§ 3 The conclusion

Letting \( J, N, b_0, b_1 \) etc. be as in §1 we prove:

**Thm 1** Let \( \beta \) be the least \( \beta > 0 \) s.t. \( J_\beta (N) \) is admissible. Then \( \langle \beta, \eta \rangle \) is Woodin in \( J_\beta (N) \).

The proof proceeds by contradiction, so assume the theorem to be false.

**Lemma 2** One of \( b_0, b_1 \) is of Type B.

**Pf.** Suppose not. Then in \( M_{b_0} \) (or \( M_{b_1} \)) there is \( \chi > \beta \) s.t. \( E_\chi \neq \emptyset \), by §1 (44).

Since \( J_\chi \models ZFC^- \), it follows that \( \langle \beta, \eta \rangle \) is Woodin in \( J_\chi \).

Hence each \( B \in \mathcal{P}(N) \cap J_\beta (N) \) is captured at some \( \chi \). Hence \( \langle \beta, \eta \rangle \) is Woodin in \( J_\beta (N) \). Contradictio!

**Q.E.D. (Lemma 2)**
Now let $b_h$ (for $h = 0$ or $1$) be a branch of type $B$. Since $N$ is a ZFC model and

$$\pi^*_{\text{j}h} : N' \to N$$

is elementary for $i \in b_h \setminus \tilde{z}_h$, we conclude that $\pi^*_{\text{j}h}$ is elementary and $N'$ is a ZFC model. Now let

$$\gamma_h = \text{the maximal } \gamma \text{-set, } N^*_h \text{ is regular in } J_{\gamma_h}(N^*_h),$$

and $J_{\nu}(N^*_h)$ is not admissible for any $\nu \leq (0, \omega)$. Set $M_h = J_{\gamma_h}(N^*_h)$. Let

$$\langle \mathbf{M}, \mathbf{\pi^*}, \mathbf{\tilde{z}_h}_b \rangle$$

be a good lifting of $\langle M^*_h, \pi^*_h, b_h \rangle$.

$$\langle \mathbf{M'}, \mathbf{\pi'}, \mathbf{\tilde{z}'_h}_b \rangle$$

for $i \in b_h \setminus \tilde{z}_h$, let $\pi^*_{\text{j}h} : b_h \to b'_h$, $\bar{\pi}^*_{\text{j}h} : \tilde{z}'_h \to \tilde{z}_h$ be the bridging maps defined in §2. Let $\pi^*_{\text{j}h} : \tilde{z}'_h \to \tilde{z}_h$. Let $\pi^*_{\text{j}h} : b_h \to b'_h$ and $\bar{\pi}^*_{\text{j}h} : \tilde{z}'_h \to \tilde{z}_h$, for $i \leq k$ in $b_h \setminus \tilde{z}_h$. We recall that

$$\pi^*_{\text{j}h} \circ \pi^*_{\text{j}k} = \pi^*_{\text{j}k} \circ \pi^*_{\text{j}h} \quad \text{and}$$

$$\pi^*_{\text{j}h} \circ \pi^*_{\text{j}k} = \pi^*_{\text{j}l}$$

for $j \leq k \leq l$ in $b_h \setminus \tilde{z}_h$.

For $b_h$ of type $A$ we simplify set:

$$\mathbf{M}_b = M^*_b, \quad \mathbf{M}_i = M^*_i, \quad \pi^*_{\text{j}h} = \pi^*_{\text{j}h}$$

and $\pi^*_{\text{j}k} \circ \pi^*_{\text{j}h} = \pi^*_{\text{j}k}$ for $j \leq k \leq l$ in $b_h \setminus \tilde{z}_h$.
Then \( \langle N_1 | i \in b \setminus \beta \rangle, \langle \pi_i^* | i \leq 1 \text{ and } b \setminus \beta \rangle \)

is a commutative system whose limit in \( N \_h \), \( \langle \pi_i^* b_h | i \in b \setminus \beta \rangle \).

From this we easily get:

**Lemma 3** Let \( B \subseteq N_1, B \in \Omega_{b_0} \setminus \Omega_{b_1} \).

Then \( B \) is strongly captured at some \( n \).

We first note \( N_1 \not\approx \text{ rng } \pi_i \) and \( \pi_i^* (N_{\beta_n}) = N \) for \( h = 0, 1 \). (If \( b \_h \) is of type A, this is immediate.)

Otherwise \( M_0 = \cup_{n \geq 0} (N_{\beta_n}) \) where \( \beta_0 \geq 1 \) and \( N_{\beta_0} \) is a ZFC model.

Now, let \( B \subseteq \text{ rng } (\pi_{\beta_0}^* b_{\beta_0}) \cap \text{ rng } (\pi_{\beta_{n+1}}^* b_{\beta_{n+1}}) \).

Then \( i | j \in b \_p, i \in b \setminus \beta \), for \( p \geq n \), we have

\[ \pi_{i+1}^* b \_p (\langle N_1 | B \_i \rangle) = \langle N_1 | B \rangle \]

where \( B \_i = \pi_{i+1}^{-1} (B) = \pi_{i+1}^{-1} \text{ image of } B \).

QED (Lemma 3)
Corollary 3.1 Let $v$ be maximal in $\mathcal{R}_x(N)$ and $J^v(N)$ is not admirable for $0 < v < x$. Then $\mathcal{F}$ is Woodin in $J^v(N)$.

Proof:

We show: $J^v(N) \subseteq M_{b_h}$ ($h = 0, 1$).

At $b_h$ is of type $A$, then

$J^v(N) \subseteq J^p(N) \subseteq M_{b_h} = M_{b_h}$,

where $p$ is least such $p > 0$ and $J^p(N)$ is admirable.

At not, then $J^v(N)$ is a segment of

$M_{b_h}$ by §2 Theorem 2.

QED (Cor 3.1)

But $J^v(N)$ is then not admirable by our assumption. Hence:

Cor 3.2 $N$ is not regular in $J^{v+1}(N)$.

Cor 3.3 $b_h$ is of type $B$ and

$M_{b_h} = J^v(N)$ is well founded ($h = 0, 1$).

Proof:

Suppose not. At $b_h$ is of type $A$,

Then $J^{v+1}(N) \subseteq J^v(N) \subseteq M_{b_h}$.
where \( p > 0 \) is least such \( \omega \in b^{h} \rangle \).

Thus \( b_{h} \) is of type B. \( \omega_{b_{h}} \) were ill-founded, then, letting \( \delta = \omega_{b_{h}} \cup \omega_{b_{h}} \), we have \( \omega_{b_{h} + 1} \langle N \rangle \subset \omega_{b_{h}} \). But \( \omega_{b_{h}} \) is admmissible. But \( N \) in regular in \( \omega_{b_{h}} \rangle \) hence in \( \omega_{b_{h}} \rangle \). Contradiction!

Thus \( \omega_{b_{h}} \rangle \) is transfinite. Call it \( M_{b_{h}}^{\kappa} \).

Let \( M_{b_{h}}^{\kappa} = \omega_{b_{h} + 1} \langle N \rangle \). Then \( \delta \subset M_{b_{h}}^{\kappa} \). But then \( \delta \subset \delta_{\kappa} \), since otherwise \( \omega_{b_{h} + 1} \langle N \rangle \subset M_{b_{h}}^{\kappa} \), where \( N \) in regular in \( M_{b_{h}}^{\kappa} \rangle \). GED (Corollary 3.3)

Thus every \( \omega_{b_{h}} \rangle \) is transfinite.

Set \( : N_{b_{h}}^* = \omega_{b_{h}} \rangle \langle N \rangle \) \( h = 0, 1 \)

\( N_{b_{h}}^* = \omega_{b_{h}} \rangle \langle i \) \( \in b_{h} \cap \omega_{b_{h}} \), \( h = 0, 1 \).

By \( \S 2 \) Lemma 2.1 \( \pi_{b_{h}}^{*} \) is the good lift of \( \pi_{b_{h}} \) and \( \pi_{b_{h}}^{*} \) is the good lift of \( \pi_{b_{h}} \) for \( i \leq k \in b_{h} \cap \omega_{b_{h}} \), \( h = 0, 1 \).
We now prove:
\[\text{Co3.4 } \quad N^* \cup \Sigma_1 \text{ - reflecting with } N^*\]

Trivially, \(N^*\) is grounded with \(N\) and \(N^* \models \theta\) is the largest cardinal.

Thus it suffices to show:

Claim Let \(p\) be a grounding parameter for \(N^*\) with \(N\). Let \(p \in \text{ rng}(\pi^*_{i_{m+1}b_{i_{m+1}}} i_{m}b_{i_{m}})\).

Let \(h_{N^*}(i_{m}, \langle \gamma, p \rangle) < \theta\) where \(\gamma < \kappa_{i_{m}}\). Then,
\[h_{N^*}(i_{m}, \langle \gamma, p \rangle) < \kappa_{i_{m}}.\]

Proof

Let \(\bar{s} = h_{N^*}(i_{m}, \langle \gamma, p \rangle)\). Then \(\bar{s} \in \text{ rng}(\pi^*_{i_{m}b_{i_{m}}} i_{m})\) whenever \(m = n\) or \(m+1\), \(i_{m} \notin b_{i_{m}} \setminus \gamma_{i_{m}}\).

Since \(\pi^*_{i_{m}b_{i_{m}}} i_{m} = \text{id}\), then, setting
\[\bar{s}_{i} = \pi_{i_{m}b_{i_{m}}}^{-1}(\bar{s}),\]
we have \(\bar{s}_{i} \in \mathcal{P}_{i_{m}}\) and \(\pi_{i_{m}b_{i_{m}}} : \langle N_{i_{m}}, \bar{s}_{i} \rangle \to \langle N_{i_{m}}, \bar{s} \rangle\). Thus \(\bar{s}_{i} \in \mathcal{P}_{i_{m}}\) is captured at \(m\).

By \(\S 1\) Corollary 4 we conclude:
\[\langle N_{i_{m}}, \bar{s}_{i} \rangle_{i_{m}}^{i_{m}} \models \gamma_{i_{m}} < \langle N_{i_{m}}, \bar{s} \rangle_{i_{m}}^{i_{m}}\]

Hence \(\bar{s}_{i} < \kappa_{i_{m}}\). QED (Co3.4)
But then every $N_i^* \in \Sigma_1$ reflecting and all of the maps $\overrightarrow{b_n}_i^h$ ($1 \leq b \leq h = \text{const}$) are $\Sigma_1$-definable by §2 Lemma 1.5.

Now let $B \in N$ be $\Sigma_1(N^*)$ in $q_i$. Let $q \in \text{rng}(\overrightarrow{3_m}_i^b_m) \cap \text{rng}(\overrightarrow{3_{n+1}}_i^b_{n+1})$.

For $j \in b_m \setminus \overline{3_m}$ (for $m = n \circ m + 1$) let $B_j$ have the same $\Sigma_1$-definition in $q_i = (\overrightarrow{1_i}^{b_i}_m, b_m)$, $B \in \Sigma_i$ Lemma 1.3(c).

We then have:

* $\langle N_i, B \rangle$ is amenable
* $\overrightarrow{b_m}_i^h$, $\langle N_i, B, B_j \rangle \overset{\Sigma_i}{\rightarrow} \langle N_i, B \rangle$ cofinally.

Hence $B$ is captured at $m$. Hence by §1 Cor 4. we conclude:

$\langle N_i, B \rangle |_{\overline{3}_m}^m \langle N_i, B \rangle$.

But $\overline{3}_m^m \in \text{acc} \in N_i$. Hence $\langle N_i, B \rangle |_{\overline{3}_m}^m = \langle \overline{J_{E_i}^N}, B \setminus \overline{J_{E_i}^N} \rangle \in \overline{ZFC}$ model. Hence $\overline{3}_m^m \in \langle N_i, B \rangle$.

By §1 Fact 3 we conclude:

$N_i$ is definably regular in $N^* = \overrightarrow{J_i}^N(N_i)$.

Hence $N_i$ is regular in $\overrightarrow{J_{i+}\overline{\eta}}^N(N_i)$, contradicting the maximality of $\overline{\eta}$.

Contradiction! QED Thm 1
Finally we note:

**Thm 2** Let \( M = J_p(N) \) where \( p \) is as in Thm 1. Then \( p_1^* = S \).

**Proof**

\( \leq \) is given. We prove \( \geq \). We must show that if \( B \) is \( \gamma \in (M) \), \( B \subset N \), then \( \langle N, B \rangle \) is amenable.

Suppose not. We derive a contradiction.

**Proof**

For \( h = 0, 1 \) we define \( \varphi_i^h \) for \( h = 0, 1 \), since otherwise \( p \in M_{b_h} \) and \( N = \gamma \in M_{b_h} \), where \( \gamma \) is a cardinal in \( M_{b_h} \).

For \( h = 0, 1 \) we define \( \varphi_i^h : M \rightarrow M_{b_h} \) and \( \varphi_i^h : M \rightarrow M_{b_h} \) for \( i \in (b_h \backslash \overline{b}_h) \) exactly as before. The bridging maps \( \overline{\varphi}_i^h : M_{b_h} \rightarrow M_{b_h} \) and \( \overline{\varphi}_k^i : M_{b_h} \rightarrow M_{b_h} \) \( (i \leq k \text{ in } b_h) \) are defined as before. Then \( M \subset M_{b_0} \cap M_{b_1} \).

For \( i \in b_h \backslash \overline{b}_h \) set \( j^* = \overline{\varphi}_i^h \) and \( M_{j} = J_{p_i^*} (N_j) = \overline{\varphi}_i^{h_j} \) \( \text{ in } M \). Then \( M_j \) is transitive and

\[ \overline{\varphi}_k^j : M_j \rightarrow M_k \text{ and } \overline{\varphi}_k^j : M_j \rightarrow M_k, \]
where $\overline{\pi}_i k = \overline{\pi}_i k \cap M_i$, $\overline{\pi}_h = \overline{\pi}_h \cap M_h$ for $i \leq h \cup b_h$.

Claim: $\overline{\pi}_i \cup \Sigma_1$ preserves $\mu_{\Sigma_1}$ hence $\mu_{\Sigma_1} \leq \overline{\pi}_i k$ for $i \leq h \cup b_h$)

proof:

Case 1 $\overline{\pi}_i = M_i$.

Let $\varphi$ be $\Sigma_1$. Then

$M \models \varphi(\overline{\pi}_i(x_1)) \rightarrow \overline{\pi}_{b_h} \models \varphi(\overline{\pi}_i(x_1))$

$\rightarrow \overline{\pi}_i \models \varphi(\overline{\pi}_i)$

Since $\overline{\pi}_i \models \overline{\pi}_i \models \Sigma_1$ - preserves $\mu_{\Sigma_1}$.

Case 2 $p_i \in \text{wfc}(\overline{\pi}_i)$.

Then

$M \models \varphi(\overline{\pi}_i(x_1)) \rightarrow \overline{\pi}_{b_h} \models \varphi(\overline{\pi}_i(x_1))$

$\rightarrow M \models \varphi(p_i)$,

Since $\overline{\pi}_i \models M \models \varphi(p_i)$,

Case 3 The above fail.

Then $p_i \in \text{wfc}(\overline{\pi}_i)$. Let

$3 \in M \setminus p_i$, Then $\overline{\pi}_i(x_1) \in \overline{\pi}_{b_h} \setminus p_i$.

3
Then \( M' \mapsto \phi'(x, i) \rightarrow (\mathcal{L}^i_{b_n i, b_n} \cup \mathcal{L}^i_{1, 3}) \mapsto \phi'(x, i) \)
\( \rightarrow (\mathcal{L}^i_{1, 3}) \mapsto \phi'(x, i) \)
\( x \in \mathcal{L}^i_{b_n i, b_n} \)
\[ \mathcal{L}^i_{b_n i, b_n} \cap (\mathcal{L}^i_{1, 3}) \leq (\mathcal{L}^i_{1, 3}) \leq (\mathcal{L}^i_{b_n i, b_n}) \]
But this holds for all \( \bar{z} \subseteq \Sigma_1 \cap p \).

Thus, if \( z = (\mathcal{L}^i_{1, 3}) \) then \( z \in \mathcal{L}^i_{1, 3} \)
\( x, t, (\mathcal{L}^i_{1, 3}) \mapsto \phi'(x, i), \) Then \( x \in \mathcal{L}^i_{1, 3} \)
Then \( \mathcal{L}^i_{1, 3} \subseteq \mathcal{L}^i_{1, 3} \) and \( \mathcal{L}^i_{1, 3} = \phi'(x, i) \)
\( \text{Q.E.D. (Claim)} \)

Now let \( \mathcal{B} \in \Sigma_1 \) \( (M) \) in \( p \) where \( (wii i, : orig.) \) \( p \) is a grounding parameter for \( M \). Let \( \mathcal{B} \in \mathcal{R}(\mathcal{L}^i_{b_n i, b_n} \cap \mathcal{R}(\mathcal{L}^i_{b_n i, b_n} \cap \mathcal{R}) \).

As follows exactly as before that if \( \bar{z} = \mathcal{L}^i_{M}(i, \langle \rho, \mu \rangle) \) \( \leq \delta \) where \( \mu \leq \mathcal{L}^i_{c_n} \)
Then \( \bar{z} \leq \mathcal{L}^i_{c_n} \).

Set:
\[ X = \mathcal{L}^i_{M}(i, \{ \rho \}), \sigma, \mathcal{M} \leq \mathcal{L}^i_{c_n} X \]
Then \( \mathcal{L}^i_{c_n} = \mathcal{R}(\sigma), \sigma : (\mathcal{L}^i_{c_n}) = \delta \),
Then \( \sigma, \mathcal{M} \rightarrow \mathcal{M} \). Hence, if \( \mathcal{B} \in \Sigma_1 \),
\( \mathcal{L}^i_{c_n} \) over \( \mathcal{M} \) in \( p = \sigma^{-1}(p) \) by the same def. as \( \mathcal{B} \) over \( \mathcal{M} \) in \( p \),
Then $B_n(N_{\mu_m}) = \overline{B} \in N$, \text{ unio } \overline{M} \in N$
and $\sigma(N_{\mu_m}) = N$, \textbf{Q.E.D (Thm 2)}