On some problems of
Mitchell, Welch and Vickers

Phillip Welch asked whether $T_\omega$ can be an initial segment of an iterate of a countable mouse. More specifically, Mitchell asked the question:

Assume $|K| = L^U$, where $U$ is a measure on $\omega_1$ (hence $U = E_\kappa$ where $\kappa = \kappa + \kappa$, $\kappa = \omega_1$). Can $<J^E_\kappa, E_\kappa>$ be an iterate of a countable mouse?

Welch stated a more difficult version of his question:

Assume that $(\times 1)\,\mathcal{O}^5$ does not exist but the reals are closed under #.

Can there be a countable mouse $M$ and a real $a$ s.t.
(a) $K_{w_1}$ is an initial segment of an iterate of $M$.

(b) $M$ is an initial segment of $K_{w_1}

Welch's student John Vickers asks whether, on the assumption (**), there can be a real a set $K_{w_1} = \{a\}$.

The answer to all these questions is yes.

We use the case model theorem of our notes "Non-Overlapping Extenders", though the argument should be quite comprehensible to those who know our notes "Measures of Order Zero". We make virtually no direct use of fine structure.
Note: The forcing method used in the proof of Thm 1 appears to have more general applications. For instance, forcing over an arbitrary ZFC model we can use it to give $\omega_2$ cofinality while preserving $\omega_1$. Thus providing an alternative to Bukovski-Nambrforcing.
I claim $V = K$. Let $E_\nu = \emptyset$, where $\kappa \geq \kappa^+$. Let $M = K_\beta$, where $\alpha \beta \geq \kappa$. Claim that $\beta^\omega$ is a cardinal of cofinality $> \omega$.

There is a set of conditions $P$ such that $G$ is $P$-generic. Then in $(L,G)$ we have $\kappa$ is regular and there are $M_0, \nu_0$ such that:

(*) $M_0$ is a mouse and $\nu_0 \leq \text{On}(M_0)$.

Let $\langle M_i, i \leq \kappa \rangle$ be the simple iteration with indices $\nu_i = \prod_{M_i} M_i$.

Then $M_\kappa = M_0$, $\nu_\kappa = \kappa$.

The proof stretches over several lemmas.

Let $\Theta = \beta^{++}$. Let $L$ be an infinitary language on $K_\theta$ with predicates $\mathcal{E}$, constants $M_0, \nu_0$ and $x (x \in K_\theta)$, and axioms:

$\text{ZFC}^+ + \theta = \omega_1 + (*)$ (with $M_i, \nu_i$ in place of $M_0, \nu_0$) and

$\forall x \in x \leftrightarrow \forall y \in y = x \ (x \in K_\theta)$.

(Note $L \vdash K_\nu = K_\nu'$.)
\[ \text{Lemma } L \text{ is consistent.} \]

\textit{Proof.}

Let \( \tau = \Theta^+ \). Let \( K_\zeta \) \((\zeta \leq \tau)\) be the iteration of \( K \) with map \( \tau \mapsto \pi_{0\zeta} \).

Let \( i \in \omega \) such that \( \pi_{0\zeta} K_\zeta \) \((\zeta \leq \tau)\) is \( \omega \)-complete.

Set \( M_i = \pi_{0\zeta} (M) \), \( \rho = \pi_{0\zeta} (\rho I) \), where \( \rho = \rho^\omega \).

Then \( \pi_{i+1 \zeta} : J_{E_i} E_{i+1 \zeta} \rightarrow J_{E_i} E_{i+1 \zeta} \).

Since \( \rho \) is regular, \( \rho I > \alpha \).

But \( \rho = \rho^\omega + M_i \), and hence \( M_i = \text{the closure of } \rho \) under \( \Sigma^\omega \) functions.

At follow that \( \pi_{i+1 \zeta} M_i : M_i \rightarrow M_i \).

Hence \( (M_i) \) is the iteration of \( M \) to \( \pi_{0\zeta} (M) \) with indies \( \zeta \leq \tau \).

Now add \( ECK \), which generically collapses \( \zeta < \tau \) to \( \omega \). Then \( (K_\zeta + LEF) \), \( M \) modules \( \pi_{0\zeta} (L) \).

Hence \( \pi_{0\zeta} \) \( L \) is consistent. Hence \( \pi_{0\zeta} \) \( L \).

\text{QED (Lemma 1).}
Let $f_c + p = \langle p_c, p_x, p_y \rangle$ s.t.

1. $p_c$ is a finite partial map $\kappa \to \kappa$.
2. $\text{dom}(p_x) = \text{dom}(p_y)$ and each $p_x(i)$ is a finite partial map $\kappa \to \kappa$.
3. $\text{dom}(p_x) \subseteq \text{dom}(p_y)$ and $p_x(i) \subseteq p_y(i)$.

Set $\bar{\ell}(p) = \ell(p) = \text{dom}(p_x)$.

Set $\alpha^p = p_c(i), \beta^p = p_y(i), \gamma^p = p_x(i)$.

Let $(P)$ be $\ell(p)$ enhanced by the axiom:

1. $\alpha^p \subseteq \beta^p \subseteq \gamma^p$, $i \in \text{dom}(P)$.
2. $\beta^p \subseteq \gamma^p$, $i \in \text{dom}(P)$.
3. $\forall i \in \text{dom}(P)$, $\langle \bar{\ell}(i), \alpha^p \rangle < \langle \bar{\ell}(i), \beta^p \rangle$.

$p$ is good iff $\ell(p)$ is consistent.

Define: Let $i, j \in \text{dom}(P)$, $i < j$.

$i$ is meet in $j$ w.r.t $p$ iff

$a_i$ is $\langle \kappa, a_j(i) \rangle$-definable from parameters in any $\langle P, \gamma^p \rangle$ for some $m < \omega$, where $a_j(i) = \left\{ z \mid \langle m, z \rangle \in a_j \right\}$.

$p$ is meet iff $i$ is meet in $j$ for all $i, j \in \text{dom}(P)$, $i < j$.
Def \( \mathcal{P} = \{ \text{the set of good, meat} \} \) ordered by:

\[
P \leq q \iff P_0 \supseteq q_0 \cap P_2 \supseteq q_2 \quad \text{and} \quad \forall \pi_i \subseteq \pi_i q_i \quad \text{for} \ i \in \text{dom}(q).
\]

We now state some lemmas on the possibility of extending conditions.

**Lemma 2.1** Let \( p \in \mathcal{P} \), let \( u \) be finite, \( u \in \text{dom}(p) \), \( u < \alpha \). There is \( p' \leq p \) s.t. \( u < \text{dom}(p') \).

**Proof.**

Let \( K[E] \) be a generic extension which makes \( \Theta \) countable. An \( K[E] \) there is a model \( M \) for \( L(p) \) which we may take to be \( \text{not} \) in \( \Theta \) so that it is well-founded and \( \text{not} \) in transitive. Then \( x = x \overline{\text{M}} \subseteq \overline{\text{M}} \) for \( x \in K \), and \( x \in \overline{\text{M}} \) whenever \( \text{nor}(x, M) \neq x \in \overline{\text{M}} \). In particular
\[ L_{p'} = \left\{ \xi \in \Lambda \mid \langle \eta_i, \xi \rangle \leq \alpha \right\} \subseteq \Lambda, \]
\[ \langle \eta_i, \xi \rangle = \langle \eta_i, \xi' \rangle \quad \forall \xi' \in \Lambda', \]
\[ \langle \eta_i, \xi \rangle = \langle \eta_i, \xi' \rangle \quad \forall \xi' \in \Lambda'. \]

Set \( p' = \langle p'_0, p'_1, p'_2 \rangle \), where
\[ p'_0 = \langle \eta_i, \xi \rangle, \]
\[ p'_1(\xi) = \begin{cases} p(\xi) & \text{if defined} \\ \emptyset & \text{if not} \end{cases} \]
\[ p' \in \mathcal{P}. \] But \( p' \) is quasi since
\[ M \] model \( L(p') \). Hence \( p' \leq p \).

\[ \text{QED (Lemma 2.1).} \]

By entirely similar proofs:

**Lemma 2.2** Let \( p \in \mathcal{P}, i \in \text{dom}(p), \xi \leq \eta \).
There is \( p' \leq p \) s.t. \( \xi \in \text{dom}(\pi^p_i) \),

( where \( \xi^p_i = \sigma(\eta_i^p) \)).

**Lemma 2.3** Let \( p \in \mathcal{P}, i \in \text{dom}(p), \xi \in \text{dom}(\pi^p_i), \gamma \leq \xi \).
There is \( p' \leq p \) s.t. \( \gamma \in \text{dom}(\pi^p_i) \),

**Lemma 2.4** Let \( p \in \mathcal{P}, i \in \text{dom}(p), \xi \in M \). There is \( p' \leq p \) s.t.
\[ \xi \in \text{dom}(\pi^p_i) \text{ for } a \xi \geq 3. \]
Lemma 2.5 Let $p \in P$, $\exists \in M$. There is $p' \leq p$ such that $\exists \in \text{rng} (\pi_i^{p'})$ for an $i \in \text{dom}(p')$.

Lemma 2.6 Let $p \in P$, $\lambda \in \text{dom}(p)$, $\exists \in \text{rng}(p')$. There is $p' \leq p$ such that $\exists \in \text{rng}(\pi_i^{p'}\lambda^{p'})$ for an $i \in \text{dom}(p')$.

Using $L \vdash \pi_i^{i+1}: M_i \rightarrow^{\forall} M_{i+1}$ we get:

Lemma 2.7 Let $p \in p$, $i, i+1 \in \text{dom}(p)$.

Let $\exists \in \text{rng}(\pi_{i+1}^p)$. There is $p' \leq p$ such that $\exists = \{\pi_{i+1}^p(\bar{x}_1), \ldots, \pi_{i+1}^p(\bar{x}_m)\}$ where $\bar{x}_1, \ldots, \bar{x}_m \in \nu_0$ and $f$ is $M$-definable in parameters from $\text{rng}(\pi_i^{p'})$.

[Note: By the form of $M_i$, $\pi_i^{M_i} \rightarrow^{\forall} M_{i+1}$, simply map that $M_{i+1}$ in the ultrafilter of $M_i$ by $\Sigma_\omega(M_i)$ functions.]
Lemma 2.8. Let \( p \in \mathcal{P} \), \( c \in \text{dom}(p) \)
and \( \bar{a} \in [\pi_c]^{<\omega} \), \( \bar{a} \in \text{dom}(\pi_c^p) \),
and \( a = \pi_c^p \bar{a} \). Let \( X \subseteq [\pi_c]^{<\omega} \)
be \( M \)-definable in parameters from \( \text{rng}(\pi_c^p) \). Then:
\[
X \subseteq \mathcal{E}/a \iff \bar{a} \in X.
\]

We also have:

Lemma 2.9. Let \( p \in \mathcal{P} \), \( c \in \text{dom}(p) \)
and let \( \bar{y} \) be \( M \)-definable in parameters from \( \text{rng}(\pi_c^p) \). There \( p' \leq p \) s.t. \( \bar{y} \in \text{rng}(\pi_c^{p'}) \).

Lemma 2.10. Let \( p \in \mathcal{P} \), \( c \in \text{dom}(p) \),
\( i \in \text{dom}(p) \), \( i > c \), \( \bar{z} \in \text{dom}(\pi_c^p) \),
There \( p' \leq p \) s.t. \( \bar{z} \in \text{dom}(p') \).
Now let $G$ be $P$-generic over $K$. By the extension lemma we can act:

$$\langle \kappa_i : i < \kappa \rangle = \bigcup_{p \in G} P_\kappa$$

$$\pi_c = \bigcup_{p \in G} \pi_i^p, \beta_i = \text{dom}(\pi_c),$$

$$i \in \text{dom}(p)$$

Then $\beta_i \geq \nu_i$ and $\pi_i^p : \beta_i \to \beta$ in order preserving. $\pi_i^p(\nu_i) = \nu$ if $\nu \leq \beta$.

Set $X_i = \text{the smallest } X \subseteq M$

such that $\text{rng}(\pi_i^p) \subseteq X$.

Then $X_i \cap \beta = \text{rng}(\pi_i^p)$ by Lemma 2.9. Set $\bar{\pi}_i : M_i \hookrightarrow X_i$.

Then $\bar{\pi}_i : M_i \subseteq M$ and $\text{rng}(\bar{\pi}_i) \subseteq \text{rng}(\pi_i^p)$ for $i \leq i'$ by Lemma 2.

Set $\bar{\pi}_i^p = \bar{\pi}_i^p, \bar{\pi}_i = \bar{\pi}_i$.

$\nu_i = \nu, M_i \subseteq M_i$. By the extension lemma it follows exactly that...
Lemma 3 \( \langle M_i \mid i \leq \alpha \rangle \) is a simple iteration of \( M_0 \) with map \( \bar{\pi}_c \) and indices \( \nu_i = \bar{\pi}_c (\nu_0) \).

Thus it remains only to show:

Lemma 4 \( \chi \) is regular in \( \mathbf{K}^\| \).

**Proof.** Let \( \delta \leq \alpha \), let \( \delta \vdash \gamma \). Let \( p \in \mathbf{K}^\| \).

**Claim.** There is \( p' \leq p \not \vdash \chi \) for an \( \delta < \gamma \).

Let \( X = \) the smallest \( X \subseteq \langle \mathbf{K}_\nu^\|, M, \pi \rangle \)

n.t. \( \beta \in X \). Let \( \nu : N \rightarrow X \)

where \( N \) is transitive. Then

\( \bar{\nu} (M, \pi) = M, \pi \).

Let \( \bar{f} = \bar{\nu} (\bar{f}_c) \).

Let \( \langle \bar{\nu}, \ldots, \bar{\nu}_n \rangle \) be a recursive enumeration of the \( N \).

- \( \bar{\nu}_0 + \not \vdash \).

\[ b_i = \{ \tilde{z} \mid N \models \bar{\nu}_i [\tilde{z}] \text{ and } \tilde{z} \in M \} \]
Set: $\beta = \exists <\alpha_2 > | \alpha \in \beta$. Then:

$a \in \mathcal{P}(\beta) \cap N = \mathcal{P}(\beta) \cap X$. If $a \in \langle M, b, \rangle$ definable in some parameters for an $i < \omega$.

Go to $K[F]$, where $F$ generically collapses $\theta$ to $\omega$ and let $\mathcal{N}$ be a solid model of $L(p)$. Since $\kappa$ is regular in $\mathcal{N}$, there must be $d < \kappa$ s.t. $d = \kappa_d > \text{dom}(p)_{1}$ and $\exists_{d} : \langle M_d, b \rangle < \langle M, b \rangle$

for some $b$. Set: $p' = \langle p_{0}', p_{1}', p_{2}' \rangle$

where $p_{0}' = p_{0} \cup \exists <d, d> \exists$, $p_{1}' = p_{1} \cup \exists <\phi, d> \exists$, $p_{2}' = p_{2} \cup \exists <b, d> \exists$, $p'$ in quod, since $\mathcal{N}$ models $L(p')$. $p'$ in neat since each $\alpha_{i}^p (i \in \text{dom}(p)) \in \langle M, b, \rangle$ definable in no parameters. Hence $p' \in$
Claim: \( p' \vdash \text{rny} (f) < \xi \).

Suppose not. Let \( q \leq p' \) s.t.

1. \( q \vdash f(\xi) \geq \xi \), where \( \xi < \xi' \).

Pick a solid \( M \) which models \( \mathcal{L}(\xi) \). Again there is \( b \in M \) s.t.

\[ \overline{\pi_d} : \langle M, b \rangle \leq \langle M, b \rangle. \]

Set:

\( Y = \text{the smallest } Y \leq N \) s.t. \( \text{rny}(\overline{\pi_d}) < \xi' \).

It follows easily that \( \forall M = \langle M, b \rangle, \text{rny}(\overline{\pi_d}) \).

Set: \( \pi : \overline{N} \preceq Y \),

where \( \overline{N} \) is transitive. Then \( \pi : \overline{N} \preceq N \) and \( \overline{N} \succ \overline{\pi_d} \).

It is easily seen that \( \overline{\pi} (M_d) = M \).

Set: \( \overline{\pi} f = \overline{\pi}^{-1} (p \circ \overline{f}) \). If we define \( \overline{P}_{\overline{N}} \) in \( \overline{N} \), \( N \) and \( \overline{P} \) was defined in \( N' = \langle K_{\theta'}, \langle M, p \circ \overline{f} \rangle \rangle \), then \( \overline{P}_{\overline{N}} = \overline{P} \cap N \).
\[ q' = \langle q_0 \cdot d, q_1 \cdot d, q_2 \cdot d \rangle. \]

Then \( q' \leq q \leq p \). But

(21) \( q' \in \text{IP}_N \)

since if \( i \in \text{dom}(q'_2) \), then \( a = a_i^{q'_2} \) is \( \langle M, b_m \rangle \)-definable in parameter from \( a \) by \( \pi \) and hence \( a \in N \).

Let \( \bar{q}' = \pi^{-1}(q') \). Since

\[ \models f : \bar{x} \rightarrow \bar{y}, \text{ there is } \bar{q}'' \leq \bar{q}' \text{ in } \text{IP}_N \]

in \( \text{IP}_N \) at (1).

(31) \( \bar{q}'' \models \pi_N \bar{f}(\bar{y}) = \bar{v} \), where \( \forall \bar{d} \in \text{IP}_N \)

and \( \bar{y} = \bar{y} \) as in (1). Hence

(41) \( \bar{q}'' \models \bar{f}(\bar{y}) = \bar{v} \),

since \( \sigma \pi' : N \geq N' \). Hence

(5) \( \bar{q}'' \), \( \bar{q} \) are incompatible.

We derive a contradiction by constructing \( q^* \leq q'' \) if
Set $\overline{L} = (\pi_{00})^{-1} (L)$. Then $\overline{L} (\overline{q})$ is consistent theory on $\mathcal{N}_{\overline{q}} = (\pi_{00})^{-1} (L)$. But $\overline{q}$ is countable in $\mathcal{M}$. Hence there is a solid $\overline{M} \in \mathcal{M}$ which models $\overline{L} (\overline{q})$, giving $\langle \overline{M}_i \mid i < \alpha \rangle$, $\langle \overline{M}_i \mid i < i < \alpha \rangle$ with indices $\overline{r}_i$.

Set $\overline{r}_i = \overline{M}_i (\overline{r}_0)$ and $\overline{M}_i, \overline{r}_i = \overline{M}_i, \overline{r}_i$. But then $\overline{M}_0, \overline{r}_0$ iterate up to $\overline{M}, \overline{r}$, the iteration being identical to that of $\overline{M}_0, \overline{r}_0$ from $\alpha$ onward. At follow worried that $\overline{M} = \langle \overline{M}_0, \overline{M}_0, \overline{r}_0 \rangle$ models $L (\overline{q})$ and $L (\overline{q})$. Thus we may assume without

(6) $\overline{M} \models L (\overline{q})$, $L (\overline{q})$. Set $\overline{q}^* = \langle \overline{q}_0^*, \overline{q}_1^*, \overline{q}_2^* \rangle$

where $\overline{q}_0^* = \overline{q}_0 \cup \overline{q}_0''$, $\overline{q}_1^* = \overline{q}_2 \cup \overline{q}_2''$

and $\overline{q}_1^* \models \overline{q}_1 \cup \overline{q}_1''$ is defined.
as follows:

$$\pi_i \cdot \beta'^* = \pi_i \beta'^* \quad \text{for} \ i < \alpha.$$ 

Since $$\bar{a}_i = a_i \beta'^* \in \bar{N}$$ for $$i \in \text{dom}(\beta'^*)$$, there are $$m < \omega$$ and a finite $$u \subseteq \beta$$ such that $$\bar{a}_i \in \langle M_m, \bar{b}_m \rangle$$ - definable in parameters from $$u$$. Hence $$a_i = a_i \beta'^* \in \langle M_m, \bar{b}_m \rangle$$ - definable in parameters from $$\pi_i \beta'^* \subseteq u$$. Assume w.l.o.g. that

$$\pi_i \beta'^* \cap \text{dom}(\pi_i \beta'^*) \subseteq u \quad \text{for} \ i \in \text{dom}(\beta'^*)$$

For $$i \in \text{dom}(\beta'^*), i \geq \alpha$$ set:

$$\pi_i \beta'^* = \pi_i \upharpoonright (\pi_i \beta'^* \cup \text{dom}(\pi_i \beta'^*))$$

This definition guarantees maximality.

But $$\beta'^*$$ is good, since $$\beta$$ model $$L(\phi^*)$$. Hence $$\phi^* \leq \beta \beta'^*$$.

Contr. QED (Thm 1)
We recall the defining property of $E$:

(1) $A \uparrow K = \mathcal{J}_\mathcal{E}^E$ and $\langle \mathcal{J}_\mathcal{E}^E, E \rangle$ in a strong mouse $\mathcal{M}$ s.t. $E \neq \emptyset$, then $F = E$.

A mouse $N = \langle \mathcal{J}^N_\mathcal{E}, E^N_\omega \rangle$ is called strong if

(2) Whenever $M$ is a premouse s.t. $N = M \uparrow^N_\mathcal{E}$ and $M$ is iterable beyond $\omega^M + \text{crit on extenders of index } > \omega^M$,

then $M$ is a mouse and $N = \text{core}(M) \uparrow^N_\mathcal{E}$

This is equivalent to

(3) $N = \text{W} \uparrow^N_\mathcal{E}$, where $\text{W}^N$ is a universal weakly.
If $K'_v \neq K_v$ for some $v$, let $\nu$ be the least $\alpha$ such a $\alpha$ exists. Let $G$ be $IP$ - generic, $W = K[G]$. Then letting $K' = K^W = J^E$, we have $J^E_v = J^E_{\omega \nu}$ and $E \neq E'$. But $K$ is universal in $W$ since successors of sufficiently large singular cardinals are preserved. Hence $E_v = \emptyset$ and $E'_{\omega \nu} \neq \emptyset$, so otherwise $\langle J^E_v, E_{\omega \nu} \rangle$ is strongly $\delta$. Hence $E_{\omega \nu} = E'_{\omega \nu}$. Now let $G \times G'$ be $IP \times IP$ - generic. Set $W' = K[G,G']$, $K'' = J^{E''} = K^W$. Then $K''_v = \langle J^E_v, E'' \rangle$. $K', K''$ are universal in $K[G \times G']$. Since sufficiently large singular cardinals are preserved, hence $\langle J^E_v, E_{\omega \nu} \rangle, \langle J^E_v, E'_{\omega \nu} \rangle$ are strongly $\delta$ in $K[G \times G']$. Hence $E_{\omega \nu} = E''_{\omega \nu} = E_{\omega \nu}$, where $K^v = K[K[G \times G']]$. 
But then $E'_w \subseteq K$, since $E'_w \subseteq K$ and $E'_w \subseteq K[G] \cap K[G']$, where $G \times G' \approx |P \times P|$ is generic. Hence $\langle J_{v'}^E, E'_w \rangle$ is strong in $K$. Hence $E'_w = E_w$. Contd.

QED (Fomma 5)

Corollary 6. Let $W = K[G]$ be the model of Theorem 1. Then $\kappa = \omega_1 W$ and $K = K W$.

Proof.
$K = K W$ is immediate. Since $E_{w_0} = \emptyset$ for $i < \kappa$, $\langle J_{v_i}^E, E_{w_0}^M \rangle$ is not strong for any $i < \kappa$. Hence $E_{w_0}^M$ is not $\omega$-complete. Hence $\Gamma (\kappa) = \omega$ for limit $\kappa < \alpha$, since otherwise $E_{w_0}^M$ is $\omega$-complete.

Hence $\kappa = \omega_1$. QED (Cor 6)

This takes care of Mitchell's problem.
Note. The question whether such anomalies can occur between $\omega_1$ and $\omega_2$ seems to be more difficult. From $\omega_1$ let $M = \langle J_{d_1}^E, E_{d_1} \rangle$ be a mouse with $\text{crit}(E_{d_1}) = \kappa$; the largest cardinal in $M$. Let $M_\beta = \langle J_{d_\beta}^E, E_{d_\beta} \rangle$ be the iteration of $M$ by the top measure. Set $\beta = \beta_M = \beta_f$ that $\beta$ is $M = \langle J_{d_\beta}^E, E_{d_\beta} \rangle$ and $d_\beta = \kappa + \kappa$. Then $\beta \leq \omega_1$ if it exists. We can show that $\beta$ can take any finite value, but do not know if $\beta = \omega_1$ is possible (we conjecture that $\beta$ can have any value $\leq \omega_1$. )
We now give the solution of Welch's problem. Assume $V = K$ and $E_v \neq \emptyset$, where $v = \kappa^+$, $\text{crit}(E_v) = \kappa$. Assume furthermore that there are arbitrarily large $\frac{2}{3}$ with $E \neq \emptyset$. Hence every set of ordinals has a $\#$. Let $\beta$ be the least $\beta > v$ such that $E_\beta \neq \emptyset$. Set $M = K_\beta = 2^{<J_{\beta}^E,E_\beta>}$. (Then $M$ has the constructibility degree of $A\#$, where $A = \langle \kappa, \Delta \rangle$, coded $2^{<J_{\kappa}^E,E_\kappa>}$.)

Let $W = K[G]$ be the extension of Thm 1 and let $M_i, \kappa_i$ be as in Thm 1. Then $M_\kappa \supseteq M_i, \kappa_i = \kappa_i$. Note that if $N_i = M_i \oplus \kappa_i$, then $N_i$ is the iteration of $N_0$ to $M_i \oplus \kappa_i = 2^{<J_{\kappa}^E,E_\kappa>}$ by the same indices.
Now let \( Q_i = \langle J_{E_i}^{\gamma}, E_{\beta_i}^{\gamma} \rangle \) be the iteration of \( Q_0 = M_0 \) by the \( \beta_i \) (i.e., only the top measure is moved). Letting \( I_i = \text{crit}(E_{\beta_i}^{\gamma}) \), we have:

\[ \bigcup_i Q_i = J_{\tilde{E}}^{\gamma} \],

where

\[ \tilde{E} = (E_0 \cup (\nu_0 + 1)) \].

[Thus \( |J_{\tilde{E}}^{\gamma}| = \aleph_\alpha L[N_\alpha] \)]. Set \( \tilde{\nu} = J_{\tilde{E}}^{\gamma} \).

Since \( \nu_0^+ \tilde{\nu} \) is countable in \( W \),

there is an \( F \in W \) which is generic over \( \tilde{\nu} \) by the condition for collapsing \( \nu_0 \) to \( \omega \). Set \( \tilde{W} = \tilde{\nu}[F] \). Then \( \tilde{W} = \tilde{\nu}[a] \), where \( \tilde{\nu} \) is \( \tilde{\nu} = \tilde{\nu}[a] \) for an \( a \in \omega \) which codes \( N_\alpha, F \). Hence \( \tilde{\nu} = \tilde{\nu} \).

and \( N_\alpha \) is an initial segment of \( \tilde{\nu} \). But \( K_{\tilde{\nu}} \) is an initial segment of \( N_\alpha = \langle J_{\tilde{E}}, E_{\nu} \rangle \),

which is an iterate of \( N_\alpha \).
Finally we note that every set of ordinals has a sharp in $\mathcal{W}$, since $\mathcal{W}$ is a set generic extension of $\mathcal{H}$. This gives a positive solution to Welch's problem.

To solve Vickers' problem, we use a theorem on coding which is stated on p. 308 of *Coeating the Universe* (Becker, Jensen, Welch).

\[
\text{(***) Let } M \text{ be an inner model of } \text{ZFC, U an ultrafilter on } \mathcal{P}(\kappa+)^M \text{ s.t.,}
\]

(a) $\langle M_{\kappa+}, U \rangle$ is amenable ($\kappa = \text{cf} \kappa + M$) and $U$ is normal on $\kappa$ in $\langle M_{\kappa+}, U \rangle$.

(b) $M$ is iterable by $U$.

(c) $\kappa^+$ is countable.

Then there is a c.c.c $\mathbf{r}$-s.t.

(i) $M \subseteq L[a]$

(ii) Cardinality and cofinalities of $M$ are preserved in $L[a]$.

The proof shows that a book "locally" like a real which is added to $M$ by the forcing.
for coding the universe. An important
idea. If \(a\) is a cardinal in \(M\), then
\[
L[a] = L[b] [\overline{a}],
\]
where
\[
L[b] \subset M \text{ and } \overline{a} \text{ is set
generic over } L[b].
\]

Now assume that in \(K\) we have
\(E_{\kappa} \neq \emptyset\), \(\kappa = \text{crit}(E_{\kappa})\), \(\nu = \kappa^+\).
Set \(M = \langle J^E_{\nu}, E_{\nu} \rangle\). Let \(W = K[G\]
be as in Thm 1, giving \(M_0, \nu_0\).
Set \(R = \bigcap_{i < \kappa} \bigcup_{i < \infty} M_i = \bigcup_{i < \infty} (M_i, ||\kappa||_i)\),
where \(M_i = \langle J^E_{\nu_i}, U_{\nu_i} \rangle\) are the
iterates of \(M_0\) with indices \(\nu_i\).
Set \(U = U_0\), \(\kappa = \kappa_0\). Then \(R, R\) satisfy the assumptions of \((***)\)
in place of \(M, \kappa, U\). Let \(L[a]\)
be as in \((iii), (iii), (iii)\). The call that
\(\kappa = \omega_1\) in \(W\). Thus it suffices
to show:

\[\text{Claim } \kappa \in L[a] \text{, } K[a] = K[\kappa] \]
Since \( K_\alpha = \overline{K}_\alpha \), this follows by:

**Claim** \( K L [a] = \overline{K} \)

Set \( K' = K L [a] \). Let \( a \) be regular in \( \overline{K} \), let \( L[a] = L[b] L[G] \), where \( G \) is not generic and \( L[b] \models G \).

Let \( b \) will code \( \overline{K} \) as an inner model of \( L[b] \). \( K' = K L[b] \) by Lemma 5.

**Claim** \( K' \models K_\alpha \) for \( \alpha < \lambda \).

Suppose not. Let \( \exists \) be the least counterexample. Let \( \overline{K}_\exists = \langle \mathcal{J}_\exists, \mathcal{F} \rangle \), \( K'_\exists = \langle \mathcal{J}'_\exists, \mathcal{F}' \rangle \), \( \overline{K} \) is universal in \( L[b] \), hence cardinals are preserved.

Thus if \( \mathcal{F} \neq \phi \), we have \( \mathcal{F} = \mathcal{F}' \), since \( \overline{K}_\exists \) is strong in \( L[b] \), \( \mathcal{F} \neq \phi \), then \( \mathcal{F} \in \overline{K} \) and \( K'_\exists \) is strong in \( L[b] \), hence in \( \overline{K} \).

Hence \( \mathcal{F}' = \mathcal{F} \). Contradiction!

**QED**