§ 2 Lift-up

Our intention is to prove that \( \mathcal{U} \) Woodin in \( \mathcal{U}_\mathbb{N} = \mathcal{U}_{\emptyset + p} \), where \( p \) is the least \( \beta > 0 \) s.t. \( \mathcal{U}_\mathbb{N} \) is admissible. By § 1 Lemma it suffices to show that each \( B \in \mathcal{P}(\mathbb{N}) \cup \mathcal{U}_\mathbb{N} \) is captured at some \( n \). This is trivial for \( B \in M_{\mathcal{B}_0} \cap M_{\mathcal{B}_1} \), but if one of the \( \mathcal{B}_n \) is of type \( \mathcal{B} \), \( M_{\mathcal{B}_0} \cap M_{\mathcal{B}_1} \) may be too small. A possible strategy for handling this is to pick a \( \delta > 0 \) s.t. \( \mathcal{V}_n \), "lift" to

\[
\mathcal{V}_n^*: \mathcal{U}_\mathbb{N}(N_{\mathcal{V}_n}) \rightarrow \mathcal{U}_\mathbb{N}(N_\mathcal{V}_n)
\]

and show that the \( B \in \mathcal{P}(\mathbb{N}) \cup \mathcal{U}_\mathbb{N} \) are captured at some \( n \). This line of attack does, in fact, lead to a proof, although we must deal with the fact that \( \mathcal{U}_\mathbb{N}(N_{\mathcal{V}_n}) \) might be lifted to an ill-founded structure.

In this section we develop the techniques for handling such lift-ups.
Def. Let $\mathcal{N} = \mathcal{J}_\varnothing^\varnothing \models \langle \mathcal{J}_\varnothing^\varnothing, \varnothing, \varnothing \rangle$ be a ZFC-model.

Set: $\mathcal{J}_\varnothing(N) = \varnothing \cup \mathcal{J}_\varnothing(N)^{\mathcal{E}_\varnothing} \cup \{ (N, \varnothing) \}$.

$\mathcal{N}$ is regular in $\mathcal{J}_\varnothing(N)$ iff $(N, \varnothing)$ is a ZFC-model for all $\varnothing \in \mathcal{J}_\varnothing(N)^{\mathcal{E}_\varnothing}$ s.t. ACN.

$\mathcal{N}$ is definably regular in $\mathcal{J}_\varnothing(N)$ iff $(N, \varnothing)$ is a ZFC-model for all $\varnothing \in \mathcal{J}_\varnothing(N)^{\mathcal{E}_\varnothing}$ s.t. ACN.

At follows easily that $\mathcal{N}$ is definably regular in $\mathcal{J}_\varnothing(N)$ iff it is regular in $\mathcal{J}_{\varnothing+1}(N)$.

Def. Let $\mathcal{M} = \mathcal{J}_\varnothing(N)$. $\mathcal{M}$ is grounded w.r.t. $\mathcal{N}$ iff there is $p \in \mathcal{M}$ s.t. $\mathcal{M}(\varnothing \cup \{ p \}) = \mathcal{M}$.

(Here $\mathcal{M}(\varnothing \cup \{ p \})$ is the closure of $\varnothing \cup \{ p \}$ under $\Sigma_1$-functions in $\mathcal{M}$. We have: $\mathcal{M}(\varnothing \cup \{ p \}) = \mathcal{M}''(\varnothing \times (\varnothing \times p^3)) \cup \{ \varnothing \}$, where $\mathcal{M}$ is the $\Sigma_1$ Skolem function for $\mathcal{M}$.)

If this case we also call $p$ a grounding parameter for $\mathcal{M}$.

Fact 1. At there is no $\varnothing$ s.t. $\varnothing \subseteq \varnothing$ and $\mathcal{J}_\varnothing(N)$ is admissible, then $\mathcal{J}_\varnothing(N)$ is grounded.
Fact 2. Let \( M \) be grounded. Then there is a \( \Sigma_1(M) \) set \( BCN \) which codes the whole of \( M \) in such a way that every \( \Delta \in \Sigma_\infty(M) \) with \( A \subseteq N \vdash \Sigma_\infty(\langle N, B \rangle) \). Moreover we can effectively assign to every first order formula \( \varphi \) a formula \( \overline{\varphi} \) such that
\[
M \models \varphi[x] \iff \langle N, B \rangle \models \overline{\varphi}[x]
\]
for all \( x \subseteq N \).

Hence:

Fact 3. Let \( N, M \) be as above, \( N \) is definably regular in \( M \) i.e. \( \langle N, B \rangle \) in a ZFC-model for every \( BCN \) which in \( \Sigma_1(M) \).

We also make use of the fact:

Fact 4. Let \( f : \bar{N} \rightarrow \Sigma_\infty \) co-finally, where one of \( \bar{N}, N \) is a ZFC-model. Then \( f : \bar{N} \rightarrow N \). (Hence both are ZFC-models.)

If \( M \) is a (possibly ill-founded) model of the extensionality axiom, we define the well-founded core of \( M \) (wfc(M)) to be the
set of \( x \) and. There is no infinite descending chain in \( E_{\theta} \) starting with \( x \).

We call \( M \) solid iff \( \mathrm{wfc}(\alpha \tau) \) is transitive and \( : x < y \rightarrow x \subseteq y \) for \( x, y \in \mathrm{wfc}(\alpha \tau) \).

Clearly, every model \( M \) of the extensionality axiom is isomorphic to a solid model.

**Fact 5** Let \( \sigma : N \rightarrow N \) finally, where \( N \) is a ZFC-model. Let \( \bar{M} = J_{\delta}(N) \). Let \( \sigma \) be a \( \mathrm{wfc} \) of \( \bar{M} \) such that \( \bar{M} \) is solid. At \( \theta \) is ill-founded and \( \delta = \mathrm{On} \cap \mathrm{wfc}(\alpha \tau) \). Then \( J_{\delta}(N) \) is admissible.

**Proof:**
Let \( M = J_{\delta}(N) \), \( u, z \in M \) and
\[
M \models \forall x \epsilon u \forall y \epsilon u \varphi(x, y, z)
\]
where \( \varphi \in \Sigma_0^1 \).

Claim: \( M \models \forall x \epsilon u \forall y \epsilon u \varphi(x, y, z) \)
for some \( u \epsilon M \).

Let \( \Psi(u, v, z) \) be the \( \Sigma_0^1 \) formula
\[
\forall x \epsilon u \forall y \epsilon u \varphi(x, y, z)
\]
Recall that \( \bar{M} = J_{\theta}^{\bar{E}} \), \( \bar{N} = J_{\theta} \).
We recall that for every \( \beta \) we have:
\[ J^\beta = \bigcup_{\nu < \omega \beta} S^\nu_{\beta}, \] where \( S^\nu_{\beta} \) (\( \nu < \omega_1 \)) is a cumulative hierarchy of transitive sets. Set \( A = \langle S^\nu_{\beta} \mid \nu < \omega \rangle \). We know that:

- \( \forall \nu \in J^\beta \) for all \( \nu < \omega \beta \)
- The formula \( y = \exists \nu \) is a \( \Sigma_0 \) condition on \( y_1 \nu \) (in the predicate \( \bar{E} \)).

The statement: \( \forall \nu \forall y \, y = \exists \nu \) holds in \( \bar{M} \).

Hence it holds in \( \bar{\omega} \). But the statement:

\[ \forall u, z_1, z_2 \left( (V^u_{3'} \leq \nu \, \exists \,(u, \lambda(z_1, z_2)) \rightarrow V^z_3 < \nu \, \forall (u, \lambda(z_1, z_2)) \right) \]

\[ \forall z_3 < \nu \, \exists \,(u, \lambda(z_1, z_2)) \]

\[ \forall A < \nu \, \exists \,(u, \lambda(z_1, z_2)) \]

also holds in \( \bar{M} \). Hence, it too holds in \( \bar{\omega} \). At is clear that if \( a \in ON \setminus \bar{\omega} \), then

\[ \bar{\omega} = \lambda \exists u \forall y \in A \left( \varphi(x_1 y_1 z) \right). \]

But by the foregoing there must be a least such \( a \) in \( \bar{\omega} \). Hence a \( \in M \) and

\[ M \models \lambda x \exists u \forall y \in S^x_{\alpha} \varphi(x_1 y_1 z), \]

\( \text{QED (Fact 5)} \)
Note that if $\sigma'$ maps $\bar{m}$ cotinally to $\mathcal{M}$, we also have \( \mathcal{M} \models \forall x \forall y \, x \in y \). This will also be the case if $\sigma' \in \Sigma_2$—preserving. Otherwise, I don't see why it would hold.

In this section we shall be very concerned with embeddings $\sigma : \bar{m} \rightarrow \mathcal{M}$ which, as in Fact 5, "lift up" an embedding $\sigma : \bar{N} < \mathcal{N}$.

We shall, in fact, prove:

**Thm 1.** Let $\sigma : \bar{N} < \mathcal{N}$ cotinally, where $\mathcal{N}$ is a ZFC-model. Let $\mathcal{M} = J_\alpha(N)$ s.t. $J_\beta(N)$ is not admissible for $0 < \beta < \alpha$. Then there exist $\bar{M} = J_{\beta'}(\bar{N})$, $\sigma' : \sigma \rightarrow \sigma'$ s.t. $\bar{M} \subseteq \mathcal{M}$, $\bar{M} \models \sigma'$, $\bar{N} \subseteq \mathcal{N}$, and $\mathcal{M} \models \sigma'$.

Hence $\mathcal{M} \subseteq J_\beta(N)$, where $\beta = \sup(\sigma(\bar{M}))$. 

For later applications, however, we shall want more precise information about $\overline{\sigma}$ and $\overline{\sigma}'$. (Finally, we shall show that $\overline{\sigma}'$ is either cofinal into $\mathcal{P}_1$ or into $\Sigma_2$—preserving.)
One type of lift up map is the $\Sigma_0$-lift up:

**Def.** Let $\sigma : \overline{N} \rightarrow N$ where $N$ is a $\Sigma_0$-model.

Let $\overline{M} = \bigcup_{d \leq \sigma} (\overline{N})$. $(\overline{M}, \sigma')$ is called a $\Sigma_0$-lift up of $(\overline{M}, \sigma)$ iff

(a) $\overline{M}$ is solid.
(b) $\sigma' \subset \sigma$ and $\sigma' : \overline{M} \rightarrow \overline{M}$ cofinally.
(c) $\overline{M}$ is the $\Sigma_0$-closure of $N \cup \text{rng}(\sigma')$ in $\overline{M}$.

Note: (c) is equivalent to

(c') Every $x \in \overline{M}$ has the form $\sigma'(f)(\exists \, z)$, where $f \in \overline{M}$, $f : \overline{M} \rightarrow \overline{M}$, $\exists < \theta$, and $\exists \in \sigma'(M)$.

It is also equivalent to:

(c'') For $x \in \overline{M}$, there $x \in \sigma'(M)$ for a $u \in \overline{M}$ such that $\exists u \in \overline{N}$ in $\overline{M}$.

**Fact.** Let $\sigma : \overline{N} \rightarrow N$, $\overline{M} = \bigcup_{d \leq \sigma} (\overline{N})$ be as above. Then a $\Sigma_0$ lift up $(\overline{M}, \sigma')$ of $(\overline{M}, \sigma)$ exists. Moreover, $\sigma'(\overline{N}) = \overline{N}$ (if $\exists > 0 \text{ on } \overline{N}$) and $(\overline{M}, \sigma')$ is unique up to isomorphism. (Hence it is unique if $\overline{M}$ is fully transitive.)
Note: The existence of \( \langle \mathcal{M}, \sigma' \rangle \) can be shown by an ultrapower-like construction.

Using the \( \Sigma_0 \)-lift-up alone will not suffice for the proof of Thm1. Under certain circumstances we can form the \( \Sigma_1 \)-lift-up of \( \langle \mathcal{M}, \sigma \rangle \), which exists whenever \( \mathcal{M} \in \Sigma_1 \)-reflecting wrt. \( \mathcal{N} \) in the following sense.

**Def:** Let \( \mathcal{N} = J^E_\theta \) be a \( \mathcal{ZFC} \)-model. Let \( \mathcal{M} = J^\theta_\eta (\mathcal{N}) \) with \( \theta > 0 \). \( \mathcal{M} \in \Sigma_1 \)-reflecting wrt. \( \mathcal{N} \) iff the following hold:

1. \( \mathcal{M} \) is grounded wrt. \( \mathcal{N} \)
2. \( \mathcal{M} \models \theta \) is the largest cardinal
3. \( \mathcal{N} \) is regular in \( \mathcal{M} \)
4. Let \( \rho \) be a grounding parameter. Then, there are arbitrarily large \( \eta \) s.t.
   whenever \( i < \omega, \exists \xi < \eta \), and
   \( h_\mathcal{M} (i; \langle \xi, \rho \rangle) < \theta \), then
   \( h_\mathcal{M} (i; \langle \xi, \rho \rangle) < \eta \).

**Def:** \( C = C_M = C_{\mathcal{M}, \rho} \) is the set of such \( \eta \).
(Note The choice of the grounding parameter \( p \) is not really important, since if \( p' \) is a second grounding parameter, then \( C = C_{M,p} \) and \( C' = C_{M,p'} \) coincide. "on a tail" — i.e., \( C \setminus \omega = C' \setminus \omega \) for some \( \omega < \theta \).)

Note \( C = C_M \) is closed in \( \theta \). Hence it is club in \( \theta \) if \( M \in \Sigma_1 \) — reflecting.

**Lemma 1.1** Let \( M \) be grounded in \( N \), where \( M = \theta \) is the largest cardinal. Let \( N \) be definably regular in \( M \). Then \( M \in \Sigma_1 \) — reflecting in \( N \).

**proof:**
Let \( p \) be a grounding parameter. Set \( F = \langle h_M(i, \langle 3, p \rangle) \mid i < \omega, 3 < \theta \land h_M(i, \langle 3, p \rangle) \in \theta \rangle \).
Then \( \langle N_i, F \rangle \) is a \( ZFC \) — model. The conclusion follows easily.

\( \square \) (Lemma 1.1)

**Lemma 1.2** Let \( M \) be \( \Sigma_1 \) — reflecting in \( N \). Let \( B \in \Sigma_1 (M) \) and \( B \subseteq N \).
Then \( \langle N, i, B \rangle \) is amenable.
We make the following definition. Let \( p \) be a grounding parameter. For \( \gamma \in C = C_{\alpha, p} \) let:
\[
X_\gamma = \text{h} \diamond (\gamma \cup \{p\}); \quad \text{Let } f_\gamma : M_\gamma \hookrightarrow X_\gamma.
\]
Then \( M_\gamma = J_{\delta_\gamma}(N_{1\delta}) \), where \( N_{1\delta} = J_{\delta_\gamma}^\mathbb{N} \).

Moreover \( f_\gamma : M_\gamma \rightarrow M \) with \( \delta = \text{crit}(f_\gamma) \) and \( f_\gamma(\delta) = \delta \). We note, however, that \( M_\gamma \in N_\delta \), since \( \delta_\gamma \leq \delta \) and:

(a) \( p_\gamma = f_\gamma^{-1}(p) \) is a grounding parameter for \( M_\gamma \) wrt \( N_{1\delta} \).

(b) \( f_\gamma \gamma = \gamma \) is the largest cardinal.

Thus \( \Theta \) would be collapsed to \( \gamma \) if \( \delta_\gamma \geq \Theta \).

Now let \( B = \Sigma_1(M) \) in the parameter \( \delta \).
Let \( \gamma \in C \) s.t. \( f_\gamma(\delta) = \gamma \). Then
\[
B \cap (N_{1\delta}) \text{ is definable over } M_\gamma \text{ by } \gamma \text{ in } M_{\gamma'}. \text{ Hence}
\]
\[
B \cap (N_{1\delta}) \in N_\delta, \text{ QED (Lemma 1.2)}
\]

Note: The choice of \( p \) was not really important to the definition of
\( \langle M_\gamma, 1 \in C \rangle \), since if \( p' \) is another grounding parameter yielding
\( \langle M_\gamma, 1 \in C \rangle \), then...
The sequences again coincide on a tail - i.e., for some \( \lambda < \Theta \) we have:
\[ C \setminus \lambda = C' \setminus \lambda, \quad M_y = M'_y, \quad f_y = f'_y \text{ for } \lambda \in C \setminus \lambda. \]

We also set:
\[ \text{Def } f_{y, y'} = f_{y', y}^{-1} \text{ for } y \leq y' \text{ in } C. \]

Then \( f_{y, y'} \in N \), since \( f_{y, y'} \) is definable from \( M_y', p_y, y' \) as \( f_y \) was defined from \( M, p, y \).

Using this machinery we define the \( \Sigma_1 \)-liftings:

\[ \text{Def } \sigma : N \rightarrow N \text{ cofinally, where } N \ni a \in \mathcal{Z} \subset C \text{ - modal. Let } \tilde{M} = \mathcal{U}_a(N) \text{ be } \Sigma_1- \text{ reflexive with } N. \text{ Define } \tilde{C} = C \setminus M \setminus \tilde{M} \text{, } \tilde{f}_{y, y'} (y \in \tilde{C}), \tilde{f}_{y, y'} (y \leq y' \text{ in } \tilde{C}) \text{ as above. Set: } \]

\[ \tilde{M}_y = \sigma(M_y), \quad \tilde{f}_{y, y'} = \sigma(f_{y, y'}). \]

Let \( \langle \tilde{M}_y, \tilde{f}_{y, y'}, \tilde{C} \rangle \) be a direct limit of \( \langle M_y, f_{y, y'}, C \rangle \), \( \langle \tilde{M}_y, \tilde{f}_{y, y'}, \tilde{C} \rangle \), where \( N \) is solid.

Define \( \sigma : \tilde{M} ightarrow N \), \( \mu \) by \( \tilde{f}_{y, y'} = \tilde{f}_{y, y'} \sigma. \)
By our above remarks on "coincidence on a tail" it is clear that \( \sigma \) is defined independently of the choice of the grounding parameter \( \bar{\rho} \), which gave \( \bar{\sigma} = \bar{\sigma}_{\bar{\rho}} \).

Any such pair \( \langle \bar{\mathcal{U}}, \bar{\sigma} \rangle \) is called a \( \Sigma_1 \)-liftup of \( \langle \bar{\mathcal{M}}, \bar{\sigma} \rangle \).

It is clear that \( \langle \bar{\mathcal{U}}, \bar{\sigma} \rangle \) is unique up to isomorphism. Hence it is unique if \( \mathcal{U} \) is well-founded (hence transitive).

We leave it to the reader to prove:

**Lemma 1.3** Let \( \langle \mathcal{U}, \sigma \rangle \) be a \( \Sigma_1 \)-liftup of \( \langle \bar{\mathcal{M}}, \bar{\sigma} \rangle \). Then

(a) \( \bar{\sigma} : \bar{\mathcal{M}} \to \mathcal{U} \)

(b) \( \bar{\sigma}(\bar{\mathcal{N}}) = \mathcal{N} \in \text{wfc}(\mathcal{U}) \)

(c) \( \bar{\sigma} \) is an \( \Sigma_1 \)-liftup of \( \langle \bar{\mathcal{M}}, \bar{\sigma} \rangle \), 

Set \( \bar{B} = \{ \bar{u} \in \bar{\mathcal{M}} \mid \exists \bar{u}' \in \bar{\mathcal{M}} : \bar{u}' \vdash \bar{u} \}, \) (Hence \( \bar{\sigma} : \langle \bar{\mathcal{N}}, \bar{B} \rangle \to \mathcal{U} \) finaly.)

Then \( B \in \Sigma_1(\mathcal{U}) \) in \( q = \bar{\sigma}(\bar{q}) \) by the same definition.
Lemma 1.4 Let \( \langle M, \sigma' \rangle \) be the well-founded \( \Sigma_1 \)-lifftup of \( \langle 
abla M, \sigma \rangle \). Then

(a) \( M \upharpoonright \Sigma_1 \)-reflective wrt \( N \)

(b) If \( \sigma' \in M \) is a grounding parameter for \( \nabla M \), then \( \sigma = \sigma'(\sigma') \) is a grounding parameter for \( M \).

We now define:

*Let \( \sigma : \nabla N \rightarrow N \) cofinally, where \( \nabla N \) is a \( \text{ZFC} \)-model.*

*Let \( M = J_{\infty}(\nabla N) \), where \( \nabla N \) is regular in \( M \).*

\( \langle M, \sigma' \rangle \) is a good lifftup of \( \langle 
abla M, \sigma \rangle \)

if either \( M \upharpoonright \Sigma_1 \)-reflective wrt \( \nabla N \) and \( \langle M, \sigma' \rangle \) is a \( \Sigma_1 \)-lifftup, or else \( M \) is not \( \Sigma_1 \)-reflective and \( \langle M, \sigma' \rangle \) is an \( \Sigma_0 \)-lifftup.

Lemma 1.5 Let \( \langle M, \sigma' \rangle \) be the well-founded good lifftup of \( \langle \nabla M, \sigma \rangle \).

If \( M \upharpoonright \Sigma_1 \)-reflective wrt \( N \),
then \( \nabla M \upharpoonright \Sigma_1 \)-reflective wrt \( \nabla N \).

(Hence \( \langle M, \sigma' \rangle \) is the \( \Sigma_1 \)-lifftup.)

proof

Suppose not. Then \( \langle M, \sigma \rangle \) is the
$\Sigma_0$ - lift up. Since $\Theta$ is the largest cardinal in $M$, it follows readily that $\Theta$ is the largest cardinal in $\bar{M}$.

Now let $\rho$ be a grounding parameter for $M$. Then $\rho = \sigma'(f)(\xi)$, where $f \in \bar{M}$, $\xi < \Theta$. Thus every $\nu \in M$ in $\Sigma_1(M)$ in $\sigma'(f)$ and parameters from $\Theta$. Hence $\sigma'(f)$ is a grounding parameter for $M$. Hence we may assume w.l.o.g. that $\rho = \sigma'(\bar{\rho})$ for a $\bar{\rho} \in \bar{M}$. But then for any $\nu \in \bar{M}$, we have:

$$\forall \xi < \Theta \forall \nu < \omega \sigma'(x) = h_M(\nu, \langle \xi, \rho \rangle)$$

Hence:

$$\forall \xi < \Theta \forall \nu < \omega \nu = h_{\bar{M}}(\nu, \langle \xi, \bar{\rho} \rangle)$$

and $\bar{\rho}$ is a grounding parameter for $\bar{M}$. We must show:

Claim: $\bar{C} = C_{\bar{M}, \bar{\rho}}$ is unbounded in $\bar{\Theta}$.

Let $\gamma < \bar{\Theta}$. We must find $\bar{s} > \gamma$ s.t. $\bar{s} \in \bar{C}$. 

Let $C = C_{M, p}$. Pick $\gamma \in C$ s.t. $\sigma(\gamma) < \delta$. Set $\bar{\gamma} = \text{the least } \bar{\gamma} \text{ s.t. } 1(\bar{\gamma}) \geq \delta$.

Claim $\bar{\gamma} \in C$.

Let $h_M(\vec{i}, \langle \bar{\gamma}, p \rangle) = \delta < \Theta$, where $\delta < \bar{\gamma}$. Then

$h_M(\vec{i}, \langle \sigma(\bar{\gamma}), p \rangle) = \sigma(\bar{\gamma}) < \Theta$

where $\sigma(\bar{\gamma}) < \delta$. Hence

$\sigma(\bar{\gamma}) < \delta \leq \sigma(\bar{\gamma})$, since $\bar{\gamma} \in C$.

Hence $\delta < \bar{\gamma}$, QED (Lemma 1.5)

We are now ready to prove Thm 1 in the stronger form:
Thm 2  Let \( \sigma: \bar{\mathcal{N}} < \mathcal{N} \), where \( \mathcal{N} = \bigcup_{\emptyset}^{E} \), is a ZFC-model. Let \( \mathcal{M} = \bigcup_{\emptyset}^{E} \mathcal{N} \), where \( \mathcal{N} \) is regular in \( \mathcal{M} \) and \( \bigcup_{\emptyset}^{E} \mathcal{N} \) is not admissible for \( 0 < \nu < \delta \). Let \( \bar{\mathcal{M}} = \bigcup_{\emptyset}^{E} \bar{\mathcal{N}} \) where \( \bar{\mathcal{N}} \) is maximal s.t. \( \mathcal{N} \) is regular in \( \bar{\mathcal{N}} \) and \( \mathcal{M} \subseteq \bigcup_{\emptyset}^{E} \bar{\mathcal{N}} \) is admissible for \( 0 < \nu < \bar{\delta} \), Let \( < \bar{\mathcal{M}}, \bar{\sigma} > \) be a good lift up of \( < \bar{\mathcal{N}}, \bar{\sigma} > \). Then \( \mathcal{M} \) is an initial segment of \( \bar{\mathcal{M}} \).

Proof:
Suppose not. Then \( \mathcal{M} \) is well founded, since otherwise, letting
\( \mathcal{S} = \{ \alpha \in \mathcal{N} \mid \mathcal{M} \prec (\mathcal{M}, \alpha) \} \), we know that
\( \mathcal{M} \) is admissible, hence contains \( \mathcal{M} \) as a segment. Let \( \mathcal{M} = \mathcal{M} = \bigcup_{\emptyset}^{E} \mathcal{N} \), Then \( \bar{\mathcal{N}} < \bar{\delta} \). Hence \( \mathcal{N} \) is regular in \( \bigcup_{\emptyset}^{E} \mathcal{N} \). Hence \( \mathcal{M} \) is \( \Sigma_1 \)-reflecting at \( \mathcal{N} \) by Lemma 1.9.
Hence \( \mathcal{M} \) is \( \Sigma_1 \)-reflecting at \( \bar{\mathcal{N}} \) by Lemma 1.5 and \( < \mathcal{M}, \bar{\sigma} > \) is the \( \Sigma_1 \)-lifting of \( < \mathcal{N}, \sigma > \). Now let
\[ \mathcal{B} \subseteq \Sigma_1(\overline{M}) \text{ s.t. } \mathcal{B} \subseteq \overline{N}, \text{ Set: } \mathcal{B} = \bigcup_{\gamma \leq \alpha} (\gamma, \overline{B}). \]

By Lemma 1.3 (c1), \( \mathcal{B} \subseteq \Sigma_1(\overline{M}) \); hence \( \mathcal{B} \subseteq \mathcal{F}_\alpha(\overline{N}) \).

But then \( \langle N, \mathcal{B} \rangle \) is a ZFC-model and
\[ \Rightarrow \langle N, \mathcal{B} \rangle \models \Sigma_0 \langle N, \mathcal{B} \rangle. \]

Hence \( \langle N, \mathcal{B} \rangle \) is a ZFC-model. By Fact 3, \( \overline{N} \) is definably regular in \( \overline{M} \), or in other words:

\( \overline{N} \) is regular in \( \mathcal{F}_\alpha(\overline{N}) \). Hence by the definition of \( \overline{M} \) we have \( \overline{M} \) is definable.

Now let \( \bar{\rho} \) be a grounding parameter for \( \overline{M} \) and let \( \overline{\mathcal{B}} \) be \( \Sigma_1(\overline{M}) \) in \( \overline{\rho} \) s.t. \( \overline{\mathcal{B}} \) codes the whole of \( \overline{M} \) with the properties given in Fact 2. Letting \( \overline{\mathcal{B}} \) be as above, \( \overline{\mathcal{B}} \) has the same \( \Sigma_1(\overline{M}) \) definition over \( \rho = \bar{\rho}(\bar{\rho}) \), which is a grounding parameter for \( M \). Hence \( \overline{\mathcal{B}} \) codes \( \overline{M} \) with the properties given in Fact 3. The statement "\( \overline{M} \) is definable" is a first order statement over \( \overline{M} \), hence is expressible by a first order statement over \( \langle N, \mathcal{B} \rangle \).

Hence \( \langle N, \mathcal{B} \rangle \) satisfies the same statement. Hence \( \overline{M} \) is definable. Hence \( M \) is a segment of \( \overline{M} \).

Contr! \ QED (Thm 2)
Now suppose that $\sigma : \bar{N} \prec N$ cofinally, where $N$ is a ZFC-model, and $\sigma = \sigma_1 \circ \sigma_0$, where $\sigma_0 : \bar{N} \rightarrow N_0$, $\sigma_1 : N_0 \rightarrow N$ are cofinal $\Sigma_0$-preserving maps. (Hence, of course, all maps are elementary and all models are $\Sigma_0$-models.)

Let $\bar{M} = \bigcup_{\bar{N}} (\bar{N})$ be grounded with $\bar{N}$ not. $\bar{M} = \bar{E}$ is the largest cardinal, (where $\bar{E} = \text{On} \cap \bar{N}$, $\Theta = \text{On} \cap N$, $\Theta = \text{On} \cap N$).

Let $< \bar{M}, \sigma >$ be a good lift up of $< \bar{M}, \sigma >$ and $< \bar{M}_1, \sigma_0 >$ a good lift up of $< \bar{M}, \sigma >$. Then there is a canonical bridge $\sigma_1 : \bar{M}_0 \rightarrow \bar{M}$ such that $\sigma_1 \circ \sigma = \sigma_0$ and $\sigma_1 \circ \sigma_0 = \sigma$. This is defined by cases as follows:

Case 1. $\bar{M}$ is not $\Sigma_1$-reflective w.r.t. $\bar{N}$.

Then $< \bar{M}, \sigma >$, $< \bar{M}_0, \sigma_0 >$ are $\Sigma_0$-lift ups and every $x \in \bar{M}_0$ has in $\bar{M}_0$ the form $\sigma_0' (f) (\bar{z})$ where $f \in \bar{M}$ and $\bar{z} < \theta_1$. We then set $\sigma_1' (\sigma_0' (f) (\bar{z})) = \sigma_1' (f) (\sigma_1 (\bar{z}))$. ...
Cure 2  Cure 1  faith.

Then \( \langle M, 0' \rangle, \langle M_0, 0' \rangle \) are \( \Sigma_1 \)-elements.

Let \( \bar{\sigma} \) be a grounding parameter, \( \bar{C} = \bar{C}_{\bar{M}, \bar{M}_0} \).

Let \( \sigma' (\bar{M}_0) = M_0, \sigma' (\bar{f}_{y, y'}) = f_{y, y'} \).

\( \sigma (\bar{M}_0) = M_0, \sigma (\bar{f}_{y, y'}) = f_{y, y'} \),

for \( y \leq y' \) in \( \bar{C} \). Then \( \sigma' f_{y} = f_{y} \sigma \) and

\( \sigma' f_{y} = f_{y} \sigma \), where:

\( \langle M, f_{y} | y \in \bar{C} \rangle = \text{the limit of} \)

\( \langle M_0, f_{y} | y \in \bar{C} \rangle, \langle f_{y}, y' \rangle | y \leq y' \in \bar{C} \rangle, \)

and accordingly for \( \langle M, f_{0} | y \in \bar{C} \rangle \).

We define \( \sigma'_1 \) by:

\( \sigma'_1 f_{y} = f_{y} \sigma \) for \( y \in \bar{C} \),

An Cure 1 we then have:

\( \sigma_1 : \mathcal{M}_0 \rightarrow \mathcal{M} \) \text{ cofinally}

and in Cure 2:

\( \sigma_1 : \mathcal{M}_0 \rightarrow \mathcal{M} \).

Now suppose that \( \mathcal{M}_0 = M_0 \) is well founded. We leave it to the reader to prove:

**Lemma 2.1** Let \( \mathcal{M}_0 = M_0 \) be well founded (hence transitive). Then:
* $(M, \sigma')$ is a $\Sigma_0$-lift up of $(M_0, \sigma_1)$ if $\bar{M}$ is not $\Sigma_1$-reflective with $\bar{N}$.

* $(M, \sigma')$ is a $\Sigma_1$-lift up of $(M_0, \sigma_1)$ if $\bar{M}$ is $\Sigma_1$-reflective with $\bar{N}$.

But if $\bar{M}$ is not $\Sigma_1$-reflective, then neither is $M_0$ by Lemma 1.8. Hence $(M, \sigma')$ is a good lift up of $(M_0, \sigma_1)$.

Now suppose $\sigma_i^j : N_i^j < N_i^j$ for finally for $i \leq i' < \gamma$, where each $N_i$ is a $\mathcal{ZFC}$-model. Let $M_0 = \bigcup_i (N_i)$ be grounded with $N_0$ and set $M_0 \subseteq \Theta_0$ be the largest cardinal, where $\Theta_i = \text{con}_i \cap M_0$. Let $(M_j, \sigma_0^j)$ be a good lift up of $(M_0, \sigma_0^j)$ for $j < \gamma$. Let $\sigma_i^j : M_i^j \to \Theta_i^j$ be the bridging map defined above. It follows from the definition that $\sigma_i^j \circ \sigma_0^j = \sigma_k^j$ for $i \leq i' \leq k < \gamma$. 


Historical Note

Thm 1 has the corollary:

(*) Let $\tilde{\sigma} : \tilde{N} \prec N$ cofinally, where $N = J^E_\kappa$ is a \ZFC\ - model. Assume that $N$ is regular in $J_\rho(N)$, where $\rho$ is least in $\kappa$, $\rho > 0$ and $J_\rho(N)$ is admissible. Let $\tilde{\rho}$ be least $\tilde{\rho} > 0$ in $\tilde{\kappa} : J_\tilde{\rho}(\tilde{N})$ is admissible. Let $J_\rho(\tilde{N}) = \varphi(\vec{x}, \tilde{N})$ where $\varphi$ is a $\Sigma_1$ formula and $\vec{x} \in \tilde{N}$. Then $J_\rho(N) \models \varphi(\sigma(\vec{x}), N)$.

To see this, note that we can indirectly assume the $\vec{a}$ in Thm 1 to be $\leq \tilde{\rho}$.

In our note [SPSC] we made heavy use of a lemma which draws the same conclusion under the assumption that $\tilde{N}$ is full. Hence (*) is a strengthening of that lemma. However, Theorem 1 and its corollary (*) predate the lemma by several decades. Our recollection is that we proved them in late 1975.

The lemma on fullness was, of course, more suitable to [ESI], since its proof involved no fine structure,