The Extended Namba Problem

§0 Introduction

Shortly after the discovery of forcing, it was noticed that any regular $\kappa > \omega_1$ can be collapsed to $\omega_1$ in a generic extension without adding real. Levy showed that, if $\kappa$ is strongly inaccessible, then every smaller cardinal can be simultaneously collapsed to $\omega_1$, so that $\alpha$ becomes $\omega_2$ in the extension. An these constructions, however, the collapsed cardinals always acquired cofinality $\omega_1$ in the extension. Namba then showed that, assuming CH in the ground model, one can collapse $\omega_2$ to $\omega_1$ in such a way that it becomes $\omega$-cofinal in the extension. A number of interesting results were obtained by combining these collapsing techniques with iterated forcing. From Shelah showed
That one can iterate Namba forcing out to a strongly inaccessible \( \kappa \) in such a way that \( \kappa \) becomes \( \omega_2 \) and every other regular \( \tau < \kappa \) acquires cofinality \( \omega_1 \). However, the answer to the following question — which we shall here call the "extended Namba problem" — remained elusive:

Assume CH. Let \( \kappa \) be strongly inaccessible. As there a generic extension in which \( \kappa \) becomes \( \omega_2 \) and in which every regular \( \tau \in (\omega_1, \kappa) \) becomes \( \omega \)-cofinal?

This seemed so difficult to us that for several years we conjectured that it was provably impossible in ZFC. Moti Gitik then refuted this conjecture by constructing a model of ZFC which admits such an extension. Gitik started with a cardinal \( \theta \) which is supercompact up to a Mahlo cardinal \( \kappa \). He then collapsed to make \( \theta = \omega_1 \)
Over the resulting model he then found a generic extension in which \( \kappa = \omega_2 \) and every regular \( \kappa \in (\theta, \kappa) \) becomes \( \omega \)-cofinal. The question remained open, however, whether the super Namba problem always has a positive solution. We now provide an affirmative answer:

**Theorem 1.** Assume CH. Let \( \kappa \) be strongly inaccessible. There is a set of conditions \( P \) of size \( \kappa \) in \( V \) s.t. \( P \) is subcomplete and whenever \( G \in \mathcal{P}_{\omega_2} \) generic, then:

- \( \kappa = \omega_2 \) in \( V[G] \).
- Every regular \( \tau \in (\omega_1, \kappa) \) becomes \( \omega \)-cofinal in \( V[G] \).
- Every stationary subset of \( \kappa \) remains stationary in \( V[G] \).

(Note: By subcompleteness, there are no new reals in \( V[G] \).)

(Note: \( P \) will satisfy \( \kappa - \text{cc} \), which by itself guarantees the last property.)
Modifying our methods slightly we can also get:

**Theorem 2** Let \( \kappa \) be inaccessible and assume GCH below \( \kappa \). Let \( A_0 \subseteq \kappa \).

There is a set of conditions \( \mathbb{P} \) of size \( \kappa \) s.t. \( \mathbb{P} \) is subcomplete and whenever \( G \) is \( \mathbb{P} \)-generic, then

- \( \kappa = \omega_2 \) in \( V[G] \)
- Let \( \tau \in (\omega_1, \kappa) \) be regular. Then
  \[
  c(\tau) = \begin{cases} 
  \omega_1 & \text{if } \tau \in A_0 \\
  \omega & \text{if not}
  \end{cases} \text{ in } V[G].
  \]
- Every stationary subset of \( \kappa \) remains stationary in \( V[G] \) (in fact, \( \mathbb{P} \) satisfies \( \kappa - \text{cc} \)).

(Note Theorem 1 can be derived from Theorem 2, though that is not the way we do it in this paper. Using \( \omega_1 \)-complete conditions we can do an Easton type collapse below \( \kappa \) s.t. GCH holds below \( \kappa \) and \( \kappa \) remains inaccessible in the extension. Since the conditions are \( \omega_1 \)-complete, every regular)}
If we then apply Thm 2 with $A_0 = \emptyset$, it is clear that every $T \in (\omega_1, \kappa)$ which is regular in $V$ will become $\omega$-cofinal.

As one might suspect, IP is obtained as the result of an iteration — albeit an iteration in a very broad sense. The most general notion of iteration is the following: $\langle IB_i : i < \lambda \rangle$ is an iteration if each $IB_i$ is a complete Boolean algebra, $IB_i \subseteq IB_{i+1}$ (i.e. $IB_i$ is completely contained in $IB_{i+1}$) for $i < \lambda$, and for all limit $\lambda < \chi$,

$IB_\lambda$ is generated by $\bigcup_{i < \lambda} IB_i$. Most often $IB_\lambda$ is taken as the direct limit or the inverse limit of $\langle IB_i : i < \lambda \rangle$, or as something intermediate between the two. In the present case, however, we shall often be forced to use a completion $IB_\lambda$ which has a much larger cardinality than either the direct or inverse limit.

(The reasons for this are explained in §2.1.) Hence we are unable to use the iteration theorem of [SPSC]. In §1 we prove a new iteration theorem
which we shall use to ensure that each $\Pi_{\lambda}^0$ in our iteration $\langle \Pi_{\lambda}^0, \alpha \leq \kappa \rangle$ is subcomplete. In §2 we open with a discussion of the constraints that the iteration must satisfy. We show first, that the first stage of the iteration, whose purpose is to give $\omega_2$ the cofinality $\omega_1$, must, in fact, make every regular $\tau \in (\omega_1, \omega_1)$ $\omega_1$-cofinal. Hence $\Pi_{\lambda}^0$ must be much larger than $\omega_2$. At the first limit point (and many others) there is a similar constraint on $\Pi_{\lambda}^1$, which is why $\Pi_{\lambda}^1$ must be larger than the direct or inverse limit. In §2.2 we then introduce the notion of "prood forcing", which will play a large role in our proof. In §2.3 we describe the proposed iteration more precisely. The iteration is intended to prove both Theorem 1 and Theorem 2; hence an arbitrary $A_0 \in \kappa$ enters into its definition. We assume GC [1] below only in the case $A_0 \neq \emptyset$. 


In §3 and §4 we then fully define the iteration and verify its properties. All of the component forcings in the proof of Theorem 1 are $L$-forcings and can be regarded as generalizations of Namba forcing. (The successor step will, in fact, be a variation on the forcing described in the appendix to [LF] §5.)

Bibliography

[LF]  $L$-Forcing

[SPSC] Subproper and Subcomplete Forcing

Unfortunately, I must assume a knowledge of both papers, which exist in the form of handwritten notes and can be found on my website.)