note The liftup $\langle \mathfrak{A}', \bar{\pi} \rangle$ is unique only up to isomorphism. But then $wfc(\mathfrak{A}')$ is unique, by solidity.)

We now weaken our earlier definition of fullness to:

Def Let N be a transitive ZFC$^-$ model s.t. $N = \langle L_\tau[A], \in, A, \ldots \rangle$. N is almost full iff there is a solid model $\mathfrak{A}$ of ZFC$^-$ s.t. $N \in wfc(\mathfrak{A})$, N is regular in $\mathfrak{A}$, and $\mathfrak{A} \models V = L(N)$.

Then by the above we have:

Fact 9 Let $N = \langle L_\tau[A], \in, A, \ldots \rangle$ be almost full. Let $\pi: N \prec N'$ cofinally. Then N' is almost full. (Moreover, if $\mathfrak{A}$ verifies the almost fullness of N and $\langle \mathfrak{A}', \pi' \rangle$ is the liftup of $\langle \mathfrak{A}, \pi \rangle$, then $\mathfrak{A}'$ verifies the almost fulness of N'.)

By Fact 6:

<u>Fact 10</u> Let N be almost full. There is δ s.t. $L_\delta(N)$ is admissible and N is regular in $L_\delta(N)$.

<u>Def</u> δ_N = the least δ s.t. $L_\delta(N)$ is admissible.

A major tool will be the following <u>transfer lemma</u>:

<u>Fact 11</u> Let $\bar{N}$ be almost full. Let $\pi : \bar{N} \prec N$ cofinally. Let $x_1, \ldots, x_n \in \bar{N}$ and let φ be a Π_1 formula. Then

$$L_{\delta_{\bar{N}}}(\bar{N}) \models \varphi(\bar{N}, \vec{x}) \rightarrow L_{\delta_N}(N) \models \varphi(N, \pi(\vec{x})).$$

proof.

Let $\bar{\mathfrak{A}}$ witness the almost fulness of $\bar{N}$ and let $\pi' : \bar{\mathfrak{A}} \prec \mathfrak{A}$ be the liftup of $\langle \bar{\mathfrak{A}}, \pi \rangle$. Obviously:

(1) $a \notin wfc(\bar{\mathfrak{A}}) \rightarrow \pi'(a) \notin wfc(\mathfrak{A})$

(2) $L_{\delta_{\bar{N}}}(\bar{N}) \subset wfc(\bar{\mathfrak{A}})$, $L_\delta(N) \subset wfc(\mathfrak{A})$ by Fact 6

Suppose not. Then there is a least $\alpha < \delta_N$ s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$. Since $L_{\delta_N}(N)$ is an initial segment of $\mathfrak{A}$, we have:

(3) $\mathfrak{A} \models \alpha$ is least s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$

(4) $\mathfrak{A} \models \bigwedge \nu \leq \alpha \; L_\nu(N)$ is not admissible.

But then $\alpha = \pi'(\bar\alpha)$, where in $\bar{\mathfrak{M}}$:

(5) $\bar\alpha$ is least s.t. $L_{\bar\alpha}(\bar N) \models \neg\varphi(\bar N, \bar x^0)$

(6) $\wedge \gamma \leq \bar\alpha$ $L_\gamma(\bar N)$ is not admissible.

But $\bar\alpha \in \mathrm{wfc}(\bar{\mathfrak{M}})$ by (1). Hence $\bar\alpha < \beta = \mathrm{On} \cap \mathrm{wfc}(\bar{\mathfrak{M}})$ and $L_\beta(\bar N)$ is admissible. Thus (5), (6) hold in $L_\beta(\bar N)$, since $L_\beta(\bar N)$ is an initial segment of $\bar{\mathfrak{M}}$. Hence (5), (6) hold outright and $\alpha < \delta_{\bar N}$. Contr!

QED (Fact 11)

<u>Note</u> Fact 11 is actually a special case of a more general theorem: If $\bar N = \langle L_\tau[A], \in, A, \ldots\rangle$ is a ZFC$^-$ model, $\pi : \bar N \prec N$ cofinally, and N is regular in $L_{\delta_N}(N)$, then the conclusion of Fact 11 holds (even if $\bar N$ is not regular in $L_{\delta_{\bar N}}(\bar N)$).

We shall not need this, however, and do not prove it here, since our proof involves a modest application of fine structure theory.

§3.2 Barwise Theory

An addition to the transfer lemma we shall make use of Barwise' theory of infinitary languages on admissible structures. In the following let M be an admissible structure satisfying choice in the form; Every set is enumerable by an ordinal. In admissibility theory the basic three notions of recursion theory are redefined as follows:

M-recursive $= \underline{\Delta}_1(M)$

M-recursively enumerable $= \underline{\Sigma}_1(M)$

M-finite $=$ element of M.

Barwise then developed an extension of first order logic involving formulae which are infinitely long but still M-finite. Thus a Barwise language on M is like predicate logic except that, whenever $\langle \varphi_i \mid i \in x \rangle \in M$ is a sequence of formulae, then $\bigwedge\limits_{i \in x} \varphi_i$ and $\bigvee\limits_{i \in x} \varphi_i$ are formulae. (Infinite blocks of quantifiers are not allowed, however.) The set of variables is M-infinite (i.e. we could have a variable v_ξ for each $\xi \in On \cap M$). A language is then specified by fixing its predicates, constants, and function symbols.

The syntax is developed internally in such a way that the basic syntactical notions (e.g. "formula", "term", "sentence") are $\Delta_1(M)$. A <u>mathematical theory</u> $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ then consists of a language $\mathcal{L}_0$ and a set $\mathcal{L}_1$ of axioms (all of which are sentences). $\mathcal{L}_1$ should be $\Sigma_1(M)$, if we wish to make use of the admissibility of M. We augment the usual predicate logical rules of inference by two infinitary rules:

$$\frac{\psi \to \varphi_i \quad (i \in x)}{\psi \to \bigwedge_{i \in x} \varphi_i} \qquad \frac{\varphi_i \to \psi \quad (i \in x)}{\bigvee_{i \in x} \varphi_i \to \psi}$$

for $\langle \varphi_i \mid i \in x \rangle \in M$.

A <u>proof</u> is then a (possibly infinite) sequence of formulae, each of which is an axiom or follows from the previous formulae by a rule of inference. If the axiom set $\mathcal{L}_1$ is $\Sigma_1(M)$, it turns out that every provable formula has a proof p which is M-finite (i.e. $p \in M$). From this we get the

M-finiteness lemma: If φ is provable in $\mathcal{L}$, then it is provable from an M-finite $u \subset \mathcal{L}_\varphi$.

A model $\mathfrak{A}$ of the language $\mathcal{L}_0$ is described by fixing its domain of individuals $|\mathfrak{A}|$ and the interpretation $s^{\mathfrak{A}}$ of each predicate symbol, constant, or function symbols, just as in finitary predicate logic. We can then straightforwardly define truth ($\mathfrak{A} \models \varphi$) for $\mathcal{L}_0$-sentences φ and satisfaction ($\mathfrak{A} \models \varphi[a_1, ..., a_n]$) for $\mathcal{L}_0$+ formulae containing only finitely many free variables. We say that $\mathfrak{A}$ models the theory $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ iff all axioms in $\mathcal{L}_1$ are true in $\mathfrak{A}$. The notion of proof is correct in the sense that, if $\mathfrak{A}$ models $\mathcal{L}$, then sentence provable in $\mathcal{L}$ is true in $\mathfrak{A}$.

The final stone in this mosaic is the completeness theorem for countable M: If M is is countable, then $\mathcal{L}$ is consistent iff $\mathcal{L}$ has a model.