§4 Improvements

§4.1 Active Mice

For the sake of simplicity we have up till now assumed that $Y$ is a truncation free iteration of a passive premise. We now drop these assumptions and show how the proof can be modified. Since both branches have at most finitely many truncations, we may willingly assume the ordinal $\alpha$ to be so chosen that no $j \leq b_n \setminus \delta_n$ is a truncation point ($n=0,1$). Thus $\varphi_1^{b_n} : M_1 \rightarrow M_{b_n}$ will still be a total function on $M_1$. However, $M_1$ may not be passive, either because $M_0$ was not passive or because a topper extender was introduced by an earlier truncation. This vitiates an important element of our proof.

We often need:
(⋆) If \( \kappa_i < \kappa < \chi_i \) and \( \kappa \) is a cardinal in \( J^{E_{\chi_i}} \), then \( E_{\chi_i} \upharpoonright \kappa \in M_{\kappa_i} \).

(Thi this became, in fact, \( E_{\chi_i} \in M_{\kappa_i} \).)

Thus when we set \( F^o = E_{\chi_i \cdot \kappa_i}, \) we knew that \( F^o \in M_{\kappa_i}. \) Since \( \lambda_{\chi_i} \leq \lambda_{\kappa_i}, \)

we also knew that \( \kappa_i = \kappa + M_{\kappa_i}, \) and

hence that \( F^o \in J^{E_{\chi_i \cdot \kappa_i}} \in N, \)

(⋆) can fail, however, if \( \kappa_i \) is the top extender of \( M_{\kappa_i}. \) This means that \( §1 (10) \) and with it §1 Lemma 1 may fail. An [NFS] §6 we developed a method of circumventing this problem, which we can also use here to prove a modified version of §1 Lemma 1.

(Note All of the proofs we have done up till now can easily be redone in Steel mice instead of \( \lambda \)-mice. Steel mice have the advantage that the above problem does not occur: Steel indices are co-
chosen that the analogue of (b) holds
even at top extenders. For $x$- mice, 
however, and for many possible 
intermediate indexing schemes
the method of [NFS786 is needed.)

In the following we shall state 
a number of facts without proof,
for which we refer the reader to 
[NFS786. We also follow the con-
vension used there of writing $E^\top_{\text{top}}$
to denote the top extender of an 
active premouse $M$. We define:

$$ND = \text{the set of } m > 0 \text{ s.t. } E^M_{\text{top}}^{i_m+1}$$

exist and crit $(E^M_{\text{top}}^{i_m+1}) \in (i_m, i_m - \xi_{i_m})$.

We then observe that in each $b_h$
there can be at most one $m$
with $i_m \in ND$. Hence we can 
with impunity suppose $d$ to be so
chosen that $b_h \cap ND = \emptyset$ ($h = 0, 1$).

In place of the sequence $F_{\alpha_m}$ of 
extenders on $r_i^\top$, we define a
new sequence \( G^m \) \((m < \omega)\) of extenders on a \( \kappa^* \) \( \nu_i < \kappa^* \) \( \nu_i + 1 \) \( \nu_i + 1 \) \( \nu_i \). We display this procedure for \( m = 0 \).

At \( E_{i_0} | \nu_{i_0} \in M_{i_0} \) we set \( G^m = F^m \) exactly as before. Now let \( E_{i_0} | \nu_{i_0} \in M_{i_0} \).

Then \( E_{i_0} \) is the top extender of \( M_{i_0} \).

Moreover \( \mathfrak{P}^m \leq \nu_{i_0} \) since \( E_{i_0} | \nu_{i_0} \) can be coded as a subset of \( \nu_{i_0} \). But

Then \( i_0 > i_1 \) since otherwise \( \nu_{i_0} \leq \lambda_{i_0} \) hence \( \mathfrak{P}^m \leq \nu_{i_0} \). Contradiction! Thus

\( i_0 \) is either a successor or limit ordinal. But then there is a \( s + 1 < i_0 \) s.t. \( \nu_{i_0} \leq \lambda_{s+1} \). Otherwise \( i_0 = \tau (i_{s+1}) = \mathfrak{P}^m \).

But, since \( \nu_{i_0} < \nu_{i_1} \), we then have

\( \text{crit} (E_{\nu_{i_0}}^{M_{i_0}+1}) = \nu_{i_0} \). Hence \( 1 \notin \text{ND} \).

Contradiction! 

Set \( s = s_{i_0} = s_0' \) the least such \( s \).

\( s' \) can then be shown to have the following properties:
(a) \( \kappa \leq \kappa' \leq \kappa' \)

(b) \( \pi_{y+1, i_0} : M_{y+1} \to M_i \) in a total map

(i.e. there is no truncation between \( y+1 \) and \( i_0 \))

\[ A^+ E_{M_i}^{\kappa_i} / \kappa' \in M_{\kappa'} \quad \text{we set} \quad G^0 = E_{M_i}^{\kappa_i} / \kappa_i \]

Then \( \kappa' + M_{\kappa'} = \kappa' + M_{\kappa_i} = \kappa' + N \) and we conclude that \( G^0 \in N \). \( A^+ E_{M_i}^{\kappa_i} / \kappa_i \in M_{\kappa_i} \)

Then \( E_{M_i}^{\kappa_i} \) is the top end of \( M_{\kappa_i} \), and we repeat the process with \( y \) in place of \( i_0 \).

In this way we obtain a descending sequence:

\[ i_0 = y^0 > y^1 > y^2 > \ldots \]

The sequence must terminate at some integer \( p \). We then set:

\[ y^p = x^p, \quad G^0 = E_{M_{y^p}}^{\kappa_i} / \kappa_i \]

Then \( G^0 \in M_{y^p} \) and, in fact, \( G^0 \in N \).

We also have \( \kappa_i \leq \kappa' < \kappa_i \) where \( \kappa_i = \kappa_{y^p} \). Moreover, each of the maps \( \pi_h \) is total on \( M_{y^p} \), where

\[ \pi_0 = \text{id} / M_{y^0}, \quad \pi_{h+1} = \pi_h / M_{y^{h+1}}, \quad \pi_h \]

for \( h < p \).
Just as before we define:

\[ G^{m+1} = \frac{1}{3m_n + 1} \left( G^n \right) | _{l_{m+1}^{+1}} \]

for \( m < \omega \). Hence \( G^m \in N \) for \( m < \omega \), verifying the strengthening of \( \kappa = \kappa_0 \in N \).

An place of §1 Lemma 1 we then get:

**Lemma 1** \( A \cap B \in N \) if captured at \( n \)

then \( \kappa_m \) is strong in \( \langle N, B \rangle \).

**Proof (sketch).**

\( B \) can be easily coded as a subset of \( S \), so suppose \( B \subseteq S \). We display the proof for \( m = 0 \), showing:

**Claim** \( \kappa_m \) (\( \text{Sh} \cap \kappa \)) = \( \text{Sh} \cap \kappa \)

where \( \kappa_m : N \rightarrow G^m N \).

The case \( m > 0 \) is exactly as before.

So let \( m = 0 \), \( A \cap \kappa = \kappa_0 \), then \( G^0 = F^0 \) and the proof is exactly as before. Now let \( \kappa \), \( \delta = \delta^+ \). Then \( \kappa = \kappa_\delta \) where \( \delta = \delta^+ \).
Let $\mathfrak{f} = T(\mathfrak{g} + 1)$. Then

$$
\overline{\mathfrak{f}}'_i, i_0 \quad (B \cap \kappa_y) = \overline{\mathfrak{f}}'_i, \beta + 1 \quad (B \cap \kappa_y).
$$

Let $\overline{\mathfrak{f}}'_i, i_0 \quad (\mathfrak{g}_r \cap \mathfrak{g}_o) \rightarrow \mathfrak{m}_0$.

Then $\overline{\mathfrak{f}}'_i, i_0 \quad \text{taken } E^{m_r} \rightarrow E^{m_o}$.

Thus, if we let:

$$
\sigma : \mathfrak{m}_0 \rightarrow \mathfrak{m}, \quad \overline{\sigma} : \overline{\mathfrak{m}_r} \rightarrow \overline{\mathfrak{m}}.
$$

Then $\overline{\mathfrak{f}}'_i, \overline{\sigma} = \sigma \overline{\mathfrak{f}}'_i, i_0$.

Thus:

$$
\overline{\mathfrak{f}}'_i, (B \cap \kappa_y) = \overline{\mathfrak{f}}'_i, (\sigma (B \cap \kappa_y) \cap \kappa_y) = \\
= \sigma (B \cap \kappa_y) \cap \lambda_y = B \cap \lambda_y.
$$

Hence $\kappa_y < \kappa_y$.

Let $\sigma : \mathfrak{m}_0 \rightarrow \mathfrak{m}$.

Hence:

$$
G^0 (B \cap \kappa_y) = \overline{\mathfrak{f}}'_i, \beta + 1 \quad (B \cap \kappa_y) \cap \kappa_y = \\
= \overline{\mathfrak{f}}'_i, (B \cap \kappa_y) \cap \kappa_y = B \cap \lambda_y \cap \kappa_y = B \cap \kappa_y.
$$

This proves the case $\overline{\sigma} = \sigma^0$. At $\overline{\sigma} < \sigma^0$,

we iterate this argument, showing,
\[
\overline{\exists \delta, \forall \gamma \forall \eta (B \land \gamma \land \eta) \land \chi \vdash B \land \chi}
\]

for \( h \leq p \), where \( \bar{\gamma} = \gamma^p \), by induction on \( h \).

Q.E.D \text{ (Lemma 1)}

(Note The full details can be read in [NFS] §6. The assumption on \( B \in N \) is weaker than here, but the proof is exactly the same.)

Lemma 2 holds as stated in §4, but its proof must be amended:

**Lemma 2** Let \( B \in N \) be captured at \( m \).
Let \( \tilde{N} = \langle N, B \rangle \). Then \( \tilde{N} \models \chi \iff \bar{\chi} \).

**Proof:**
Let \( \phi \in \cup \mathcal{E}_m \) and \( \tilde{N} \models \phi(x) \); when \( \phi \in \mathcal{E}_m \),

Claim \( \tilde{N} \models \chi \iff \phi(x) \).

Arguing as in §7 but using the amended form of Lemma 1, we find yet:
(1) $\overline{N} | \overline{r}_m \models \varphi(x)$.

But $\overline{r}_m < r_{i_{m+1}} < \lambda_{i_{m}}$. Since $\overline{N} | \overline{r}_m \leq \sum \overline{N} | \lambda_{i_{m}}$, we conclude:

(2) $\overline{N} | \lambda_{i_{m}} \models \varphi(x)$.

But $\prod_{i_{m}}^{i_{m+1}} (\overline{N} | \lambda_{i_{m}}) = \overline{N} | \lambda_{i_{m}}$

since $\text{crit}(\overline{r}_m, b_m) \geq \lambda_{i_{m}}$.

Hence $\overline{N} | r_{i_{m}} \leq \overline{N} | \lambda_{i_{m}}$ and

(3) $\overline{N} | r_{i_{m}} \models \varphi(x)$. \quad \text{(QED (Lemma 2))}

Lemma 3 and Cor. 4 of § 1 then follow exactly as before.

The proofs in § 3 go through virtually unchanged.
§ 4.2  \textit{E - Woodinness}

\textbf{Def} Let \( M = \mathcal{J}_\theta (N) = \mathcal{J}_\theta^E \), where \( N = \mathcal{J}_\theta \),
\( \theta \) is a premouse, \( \Theta \in E - \text{Woodin in } M \) if \( \Theta \) is Woodin as instantiated by \( \mathcal{J}_\theta \),
lying on the sequence given by \( E \).

More precisely:

\textbf{Def} Let \( F \in N \) be an extender of
length \( \kappa \), \( F \) conforms to \( N \) if there
\( \nu \in N \) s.t.
\( F = E_{\nu \mid \kappa} \)
where \( \kappa < \mu < \lambda \),
\( \nu = \text{crit} (E_{\nu \mid \kappa}) \), \( \lambda = E_{\nu \mid \kappa} (\kappa) \).

\( F \) generates \( E_{\mu} \), i.e., \( E_{\mu} = \mathcal{H}(\text{dom} J^E_{\lambda}) \)
where \( \tau = \kappa + J^E_{\lambda} \) and \( \tau : J^E_{\lambda} \rightarrow J^E_{\mu} \).

\textbf{Def} \( \kappa \in E - \text{strong in } \tilde{N} = \langle N, B \rangle \) if
for arbitrarily large \( \lambda \in N \) there
\( \nu \in F \in N \) which is strong w.r.t. \( \tilde{N} \)
and conforms to \( N \).

\textbf{Def} \( \Theta \in E - \text{Woodin in } M = \mathcal{J}_\theta (N) \) if \( B \in M \) for every \( B \in N \) s.t. \( B \in M \) there is
\( \kappa \in N \) which is \( E - \text{strong w.r.t. } \langle N, B \rangle \).
We can strengthen the conclusion of Thm 1 §3 from "Woodin" to "E-Woodin" if we assume that the $M_0$ which we are iterating is not only a premouse, but also "mouse-like" in the sense that it internally satisfies the condensation lemma of [EPR] §8 Lemma 4'. An fact, the only consequence of those lemmas we need is:

(*) Let $E_\gamma \neq \emptyset$. Let $\kappa = \text{crit}(E_\gamma)$, $\kappa < \gamma < \lambda = E_\gamma(\kappa) \Rightarrow \gamma$ is a limit cardinal in $J_\lambda^E$. Then $E_\gamma, \lambda$ conforms to $M_0$.

Since (*) holds in $M_0$, it also holds in $N$. But that is enough to tell us that the $F^m (m < \omega)$ which verified the $< N, \beta >$-strongness of $\kappa_0$ in the proof of §1 Lemma 1
are all $N$-conforming. Similarly it tells us that the extenders $G^n(m,c,w)$ used in the revised version of in §4.1 are $N$-conforming.