§3 The successor case

We are given \(<IB_i \mid i \leq \mu>\) satisfying (a)-(b) and wish to construct \(IB_{\mu+1}\) s.t. (a)-(b) continue to hold.
Let \(\gamma = \gamma_{\mu+1}\). Then \(\gamma = w_2\), \(\gamma = \gamma_1\) and \(\gamma = \beta^+\) if \(\beta^+ \in V\). Otherwise \(\gamma = \gamma = \gamma\) is inaccessible in \(V\). An either case we know that \(IB_\mu \subseteq H_\gamma\).

§3.1 The first successor case

Suppose that \(\gamma \in A_0\). Then \(\text{GCH}\) holds below \(\kappa\) and we wish to make \(\gamma\) \(\omega_1\)-cofinal without collapsing \(\beta^+\). To this we force with \(\text{coll}(\omega_1, \omega_2)\) over \(V[G]\), where \(G\) is \(IB -\) generic. An otherwise we let \(\gamma = IB * IB\), where \(IB = \text{coll}(\omega_1, \omega_2)\), \(\mu^+ = IB\), and \(\mu = IB\).

We verify (a)-(b), (a) is straightforward, (b) holds, i.e. \(\text{coll}(\gamma, \gamma)\) is subcomplete, (c) is immediate, since \(\mu+1\) is not a limit point and (a)-(b) hold of \(<IB_i \mid i \leq \mu>\). Similarly for (d), (e) follow by §2 Lemma 4.1, (f) follow by §2 Lemma 4.2. We now sketch the proof of (f). Let \(B\) be \(IB_\mu\) - generic, where \(h \leq \mu\).
Set \( \tilde{B}_i = B_i / B \) for \( h \leq i \leq \mu + 1 \). (Then \( \langle \tilde{B}_{h+i} \mid 1 \leq \mu + 1 - h \rangle \) \( \in \) the new iteration in \( V[B] \).) We first note that if \( B' \) \( \in \) \( B \)-generic over \( V \), then \( B / B' \cong \tilde{B} \tilde{B}' = BA(\text{coll}(\omega_1, \omega_2)) \), \( \mu + 1 \) \( \in \) \( V[B'] \). (Recall that \( \tilde{B} \cong \tilde{B} * \tilde{B} \).

where \( \text{coll}(\omega_1, \omega_2) \).

Now let \( \tilde{B} \) be \( \tilde{B} \)-generic over \( V[B] \).

Then \( \tilde{B} / \tilde{B} = (B / B) / \tilde{B} \cong B / B' \), \( \mu + 1 \) \( \in \) \( V[B] \), \( \tilde{B} \) \( \in \) \( B \)-generic over \( V \). Hence there \( \tilde{B} \in \tilde{B} \in \tilde{B} \) \( \in \) \( V[B] \).

\( \text{coll}(\omega_1, \omega_2) \).

in \( V[B] \), \( \tilde{B} \) being the canonical generic name). Let \( \tilde{B} = BA(\text{coll}(\omega_1, \omega_2)) \).

Define \( \sigma : \tilde{B}_{\mu+1} \cong \tilde{B} + \tilde{B} \) \( \in \) \( V[B] \).

Let \( \sigma(a) = \text{that } a' \text{ s.t. } \tilde{B} \sigma(a) / \tilde{B} \).
Then for \( b \in \tilde{\mathcal{B}} \), we have:

\[ \sigma \left( b \right) = b' \text{ where } \begin{cases} 1 \text{ if } b' \in \mathcal{B} \\ 0 \text{ if } b' \in \tilde{\mathcal{B}} \end{cases} \]

- i.e. \( \sigma \circ \tilde{\mathcal{B}} \) is the natural injection.

Thus \( \tilde{\mathcal{B}}_{+1} \) satisfies precisely those conditions which we had placed upon \( \mathcal{B}_{+1} \) in \( \mathcal{T} \). Thus we can carry out all of our proofs in \( \mathcal{T} \left[ \mathcal{B} \right] \) with \( \left< \tilde{\mathcal{B}}_{h+j} \mid j \leq \kappa+1-h \right> \) in place of \( \left< \mathcal{B}_{h} \mid i \leq \kappa+1 \right> \).

Finally, we note that, since GCH holds below \( \kappa \), elementary consideration gives us: \( \mathcal{B} \times \mathcal{B} \) has cardinality \( \leq \aleph^+ \). Hence we can choose \( \mathcal{B}_{+1} \) s.t. \( \mathcal{B}_{+1} \subseteq \mathcal{H}_{\aleph^+} \).

This completes the first successor case. The second will be much harder, since we shall need \( \mathcal{B} = \mathcal{B} \times \mathcal{B} \) for a \( \mathcal{B} \) which has yet to be defined.
The second successor case

Now suppose \( \kappa \notin A_0 \), where \( \kappa = \kappa_{a+1} = \omega_2 \).

Then \( \kappa \) must acquire cofinality \( \omega \) at the next stage. But then all regular cardinals in \( [\kappa, \beta_{a+1}] \) must become \( \omega \)-cofinal. Recall that \( \beta_{a+1} \) is the least \( \beta \) such that either \( \text{cf}(\beta) = \omega \) and \( 2^{\text{cf} \beta} = \beta \), or \( \beta \in A_0 \) is regular.

In the latter case \( \beta = \beta^+ \) in a successor cardinal, \( \beta \geq \kappa \), and \( 2^{\text{cf} \beta} = \beta \), since GCH holds below \( \kappa \) i.e. \( A \neq \emptyset \). From now on let \( \beta \) be defined by:

\[
\beta = \beta_{a+1} \quad \text{if} \quad \text{cf}(\beta_{a+1}) = \omega,
\]

\[
\beta^+ = \beta_{a+1} \quad \text{if} \quad \text{not}.
\]

where \( \beta \) is a cardinal.

Now let \( B \) be \( IB \)-generic. We work in \( V[B] \) to define a set of conditions \( P = P_B \) which collapses all regular \( \in [\kappa, \beta] \) to \( \omega \). If \( \text{cf}(\beta) = \omega \), then \( \beta^+ \) becomes \( \omega_2 \) in \( V[B]^{P_B} \). Otherwise \( \beta^+ \) becomes \( \omega_2 \). We then take:

\[
\text{IB}^{a+1} = \text{IB} \times \text{IB} \quad \text{where} \quad \text{IB} = \text{BA}(P_B),
\]

\[
\beta^+ = \beta_{a+1} \text{ }^{IB} \quad \text{if} \quad \text{not},
\]

where \( B \) being the canonical generic name.
Let $A \in V$ n.t. $A \subseteq H_\gamma$ and $H_\gamma^V = L_\gamma^V$ \(\star\)

whenever $\gamma \leq \beta$ n.t. $2^\beta = \gamma$.

Set:

$$M = L_{\beta_\gamma}^{A \cup B_\mu}, \quad N = \langle H_{\beta_\gamma}^V, M, \langle, \rangle \rangle,$$

where $\langle, \rangle$ is a well ordering of $N$.

$$Q = H_{\gamma_\gamma}^V.$$

Def. Working in $V[B]$ where $B \cup B_\mu$ is a generic set:

$$M_B = L_{\beta_\gamma}^{A \cup B \cup B_\mu}, \quad N_B = \langle H_{\beta_\gamma}^V, M_B, \langle, \rangle \rangle,$$

(Note $M_B$ has the same set as $H_{\beta_\gamma}^V$)

$$Q_B = Q[B] = \bigcup_{x \in Q} L_{\gamma_\gamma}^V [x, B].$$

(Note $Q_B = H_{\gamma_\gamma}^V$ since $\gamma = \omega_2$ in $V[B]$ is regular in $V[B]$. Hence if $B, B'$ are $|B|_\mu$-generic and $V[B] = V[B']$, then $Q_B = Q_B'$.)

Working in $V[B]$ we now define:

$\Gamma^*_\star$ = the collection of $<S, C>$ n.t.

- $S$ is a transitive set
- $S \subseteq (\mathbb{ZFC}^- + \omega_1 \text{ is the largest cardinal})$
- $C < S$ cofinally
- $C$ is countable

(Recall that "$C < S$ cofinally" means $UC = S$.)
Def. For \( u = \langle S_u, C_u \rangle \) \( \forall v = \langle S_v, C_v \rangle \in \Gamma^* \) let:
\[
\pi: u \triangleleft_\pi v \iff (\pi: S_u \triangleleft S_v \land \pi: C_u = C_v)
\]

Def. \( u \triangleleft_\pi v \iff \forall \pi: u \triangleleft_\pi v \)

Def. For \( u = \langle S_u, C_u \rangle \in \Gamma^* \) let:
\[
u_u = s_u = \omega_u S_u.
\]

The following facts are readily verified and will be stated here without proof:

Fact 1. Let \( \langle S, C \rangle \in \Gamma^* \) \( i = \omega_S \). Then:
\[
S = \{ f(v) \mid f \in C \land v < \omega_S \}
\]

Fact 2. \( \pi: u \triangleleft_\pi v \), then \( \nu_u \leq \nu_v \) and
\[
\text{any}(\pi) = \{ f(v) \mid f \in C_v \land v < \nu_u \}
\]

Hence:

Fact 3. For any \( \nu \leq \nu_v \) there is at most one pair \( \langle u, \pi \rangle \) s.t.
\[
\pi: u \triangleleft_\pi v \text{ and } \nu_u = \nu.
\]

Hence:

Fact 4. Let \( u \triangleleft_\pi v \). There is exactly one \( \pi \) s.t. \( \pi: u \triangleleft_\pi v \).

Def. \( \pi u v \triangleleft^{\pi} \) that \( \pi \) s.t. \( \pi: u \triangleleft_\pi v \).
Fact 5: \( \langle \forall u \mid u \leq \ast \rangle \) is a continuous commutative system.

Note: "continuous" means that if \( u_i \leq \ast, u_j \leq \ast \) for \( i \leq j \leq \lambda \), then the transitively directed limit \( \langle u_i, \langle \forall u_i \mid i \leq \lambda \rangle \rangle \) of \( \langle u_i \mid i \leq \lambda \rangle, \langle \forall u_i \mid i \leq \lambda \rangle \rangle \) exists and there is \( \pi : u \leq \ast \) defined by \(~\pi \pi u_i u = \pi u_i \rangle \langle i \leq \lambda \rangle.\)

Hence:

Fact 6: \( \exists u \mid u \leq \ast \wedge u \neq v \rangle \rangle \) is closed in \( \ast \).

We now define:

\[ \text{Def } R \text{ is a smooth model if: } \]
\[ R = \frac{A}{B} \text{ for some } A \models \text{ZFC, } B \text{ a Zermelo set theory}. \]
\[ \langle E \mid [A^*] = H^R \rangle \text{ whenever } H^R = \gamma \text{ in } R. \]

\[ \text{Def } R = \text{the set of } \langle R, c \rangle \text{ not,} \]
\[ R \text{ is a smooth model} \]
\[ \langle Q, c \rangle \in \Gamma \text{ where } Q = H^R \omega_1. \]

We also write \( Q = Q_{(R, c)} = H^R_{\omega_1}. \)
**Def** Let \( \mu = \langle \mathcal{R}_u, \mathcal{C}_u \rangle, \nu = \langle \mathcal{R}_v, \mathcal{C}_v \rangle \in \mathcal{P}. \)

\( \pi \vdash \mu \leq \nu \) iff

- \( \pi : \mathcal{R}_u \leq \mathcal{R}_v \)
- \( \pi \mathcal{R}_u \mathcal{C}_u \leq \pi \mathcal{R}_v \mathcal{C}_v \)
- There is \( \mathcal{R}_u \mathcal{C}_u \) s.t. \( \langle \mathcal{R}_u, \pi \mathcal{C}_u \rangle \) is the
  lift-up of \( \langle \mathcal{R}_u, \mathcal{C}_u \rangle \)

(Hence the map \( \pi \) is wholly determined by \( \pi \mathcal{C}_u \).)

**Def** \( \mu \leq \nu \) \( \vdash \forall \pi : \mu \leq \nu \).

At follows easily that:

**Fact 7** Let \( \mu \leq \nu \). There is exactly one
\( \pi \) s.t. \( \pi : \mu \leq \nu \).

**Def** \( \Pi_{\mu \nu} = \) that \( \pi \) s.t. \( \pi : \mu \leq \nu \)

**Fact 8** \( \langle \Pi_{\mu \nu} (\mu \leq \nu) \rangle \) is a continuous
commutative system.

However, the analogue of Fact 3 does not hold for \( \mathcal{P} \), since \( \langle \mu \leq \nu \rangle \) need
not be linearly ordered by \( \leq \).

None the less we do have:
Fact 9. Let \( u, w < v \), \( \text{rng}(u, v) \cap \text{rng}(w, v) \). Then \( u < w \) and \( \pi(u, v) \cap w = \pi(u, v) \).

The following fact will often be used tacitly:

Fact 10. Let \( \pi : \langle R, C \rangle \rightarrow \langle R', C' \rangle \). Let \( \gamma \in R \) s.t. \( \overline{Z} = \gamma \in \overline{R} \) and \( \gamma \geq \omega_2 \).
Let \( \pi(\gamma) = \gamma' \), \( R = L_{\beta}^{\overline{A}} \), \( R' = L_{\beta'}^{\overline{A}'} \).
Let \( \pi = \pi : \langle R, C \rangle \rightarrow \langle R', C' \rangle \).

Then \( \pi : \langle \overline{R}, C \rangle \rightarrow \langle \overline{R}', C' \rangle \).

Proof. Clearly \( \pi : \overline{R} \leq \overline{R}' \) and \( \overline{R} \) models \( ZFC^- \) or Zermelo. Moreover \( \pi^"C = C" \) and \( \pi^"CR = CR \). Hence \( \pi^"CR : \langle CR, C \rangle \rightarrow \langle CR, C' \rangle \).

Claim. Let \( \pi : \overline{R} \rightarrow \overline{R} \) cofinally. Then \( \langle \overline{R}, \pi \rangle \) in the lifting of \( \langle \overline{R}, \pi \rangle \).

\( \triangleright \)
We must show that \( \pi : \overline{R} \rightarrow \overline{R} \) is \( \omega \overline{R} \)-cofinal. Let \( x \in \overline{R} \). Then \( x \in \overline{R} \) where \( x \in \overline{R} \) where \( x \in L_{\pi(w)}^{\overline{A}} \) where \( \pi(w) \). Let \( \pi(w) \). Then \( z \in \overline{R} \), \( z \in \overline{R} \) where \( x \in L_{\pi(w)}^{\overline{A}'} \) where \( \pi(w) \). Let \( \pi(w) \) and \( x \in \pi(z) = \pi(w) \neq L_{\pi(w)}^{\overline{A}'} \).

\( \Box \) (Fact 10)
We return now to $Q^B, M^B, N^B$ as defined above. We shall use an
infinitary language $L^B$ on $N^B$ to define an $L$-forcing $IP^B = P^L^B$
in $V^B$. $IP^B$ is intended to add
a set $(Q^B, C) \in \Gamma^*_\beta$ without adding
new reals. However, $C$ should make
not only $\beta$ co-final, but every
regular $\alpha \leq \beta$.

$L = L^B$ is the infinitary language on
$N^B$ with:

**Predicate** $\in$, **Constants** $x, (x \in N^B)$, $c$

**Axioms**:
1. $ZFC^-$, $\forall u (u \in x \iff x \in x)$
2. For $x \in N^B$, $H_{\omega_1} = H_{\omega_1}$, $\check{c} \leq M^B$, and

\[
(\exists ! x) \forall x \in M^B \forall u \in H_{\omega_1} \forall \pi (\pi \upharpoonright u \leq \langle M^B, \check{c} \rangle) \\
\land x \in \text{rng}(\check{\pi}) \land \Psi(\check{\pi})^x
\]

(This says, in particular, that every
$x \in M^B$ can be found in the lifting
of a countable $\check{M}$ by a $\check{\pi}$.)

$\check{\pi} : \langle Q^B, \check{c} \rangle \leq^* \langle M^B, \check{c} \rangle$,

where $\Psi(\check{\pi}) = \begin{cases} \text{true} & \text{if } \beta \text{ is regular,} \\
\pi = \check{\pi} & \text{if not} \end{cases}$
Lemma 1. Let $L$ be a consistent.

Proof.

Let $\sigma: \tilde{N} \rightarrow NB$ where $\tilde{N}$ is countable and transitive. Set $\tilde{\alpha} = \mu_{\tilde{N}}$. Let 
$\sigma: \tilde{\alpha} \rightarrow \tilde{\alpha}$ cofinally. Let $\langle \tilde{N}, \tilde{\alpha} \rangle$ be the lift-up of $\langle N, \sigma \tilde{\alpha} \rangle$. Let 
$k: \tilde{N} \rightarrow NB$ s.t. $k \tilde{\alpha} = \sigma$ and $k \tilde{\alpha} = \tilde{\alpha}$. Let $\tilde{L}$ be defined on $\tilde{N}$ like $L$ on $N$.

It suffices to show:

Claim: $\tilde{L}$ is consistent.

since this is a $\Sigma_1(\tilde{N})$ statement.

We show that $\langle \mu_{\tilde{N}}, \tilde{\alpha} \rangle$ models $\tilde{L}$ where $\tilde{\alpha} = \gamma_{\tilde{N}}(\sigma \tilde{\alpha})$. (Note that $\tilde{N} \in \mu_{\tilde{N}}$.)

All axioms other than

$\ast 1$ are trivial. We verify $(\ast)$. Set:

$D$ is the set of $\tilde{\alpha} \in \omega_1$ s.t. there is a $u_\tilde{\alpha} = \langle \alpha_\tilde{\alpha}, c_\tilde{\alpha} \rangle$ with $\langle \alpha_\tilde{\alpha}, c_\tilde{\alpha} \rangle \subseteq \langle \tilde{\alpha}, \tilde{\alpha} \rangle$

Then $D$ is club in $\omega_1$ and $\delta_0 = \omega_1$ is minimal in $D$ with $u_{\delta_0} = \langle \tilde{\alpha}, \tilde{\alpha} \rangle$.

Since $\langle \tilde{N}, \tilde{\alpha} \rangle$ is the lift-up of $\langle N, \sigma \tilde{\alpha} \rangle$ and $\sigma \tilde{\alpha} = \tilde{\mu}_{\delta_0}, \langle \tilde{\alpha}, \tilde{\alpha} \rangle$ we see that $\langle \tilde{N}, \tilde{\mu}_{\delta_0}, u_{\delta_0} \rangle$ has a transitive
lift up $\langle N_d, \tilde{c}_{\alpha, d} \rangle$. Moreover, there is a map $\tilde{c}_{\alpha, d, \omega_1} : N_d \times \tilde{N}$ defined by

$$\tilde{c}_{\alpha, d, \omega_1} (\tilde{c}_{\alpha, \omega_1} (f) (\omega_1)) = \tilde{c} (f) (\omega_1),$$

where $\omega_1 < \omega$, and $\tilde{c}_{\alpha, \omega_1} : f : \omega_1 \rightarrow \tilde{N}$.

Set: $\tilde{c}_{\alpha, \omega_1} = (\tilde{c}_{\alpha, \omega_1})^{-1} \tilde{c}_{\alpha, \omega_1}$ for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \xi \omega_1^3$. At $\omega_1$ clearly there is a definition that:

$$\langle N_\beta, \tilde{c}_{\alpha, \omega_1} \rangle = \text{the lift up of } \langle N_\beta, \tilde{c}_{\alpha, \omega_1} \rangle$$

for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \xi \omega_1^3$, with $\omega_1 = \langle \tilde{c}_{\alpha, \omega_1} \rangle$.

Now set: $\tilde{M} = \tilde{c}^{-1} (M)$, $M_d = \tilde{c}_{\alpha, \omega_1} (\tilde{M})$

for $\alpha \in D \cup \xi \omega_1^3$. Then $M_\omega = \tilde{M} = \tilde{c} (M)$.

Set $\tilde{c}_{\alpha, \omega_1} = \tilde{c}_{\alpha, \omega_1} \uparrow M_\omega$ (for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \xi \omega_1^3$).

Clearly $\tilde{M} = \bigcup_{\alpha \in D} \text{rg} (\tilde{c}_{\alpha, \omega_1})$, so it suffices to show:

Claim $\tilde{c}_{\alpha, \omega_1} : \langle M_\omega, c_\lambda \rangle \triangleleft \langle \tilde{M}, \tilde{c} \rangle$.

Clearly $\tilde{c}_{\alpha, \omega_1} : M_\omega \rightarrow \tilde{M}$ and $\tilde{c}_{\alpha, \omega_1} : c_\lambda = \tilde{c}$. Now let:

$$\tilde{c}_{\alpha, \omega_1} : M_\omega \rightarrow \tilde{M}_{\omega}$$

finally,
Then \( \bar{M} = \bar{M} \) if \( \beta \) is regular.

But \( \sigma_{\beta, \omega_1} : \omega_2^{M \bar{M}} \to \omega_2^{\bar{M}} \) is cofinal, since
\( \bar{s}_{\alpha, \omega_1} : N_\alpha \to \bar{N} \) is cofinal. (To see this let \( x \in \bar{M} \). Then \( x \in \bar{s}_{\alpha, \omega_1}(a) \)
where \( \alpha \in N_\alpha \), \( \bar{\alpha} \in \omega_2 \) in \( N_\alpha \). Since
\( \sigma_{\beta, \omega_1} : \omega_2^{M \bar{M}} \to \omega_2^{\bar{M}} \) cofinally, there
\( \bar{\beta} \in \bar{M} \) s.t. \( x \in \bar{s}_{\beta, \omega_1}(b) \). But
then \( \alpha \bar{\beta} \in \bar{M} \) and \( x \in \bar{s}_{\alpha \bar{\beta}, \omega_1}(\alpha \bar{\beta}) \).

We shall make heavy use of the
following lemma:

**Lemma 2** Let \( M \) be a solid model of \( L \). Let \( \langle A_m \mid m < \omega \rangle \in \mathcal{M} \) s.t. \( A_m \subseteq M \) for
\( m < \omega \). Then there is \( u = \langle s, c \rangle \in \mathcal{M} \) s.t.
and \( \bar{u} \in \bar{M} \) s.t.
- \( \pi : \langle s, c \rangle \triangleleft \langle M^B, \bar{c}^M \rangle \)
- \( \bar{\pi} : \langle s, \bar{A}_m \rangle \triangleleft \langle M^B, \bar{A}_m \rangle \) for \( m < \omega \)

where \( \bar{A}_m = \rho \bar{\pi}^{-1} \bar{A}_m \). 


proof of Lemma 2
Set \( M^* = \langle MB, A_1, A_2, \ldots \rangle \). Then \( M^* \in \mathcal{G}_\mathcal{M} \).

Working in \( \mathcal{G}_\mathcal{M} \) we successively pick
\[ X_i < M^* , \; \pi_i \mid u_i \leq \langle MB, \mathcal{C}_\mathcal{M} \rangle \text{ s.t. } u_i \leq \mathcal{H}_\mathcal{M} \]
as follows: Let \( \mathcal{L} \) be well ordered \( \mathcal{H}_\mathcal{M} \):
\[ X_0 = \text{the smallest } X < M^* \]
Let \( \langle x^*_m \mid m < \omega \rangle \) enumerate \( X^*_1 \):
\[ u_i = \text{the } \langle - \text{ least } u \in \mathcal{H}_\mathcal{M} \text{ s.t. } \]
\[ u < \langle MB, \mathcal{C}_\mathcal{M} \rangle \text{ and } x^*_m \in \text{ ring } (\pi_i) \]
for \( i, m < i' \) where \( \pi_i = \langle \mathcal{M}, \mathcal{C}_\mathcal{M} \rangle \).
\[ X_{i+1} = \text{the smallest } X < M^* \text{ s.t. } \]
\[ X_i, U \text{ ring } (\pi_i) \subset X, \]
Let \( X = \bigcup X_i \). Then \( X = \bigcup \pi_i \cdot (\pi_i) \).

Let \( \pi^*: M^* \to X \) where \( M^* \) is transitive.
Hence \( M^* \in \mathcal{H}_\mathcal{M} \). Let \( \mathcal{M}^* = \langle \mathcal{M}, A_1, A_2, \ldots \rangle \).
At sufficient to show:

Claim \( \pi : \langle \mathcal{M}, \mathcal{C} \rangle \leq \langle MB, \mathcal{C}_\mathcal{M} \rangle \),
where \( \pi = \pi^* \cdot \mathcal{M} , \; \mathcal{C} = \mathcal{C}_\mathcal{M} \).

Proof:
Clearly \( \mathcal{C} \subset \text{ ring } (\pi_0) \subset X \). Hence
\[ \pi^* (\mathcal{M}) : \langle \mathcal{M}, \mathcal{C} \rangle \leq \langle MB, \mathcal{C}_\mathcal{M} \rangle \text{ where } \pi^* (\mathcal{M}) = \mathcal{M}, \]
Now let \( \pi : \mathcal{M} \to \mathcal{M}^* \) cofinally.
At sufficient to show:
Claim \( \langle \overline{M}, \overline{\alpha} \rangle \) is the lift up of \( \langle M, \pi, \overline{\alpha} \rangle \).

Proposition 1.6.

We must show that \( \pi: \overline{M} \rightarrow \overline{\alpha} \) in \( \omega_2 \) cofinal. Let \( x \in \overline{M} \). Then \( x \in \overline{\pi}(\alpha_i) \) for an \( i < \omega \), where \( \alpha_i < \omega \) in \( M_{\alpha_i} \), and \( \overline{\alpha} = \bigcup \overline{\pi}(\alpha_i) \). Hence \( \alpha = \overline{\pi}(\alpha_i) \in X \) and \( x \in \overline{\alpha} \) in \( \overline{M} \).

Hence \( x \in \overline{a} = \overline{\pi}(\overline{a_i}) \) for an \( \overline{a} \in \overline{\alpha} \) and \( \text{card}(\overline{a_i}) \leq \omega_1 \) in \( \overline{M} \). Q.E.D.

We are now ready to define the set of conditions \( \mathcal{P}_L = \mathcal{P}_L \).

We first set:

**Definition.** \( \overline{\mathcal{P}} = \) the set of \( \overline{p} = \langle p_0, p_1 \rangle \) s.t.

- \( p_0 = \langle M_p, c_p \rangle \in \Gamma \cap M_{\omega_1} \)
- \( p_1 = F_p \) in an at most countable set of pairs \( \langle a, \overline{a} \rangle \) s.t. \( a \in M_p \) and \( \overline{a} \in M_B \).

**Definition.** For \( p \in \overline{\mathcal{P}} \), let \( \varphi_p \) be the conjunction of:

- \( p_0 \leq \langle M_B, c \rangle \) (Let \( \overline{p}_p = \pi_{p} : \overline{p}_p = \overline{M}, c \) \)
- \( \overline{p}_p : \overline{p} \rightarrow \overline{M}, a \) for all \( \langle a, \overline{a} \rangle \in F_p \)
- \( \overline{p}_p : M_p \rightarrow M_B \) cofinally if \( \beta \) is regular

**Definition.** \( \mathcal{L}(p) = L + \varphi_p \).
\[ \text{Def: } R^p = \text{rng}(F^p), \quad D^p = \text{dom}(F^p) \]

\[ \text{Def: IP = IP}_B = IP_{B^p} = \rho + \sum_{p \in \overline{D}^p} \text{if } \rho \text{ is consistent} \]

For \( p, q \in IP \) let:
\[ p \preceq q \text{ iff the following hold:} \]
- \( R^q \subseteq R^p \)
- \( \pi_{B^q} \rho_{B^q} < \rho \)
- \( \pi_{B^q} p_{B^q} < \pi_{B^p} p_{B^p} \quad \text{whenever} \quad \langle a, \overline{m}_q \rangle \in \text{rng}(\pi_{B^q}) \text{ and } \langle a, \overline{m}_p \rangle \in \text{rng}(\pi_{B^p}). \)

**Lemma 3.1** Let \( p, q \in IP \). Then \( p \preceq q \) iff
- \( R^q \subseteq R^p \)
- \( F(p) \models (L(q) \land \text{rng}(\pi_{B^q}) \subseteq \text{rng}(\pi_{B^p})). \)

\[ \text{Proof.} \]

(\( \rightarrow \)) Let \( p \preceq q \). Let \( \mathcal{M} \) be a solid model of \( L(p) \). \( \mathcal{M} \) follows easily that \( \text{rng}(\pi_{B^q}) \subseteq \text{rng}(\pi_{B^p}) \) and \( \mathcal{M} \models L(q) \).

(\( \leftarrow \)) Let \( \mathcal{M} \) be a solid model of \( L(p) \). Then \( \text{rng}(\pi_{B^q}) \subseteq \text{rng}(\pi_{B^p}) \). Hence by Fact 9, \( \pi_{B^q} \rho_{B^q} < \rho \) and \( \pi_{B^q} = \pi_{B^q} \rho_{B^q} \).

Since \( \mathcal{M} \models L(q) \), we then have:
\( \pi_{B^q} \rho_{B^q} = (\pi_{B^q} \rho_{B^q})^{-1} \pi_{B^q} \rho_{B^q} = \langle \overline{m}_q, \overline{a} \rangle < \langle \overline{m}_p, \overline{a}' \rangle \)

for \( \langle a, \overline{a} \rangle \in \text{rng}(\pi_{B^q}), \langle a, \overline{a}' \rangle \in \text{rng}(\pi_{B^p}). \)

QED (3.1)
We set \( \pi_p^0 = \pi_p \) if \( p \leq q \).

Exactly as in [LiF] \( \epsilon 0.1 \) we prove:

**Lemma 3.2** Let \( p \in \mathcal{P} \), Then

- \( (F^p)^{-1} \) is a function
- \( \mathcal{P} \) is closed under set difference,
  - then \( F^p : \mathcal{D}^p \leftrightarrow \mathcal{P}^p \)
- \( \pi_p^0 = \pi_p^0 \mathcal{P} \) is injective into \( \mathcal{M}^p \).

The following lemma expresses a strong form of "reversibility" in the sense of [LiF],

**Lemma 3.3** Let \( p \in \mathcal{P} \), Let \( C \subseteq \mathcal{M}_p \)

cofinally. Then \( p \in \mathcal{P} \) where:

\[
p_0^p = \langle \mathcal{M}_p, C \rangle, \quad p_1^p = p_1^p.
\]

**Proof**

Let \( \mathcal{M} \) be a solid model of \( \mathcal{L}(p) \). Form \( \mathcal{M}' \) by replacing \( C \) with \( C' = \pi_p^0 \) "\( C \)."

**Claim** \( \mathcal{M}' \models \mathcal{L}(p') \)

We first show: \( \mathcal{M} \models \mathcal{L} \).

**Note** that if

\[
u = \langle \mathcal{O}_u, \mathcal{C}_u \rangle \preceq \langle \mathcal{O}_v, \mathcal{C}_v \rangle \text{ and } \mathcal{O}_u \geq \mathcal{O}_v,
\]

Then \( \langle \mathcal{O}_p, \mathcal{C}_p \rangle \preceq \langle \mathcal{O}_v, \mathcal{C}_v \rangle \)

and

\[
\sigma \in \theta \pi_u, \langle \mathcal{O}_v, \mathcal{C}_v \rangle \Rightarrow \sigma \in \theta \pi_p^0 \langle \mathcal{O}_p, \mathcal{C}_p \rangle
\]

(Where, of course, \( \mathcal{O}_p = H_{\omega_2}^p \)).
Set: \( u' = \langle Q_{u'}, C_{u'} \rangle \) where \( C_{u'} = \overline{u}, \langle Q^B, c_M \rangle \)

\[ \overline{u}, \langle Q^B, c'_M \rangle = \overline{u}, \langle Q^B, c''_M \rangle, \text{ as is easily seen. But this means that if} \]

\( u = \langle S_u, C_u \rangle \sqcup \langle M^B, c^M \rangle \) with \( d_u \geq d_p, \)

then \( u'' = \langle S_{u'}, C_{u'} \rangle \sqcup \langle M^B, c'_M \rangle \)

where \( C_{u'} = \overline{\langle Q_p, C_p \rangle, \langle Q_u, C_u \rangle, c', c''_M} \)

with \( \overline{u'}, \langle M^B, c'_M \rangle = \overline{u}, \langle M^B, c''_M \rangle \)

since \( \overline{u'}, \langle M^B, c'_M \rangle \) is uniquely determined by \( \overline{u}, \langle Q_u \rangle \). Thus (\( \ast \)) continues to hold in \( M' \). The other axioms are trivial.

Our argument shows, in particular, that \( \overline{\langle M^B, c'_M \rangle} = \overline{\langle M^B, c''_M \rangle} \).

Hence \( \overline{\langle M_p, c \rangle, \langle M^B, c'_M \rangle} \cdot \langle M_p, \bar{a} \rangle \langle M^B, a \rangle \)

whenever \( \langle a, \bar{a} \rangle \in \mathbb{E} = \mathbb{E}' \).

Thus \( M' \models \mathcal{L}(p') \). Q.E.D. (3.3)
We now prove the main lemma on extendability of conditions.

**Lemma 3.4** \( IP \neq \emptyset \). Moreover, if \( p, q \in IP \) and \( L(p) \cup L(q) \) is consistent, then there is an \( n \) s.t. \( n \leq \text{p} \text{.q} \). At \( RC \neq (\text{MB}) \) any countable set we may, in fact, choose \( n \) s.t. \( R = R^n \).

**Proof.** To see \( IP \neq \emptyset \), let \( \mathcal{U} \) be a solid model of \( L \). Let \( u \in <\text{MB}, \text{c}, \omega> \), \( u \in H_{\omega_1} \). Then \( p \in IP \) where \( p_0 = u \), \( p_\omega = \emptyset \).

Now let \( \mathcal{U} = L(p) \cup L(q) \). Set:

\[
X = \text{ring}(\pi_{p}^{\mathcal{U}}) \cup \text{ring}(\pi_{q}^{\mathcal{U}}) \cup F \cup F^c \cup R.
\]

Then \( X \in \mathcal{U} \) is countable in \( \mathcal{U} \) with \( X \subseteq \mathcal{U}(M) \). By Lemma 2 there is \( <\text{m}, \text{c}> \subseteq <\text{MB}, \text{c}, \omega> \) s.t. \( <\text{m}, \text{c}> \in H_{\omega_1} \) and \( \pi_r : <\text{m}, \text{A}> \subseteq <\text{M}, \text{A}> \) for all \( A \in X \), where \( \pi_r = \pi_{<\text{m}, \text{c}>}^{<\text{M}, \text{A}>} \) and \( \text{A} = \pi_r^{-1} \text{"A} \).

Define \( \mathcal{U} \) by:

\[
\mathcal{U}_0 = <\text{m}, \text{c}>, \quad \mathcal{U}_1 = \text{the set of } <A, \text{A}> \text{ s.t. } A \in R^n \cup R^c \cup R
\]

Then \( \mathcal{U} = L(\mathcal{U}) \). Hence \( \mathcal{U} \in IP \). But

\[
\pi_{p}^{\mathcal{U}}, p_0 \neq <\text{MB}, \text{c}, \omega>, \pi_{r}^{\mathcal{U}}, r_0 \neq <\text{MB}, \text{c}, \omega>, \text{ and } \text{ring}(\pi_{p}^{\mathcal{U}}) \subseteq \text{ring}(\pi_{q}^{\mathcal{U}}).
\]
Hence \( x \in M_0 \) where \( x = x^M(\chi_p, \omega) \).

By Fact 9. But then, if \( \langle a, \bar{a} \rangle \in E^p \), \( \langle a, a' \rangle \in E^{\omega} \), we have:

\[ \pi: \langle M_p, \bar{a} \rangle = \langle M_\omega, a' \rangle. \]

Hence \( r \leq p \) with \( \pi = \pi_p, \).

Similarly \( r \leq q \), QED (3.4)

**Cor 3.5** \( p, q \) are compatible in \( \mathcal{M} \).

\( L(p) \cup L(q) \) is consistent.

**Proof**

\( (\leftarrow) \) by Lemma 3.4

\( (\rightarrow) \) if \( r \leq p, q \), then \( L(r) \models L(p) \cup L(q) \), QED (3.5)

**Cor 3.6** Let \( p \in IP, R \subseteq \mathcal{P}(M^R) \) where \( R \) is countable. There is \( q \leq p \) s.t. \( R \subseteq R^q \).

**Cor 3.7** Let \( p \in IP, u \subseteq M^R \), where \( u \) is countable. There is \( q \leq p \) s.t. \( u \subseteq \text{rng}(\pi^q) \).

**Lemma 3.8** Let \( p \in IP, u \subseteq M^R, u \) finite.

There is \( q \leq p \) s.t. \( q_0 = p_0 \) and \( u \subseteq \text{dom}(\pi^q) \).

**Let \( \mathcal{M} \) be a solid model of \( L(p) \).

Set: \( q_0 = p_0, F^\mathcal{M} = F^p \cup \{ \omega \}^M \cup \{ M \} \).

QED (3.8)
Lemma 3.9 Let $G$ be $IP$-generic. Then
(a) $\langle p_0 \mid p \in G \rangle \downarrow \langle \bar{p}_p \mid q \leq p \text{ in } G \rangle$ is a directed system with the limit $\langle M^b, c_G \rangle$, $\langle \bar{p}_p \mid p \in G \rangle$
(More over $\bar{p}_p = \bigcup \{ p_0 \mid q_0 = p_0, q \leq p, q \in G \}$)
(b) $\bar{p}_p : p_0 \downarrow \langle M^b, c_G \rangle$ for $p \in G$
(c) $\bar{p}_p : \langle M_{p_0}, \bar{a} \rangle \triangleleft \langle M, \bar{a} \rangle$ whenever $\langle a, \bar{a} \rangle \in EP$.
The proof is left to the reader.

Using the "reversibility" lemma 3.3 we get:

Lemma 3.9 $IP$ adds no reals.

Proof.
Let $\Pi : f : \omega \to 2$. It is enough to show:
Claim The set $\Delta$ of $p$ s.t. $Vf p \uparrow f = f$
is dense in $IP$.
Let $s \in IP$. We construct $q \leq s$ s.t. $q \in \Delta$. Let:
$N^* = \langle H_{G}, N^b, 1, f, IP, r_1, \ldots \rangle$ where $\theta \geq 2^\omega$
and $\triangleleft$ well orders $N^*$.
Let \( p \in \mathcal{P} \) conform to \( N^* \) (as defined in [17] §3.1). Set:
\[
\overline{N}^* = \overline{N}^* (p, N^*) = \langle H^*, N^*_B, \langle f, \zeta \rangle, IP, x, \ldots \rangle
\]
Pick \( G' \exists x \) which \( u (IP') \) generic over \( \overline{N}^* \).
Define \( \varphi \) by \( \varphi = \langle \mu_B, G^* \rangle \), \( \varphi_A = \mu_A \).

Then \( \varphi \in \mathcal{P} \) by the reducibility lemma.

Let \( \bar{f} = f' \otimes G' \). It suffices to show:

Claim \( \varphi \models \bar{f} = f \) and \( \varphi \) is compatible with \( \overline{N}^* \).

We first show \( \varphi \models \bar{f} = f \). Suppose not.

Then there is \( \varphi' \leq \varphi \) s.t. \( \varphi' \models f(m) \neq \bar{f}(\overline{m}) \)
for some \( m \). Let \( \mathfrak{M} \) be a solid model of \( L(\varphi') \). Let \( \pi^* \models \overline{N}^* \). Let \( \pi \models L(\varphi) \).

\( \overline{N}^* \models \overline{N}^* < N^* \). Let \( \lambda' \models G \) s.t.
\( \lambda' \models f(\overline{m}) \neq \bar{f}(\overline{m}) \). Set \( \lambda = \pi^*(\lambda') \).

Then \( \varphi \models f(\overline{m}) = \bar{f}(\overline{m}) \). Hence \( \varphi \) is incompatible. We obtain a contradiction by proving:

Claim \( \mathfrak{M} \models L(\varphi') \cup L(\lambda) \).

\( \mathfrak{M} \models L(\varphi') \cup L(\lambda) \) is trivial. We prove \( \mathfrak{M} \models L(\lambda) \).

Note that \( \lambda_0 = \lambda_0' \) and \( F_{\lambda_0} = \) the set
set of \( \langle a, \overline{a} \rangle \) s.t. \( \overline{a} = \pi^*(a') \) and
\( \langle a', \overline{a} \rangle \in F_{\lambda'} \) for some \( a' \). Clearly
\( \overline{a}' : \lambda_0 \models \langle M_B', G^* \rangle \), since \( \lambda' \models G' \).
But \( \tilde{q}_0 = \langle M, B', C^G \rangle \) and \( \mathcal{M} \models \tilde{q}_0 \models \langle M^G, \tilde{c}^{\mathcal{G}} \rangle \).

Set \( \mathcal{M} = \tilde{q}_0, \mathcal{M}^G = \tilde{q}_0 \models \tilde{c}^{\mathcal{G}} \). Then \( \mathcal{M} \in \mathcal{M} \) and \( \mathcal{M} \models \langle M^G, \tilde{c}^{\mathcal{G}} \rangle \). It remains only to show:

Claim \( \mathcal{M}, a \rangle < \langle M^G, a \rangle \) for \( \langle a, \bar{a} \rangle \in F^a \).

Since then \( \mathcal{M} \models \mathcal{M}^G \models \tilde{a}^{\mathcal{G}} \).

Let \( a = \tilde{a}^{\mathcal{G}} \). Then \( \langle a, \bar{a} \rangle \in F^a \).

Then \( \mathcal{M}^G: \langle M, a \rangle < \langle M^G, a \rangle \) and

\( \mathcal{M}^G: \langle M, a \rangle < \langle M^G, a \rangle \) since \( \tilde{a}^{\mathcal{G}} = \langle M^G, a \rangle \).

This proves \( \mathcal{M}, f = \tilde{f} \). But the last part of the proof shows that for every \( L' \in \mathcal{L} \), \( \tau = \tilde{\tau} < (2) \) is compatible with \( \tilde{q} \) for any \( \tilde{q} \leq q \), i.e., \( \mathcal{M} = L(\bar{q}) \).

But \( \tilde{\tau} \in \mathcal{L} \) and \( \tau = \tilde{\tau} < (2) \) since \( \tilde{\tau}^{\mathcal{G}} < \tilde{\tau}^{\mathcal{G}} \). Hence \( \tau \) is compatible with \( \tilde{q} \). \( \mathcal{QED} (3.91) \)

An immediate corollary is:

Cor 3.10 Let \( \mathcal{G} \models 2^\mathcal{G} \) be regular. If \( \mathcal{G} \models \mathcal{P} \)

is \( \mathcal{P} \)-generic and \( C = C^G \), then

\( \langle H[V[B][G], C] \rangle = \mathcal{L}(\mathcal{P}) \).

Proof:
The only problematical axiom was

\( \mathcal{H}_\theta = \tilde{\mathcal{H}_\theta} \), which is now seen to hold. \( \mathcal{QED} (3.10) \)
Defn. Let $C \prec MB$ be countable and cofinal. $G^C = \{ p \in \beta \mid \exists p \in \beta \text{ s.t. } p \in <MB, C> \text{ and } \pi(p) = \pi(p), <MB, C> \}$, we have:

$\pi : <MP, \bar{a}> \prec <MB, a>$ whenever $a, \bar{a} \in F$.

and $\pi : MP \rightarrow MB$ is cofinal if $B$ is regular.

**Lemma 3.11** Let $G$ be IP- generic. Then $G = G^C$ where $C = C^G$.

**Proof**

$G \subseteq G^C$ is trivial. We prove $(\supset)$

Let $p \in G^C$, $\forall p \notin G$ there is $q \in G$ which is incompatible with $p$. But then $<H^q[G], C> = L(p) \cup L(q)$. for regular $G \geq 2^\beta$. QED (3.11)

**Lemma 3.12** Let $G$ be IP- generic. Then $\beta \leq \omega_1$ in $V[G][\bar{G}]$.

**Proof**

For each $\bar{z} < \beta$ there is $<\bar{m}, \bar{c}, \bar{z}> \in H_{\alpha}[\bar{G}]$ s.t. $<\bar{m}, \bar{c}> \prec <MB, G>$ and $\pi(\bar{z}) = \bar{z}$ whenever $\pi = \pi(<\bar{m}, \bar{c}>, <MB, G>)$. This maps a subset of $H_{\alpha}[\bar{G}]$ onto $\beta$. QED (3.12)
Lemma 3.13 Let $G$ be $\mathcal{P}$-generic. If $\omega_1 < \tau \leq \beta$ and $\tau$ is regular in $V[B][G]$, then $\text{cf}(\tau) = \omega$ in $V[B][G]$.

Proof.
If $\tau = \beta$, then for any $p \in G$ we have
$$\sup\pi_p^G(\beta) = \beta$$
where $\beta_p$ is countable.

Now let $\tau < \beta$. Let $p \in G$ s.t. $\pi_p^G(\tau) = \tau$.
Then each $\zeta < \tau$ lies in $\pi_p^G(u)$ for a $u \in M_p$ s.t. $\bar{u} \leq \omega_1$ in $M_p$. But the set $U$ of such $u$ is countable. Set
$$M_u = \sup\pi_p^G(u)$$
and $E_p^G(M_u) | u \in U$ is cofinal in $\tau$.

QED (3.13)

Cor. 3.14 $\vdash \omega_1 < \delta \leq \beta$ and cf$(\delta) = \omega_1$,
then $\text{cf}(\delta) = \omega$ in $V[B][G]$. We now recall [LF] 5.4 Lemma 4.1 which says:

Fact 11 Let $\beta$ be a cardinal in an inner model $W$ s.t. $2^\beta = \beta$ in $W$. Let $\delta = 2^\beta$ in $W$. Assume that in $V$ we have:
$2^\omega = \omega_1$, $\beta = \omega_1$, $\text{cf}(\beta) = \omega$. Then $\delta \leq \omega_1$ in $V$.

* We are working over $V[B]$, so statements like $\text{cf}(\delta) = \omega_1$ are understood to be in the sense of $V[B]$. 
Hence:

Cor 3.14.1 If \( \text{cf}(\beta) \neq \omega_1 \), then \( \text{cf}(\beta) = \omega_1 \) in \( V[\beta][\mathcal{G}] \), where \( \mu = 2^{\beta} \).

\( \mu^+ \) remains a cardinal; however, \( \mu^+ \leq \mu \). Hence if \( 2^{\omega_1} = \mu \), we can conclude \( \text{cf}(\mu) = \omega_1 \), since otherwise \( \mu^+ \) would be collapsed by Fact 11.

An particular, \( \mu = \beta^+ + \text{cf}(\mu) = \omega_1 \) in \( V[\mathcal{G}] \) if \( \text{GCH} \) holds in \( V \).

The case \( \text{cf}(\beta) = \omega_1 \) is quite different as shown by:
Lemma 3.15 Let \( \beta = \omega_1 \) in \( V[b] \). Then \( \beta^+ \) remains a cardinal in \( V[B][G] \). (Hence \( \beta^+ = \omega_2 \) in \( V[B][G] \).)

Proof: We imitate the proof of [LJ] §4 Lemma 3.1 to show:

Sublemma 3.15.1 \( BA(1P) \) has a dense subset of \( \beta \) in \( V[B] \).

Set \( H = H^{(B)} \). Then \( \langle H[G], c \rangle \rangle \)
models \( L \) whenever \( G \) is \( 1P \)-generic (interpreting \( \mathcal{V} \) by \( \mathcal{X} \)). Let \( H \models c = c^G \)
where \( c^G \) is the canonical generic name.

We can give every \( L \) sentence \( \varphi \) an interpretation \( [[\varphi]] \in \mathcal{B} = BA(1P) \) in \( H^{1P} \),
interpreting \( c \) by \( c^G \) and \( \mathcal{X} \) by \( \mathcal{X} \).

We then have:

\[ \langle H[G], c \rangle \models \varphi(\bar{x}_1, \ldots, \bar{x}_n) \iff \]

\[ \iff [[\varphi(\bar{x}_1, \ldots, \bar{x}_n)]] \cap G \neq \emptyset \]

for \( \bar{x}_1, \ldots, \bar{x}_n \in \mathcal{N} \) and \( G \) a \( 1P \)-generic set.

Thus it suffices to prove:

Claim For each \( p \in 1P \) there is an \( L \)-sentence \( \varphi \in M^{B} \) s.t. \( [[\varphi]] \neq 0 \) and

\[ [[\varphi]] \subset [p] \cup \{ [p] \} \text{ (}[p] \text{ being the smallest}
\]

\( b \in B \text{ s.t. } p \leq b \).
At $G \in \text{IP}-\text{generic}$, we have:

$[p] \cap G \neq \emptyset \quad \text{if} \quad p \in G \quad \iff \quad \langle H, [G], c^g \rangle \in \Phi_p$

$\iff [\Phi_p] \cap G \neq \emptyset$. Hence

$[p] = [\Phi_p]$ and it suffices to show:

Claim $1\text{IP}\psi \rightarrow \Phi_p$ for $\psi \in M^B \cap [\psi] \neq 0$

Set $N^x = \langle H, N^B, < \rangle$ where $< \text{ well orders } H$. We may assume without loss of generality that $p$ contains $N^x$, since the set of such $p$ is dense in IP. Let $G$ be $\text{IP}-\text{generic}$ with $p \in G$. Let $\beta = \bar{\beta}_p \bar{\beta}_p$.

Then $\beta < \bar{\beta}$ since $\bar{\beta}$ is $\omega$-cofinal.

Set $\tilde{M} = \bar{\beta} \setminus M$. For $\alpha \in \text{IP}^p$, let $\tilde{\alpha} = \alpha \cap \tilde{M}$.

Then $\pi^G_{\tilde{\alpha}}: \langle \tilde{M}, \tilde{\alpha} \rangle \rightarrow \langle \tilde{M}, \tilde{\alpha} \rangle$ is $\omega$-cofinal and $\Sigma_0$-preserving whenever $\langle \alpha, \tilde{\alpha} \rangle \in \text{IP}^p$.

But then

$\forall \tilde{\alpha} \in \pi^G_{\tilde{\alpha}}(\exists \pi_{\tilde{\alpha}})$.

Let $\langle \alpha_i | i < \omega \rangle$ enumerate $\text{IP}^p$ in $V$.

Then $\langle \alpha_i | i < \omega \rangle \in H_{\omega_1}$, where $\langle \alpha_i, \alpha_i \rangle \in \text{IP}^p$.

Moreover $\langle \tilde{\alpha}_i | i < \omega \rangle \in M$, since $\tilde{\alpha}_i \in M$.

Let $\forall \theta$ be the sentence: $\langle \tilde{\alpha}_i | i < \omega \rangle$: $\tilde{\alpha}_i \in M$. Let $\psi$ be the sentence.
There are \( \pi, \sigma \) s.t. \( \pi: \langle q_p, c^p \rangle \subseteq_x \langle q, c^g \rangle \wedge \langle \tilde{\pi}, \pi \rangle \) \( \hat{\pi} \) the lift up of \( \langle M_p, \sigma \rangle \wedge \wedge \bigoplus \widehat{a_i} = \bigcup \pi \left( \zeta \cap \hat{a}_i \right) \).

Clearly \( \forall \in M \). Moreover,

(2) \([\forall] \neq 0\), since \( \langle H[G], c^G \rangle \models \forall \).

Since then \( \forall \) holds with \( \pi = \pi_p \cap \pi_q \), \( \bar{\pi} = \pi_p \).

We show:

(3) \( \langle H[G], c^G \rangle \models \forall \rightarrow \forall \).

Whenever \( G \) is \( \Pi^1_1 \) generic.

Let \( \langle H[G], c^G \rangle \models \forall \). Let \( \pi: \langle q_p, c^p \rangle \subseteq \langle q, c^g \rangle \)
and \( \langle \tilde{\pi}, \pi \rangle = \) the lift up of \( \langle M_p, \sigma \rangle \).

It remains only to show:

\( \pi: \langle M_p, \tilde{\pi} \rangle \models \langle M^3, \pi \rangle \) whenever \( \langle a, \tilde{a} \rangle \in F^p \), since then we have

\( \pi: \langle M_p, c^p \rangle \subseteq \langle M^3, c^G \rangle \) and

hence:

\( P \in G^c \subset G \) with \( \tilde{\pi} = \pi_p \).

Let \( b = \tilde{\pi} \in M \mid \langle M_p, a \rangle \models \chi (\tilde{\pi}) \).

Then \( b \in F^p \)

by the \( N^* \)-conformity of \( \pi \).

Let \( \langle b, \tilde{b} \rangle \in F^p \). Then by \( N^* \)-conformity:

\( \tilde{b} = \tilde{\pi} \in M_p \mid \langle M_p, \tilde{a} \rangle \models \chi (\tilde{\pi}) \).

Hence \( \langle M_p, \tilde{a} \rangle \models \chi (\tilde{\pi}) \) \( \implies \tilde{\pi} \in b \) \( \implies \)

\( \implies \pi (\tilde{\pi}) \subseteq \tilde{b} = b \cap \tilde{\pi} \implies \langle M, a \rangle \models \chi (\pi (\tilde{b})) \).

Since \( \tilde{b} = \bigcup_{u \in M_p} \pi (u \cap \tilde{b}) \).

Q.E.D. (3.15)
Note \( cf(\beta) = \omega_1 \) if the only case to consider is \( A_0 = 0 \).

We also note that we could have defined \( L \) (and hence \( IP = IP_L \)) somewhat differently: Let \( L' \) be like \( L \) except that

in \((\ast)\) we omit \( i \) \( \sum \langle \alpha i,1 \rangle = M \) if \( \beta \) is regular, and instead add the axiom:

\((\ast)\) \( \beta \) is regular. Then whenever \( u \in H_{\omega_1} \) and \( \pi ; u \in \langle \mathbb{M}, \mathbb{C} \rangle \), we have:

\( \sup \text{on} \{ n \mid n < \beta \} \text{ is also consistent}. \)

At \( IP = IP_L \) and \( \beta \) is regular, we can modify the proof of Lemma 3.1. To get:

\( IB' = BA (IP') \) contains a dense subset of \( \eta \), \( \beta \) is regular. Hence \( \beta^+ = \omega_2 \) and \( cf(\beta) = \omega_1 \)

in \( V[G'] \), where \( G' \in IP' \) generic.

We omit the proofs, since this is not relevant to the present paper.

We are now ready to prove that \( IP \) is subcomplete. Since we are working in \( V[G] \) we shall again write \( V \) for \( V[G] \) and - for the sake of simplicity - we also write \( A, M, N \) for \( A^G, M^G, N^G \).
Lemma 4. IP is sub-complete.

Proof. (We work in $\mathcal{V}[\mathcal{B}]$

Let $\mathcal{W} = \mathcal{L}_{\mathcal{B}}$ where $2^{\beta} < \theta < \omega_1$, $\mathcal{B}$ is regular, and $\mathcal{H}_0 \subseteq \mathcal{W}$. Let $\sigma : \mathcal{W} \rightarrow \mathcal{W}$ be a countable and full with:

$$\sigma(\overline{\theta}, \overline{P}, \overline{\mathcal{B}}, \overline{\lambda}, \overline{\lambda}_i) = \theta, \overline{P}, \mathcal{B}, \overline{\lambda}, \overline{\lambda}_i \quad (i = 1, \ldots, m)$$

where $\overline{P} \in \mathcal{H}_{\overline{\lambda}_i}$ (hence $\mathcal{N} \in \mathcal{H}_{\overline{\lambda}_i}$), $\lambda_i < \theta$, and $\lambda_i$ is regular for $i = 1, \ldots, m$. Let $\overline{G}$ be $\overline{P}$-generic over $\mathcal{W}$.

Claim. There is $\varphi \in \mathcal{P}$ s.t. whenever $G \supseteq \varphi$ is $\mathcal{P}$-generic, then there is $\sigma_0 \in \mathcal{V}[\mathcal{G}]$ with:

(a) $\sigma_0 : \overline{\mathcal{W}} \subseteq \mathcal{W}$

(b) $\sigma_0(\overline{\theta}, \overline{P}, \overline{\mathcal{B}}, \overline{\lambda}_i) = \varphi, \overline{P}, \mathcal{B}, \overline{\lambda}, \overline{\lambda}_i \quad (i = 1, \ldots, m)$

(c) $\lim_{\sigma_0} \overline{\overline{\lambda}_i} = \lim_{\sigma_0} \overline{\lambda}_i \quad (i = 0, \ldots, m)$

where $\overline{\lambda}_0 = \text{Onn} \overline{\mathcal{W}}$

(d) $\sigma_0 \overline{G} \supseteq G$.

We first show by standard methods:

Sublemma 4.1. Let $\varphi$ be least s.t. $\mathcal{L}_\varphi(\mathcal{W})$ is admissible. The following language $\mathcal{L}_{\mathcal{L}_\varphi(\mathcal{W})}$ is consistent:

Predicate $\in$, Constant $\# (x \in \mathcal{L}_\varphi(\mathcal{W}))$, $\varphi$

Axiom 1: ZFC$^-$, $\forall x (\in \rightarrow F \in \cup \overline{\mathcal{W}} \overline{\mathcal{W}})$ for $x \in \mathcal{L}_\varphi(\mathcal{W})$,

$\sigma_\varphi : \overline{\mathcal{W}} \subseteq \mathcal{W}$, $\sigma_\varphi(\overline{\theta}, \overline{P}, \overline{\mathcal{B}}, \overline{\lambda}_i) = \varphi, \overline{P}, \mathcal{B}, \overline{\lambda}, \overline{\lambda}_i \quad (i = 1, \ldots, m)$

$\lim_{\sigma_\varphi} \overline{\overline{\lambda}_i} = \lim_{\sigma_\varphi} \overline{\lambda}_i \quad (i = 0, \ldots, m)$, and

$\sigma_\varphi(\overline{\mathcal{B}}) = \overline{\mathcal{B}} < \mathcal{B}$ cofinally (where $\sigma_\varphi(\mathcal{B}) = \mathcal{B}$),

$\langle \mathcal{N}, \sigma_\varphi \overline{\mathcal{B}} \rangle$ is the lifting of $\langle \mathcal{N}, \sigma \overline{\mathcal{B}} \rangle$. 


Note $\mathcal{L}^*$ does not point that $H_{\lambda_0} = H_{\lambda_1}$. 

**Proof (sketch) of 4.1**

Let $L_0$ be like $L^*$ except that the axiom

$$\sup_{\omega} \sigma^* \bar{\lambda}_i = \sup_{\omega} \sigma^* \bar{\lambda}_i \quad (i = 0, \ldots, m)$$

is replaced by:

$$\sup_{\omega} \sigma^* \bar{\lambda}_i = \lambda_i \quad (i = 0, \ldots, m) \quad (\text{where } \lambda_0 = \sigma \bar{\lambda})$$

Let $\sigma \bar{\lambda} : \bar{\lambda} \leq \tilde{\lambda}$ cofinally ($\bar{\lambda} = H_{\bar{\lambda}}$) and let $\tilde{\sigma} : \tilde{\lambda} \leq \lambda$ be the lift up of $\tilde{\lambda}$ by $\sigma \bar{\lambda}$.

Let $k : \tilde{\lambda} \leq \lambda$ s.t. $k \bar{\lambda} = \text{id}$ and $k \tilde{\sigma} = \sigma$.

Let $\tilde{L}_0$ be defined on $L_0(\tilde{\lambda})$ like $L_0$ on $L_0(\lambda)$ in the obvious sense, where $\tilde{\sigma}$ is least s.t. $L_0(\tilde{\lambda})$ is admissible. (More precisely, $\tilde{L}_0$ is defined in the parameter $\tilde{\lambda}$ and parameters $\bar{\lambda}, M, N, \theta, x, \lambda, \ldots \leq \tilde{\lambda}$.)

$\tilde{L}_0$ has the same definition over $L_0(\tilde{\lambda})$ in the parameter $\tilde{\lambda}$ and the parameters $k^{-1}(\bar{\lambda})$, $k^{-1}(M)$, $k^{-1}(N)$, $k^{-1}(\theta)$, $k^{-1}(\lambda)$ ($i = 1, \ldots, m$).

Then $<H_{\lambda_2}, \tilde{\sigma}>_0$ models $\tilde{L}_0$. Assume $\lambda_0, \ldots, \lambda_m$ and let

$$\sigma \bar{\lambda} \tilde{\lambda}_m : H_{\tilde{\lambda}_m} \leq H, \text{ cofinally, }$$

(Here $H_{\tilde{\lambda}_m} = \tilde{\lambda}_m$ if $m = 0$.) Let $\sigma' : \tilde{\lambda} \leq \lambda'$ be the lift up of $\tilde{\lambda}$ by $\sigma \bar{\lambda} \tilde{\lambda}_m$. Let $k' : \lambda' \leq \lambda$ s.t. $k' \bar{\lambda} = \text{id}$ and $k' \tilde{\sigma} = \sigma$.

Then $\tilde{\sigma} : \tilde{\lambda} \leq \lambda'$ s.t.
\[ \kappa \ \tilde{\alpha} = \text{id} \text{ and } \kappa \tilde{\alpha} = \sigma' \] We then have 
\[ k' \kappa = k. \] Let \( \delta' \) be least s.t. \( L_{\delta'}(W') \) is admissible and let \( L' \) be defined over \( L_{\delta'}(W') \) or \( L' \) was defined over \( L_{\delta'}(W') \) (in the obvious sense). The statement that \( L' \) is consistent in \( \text{Th}_n(L_{\delta'}(W')) \) in the parameter \( W' \) and parameters \( \delta' \in \tilde{W} \). The statement that \( L' \) is consistent is then \( \text{Th}_n(L_{\delta'}(W')) \) in \( W' \) and \( k'(\delta') \). Hence \( L' \) is consistent. Note that \( k'(\mathbb{N}) = \mathbb{N} \). Let \( M \) be a solid model of \( L' \) which lies in some generic extension \( V[G] \) of \( V \). Let \( \mu > \omega \) be regular in \( V[G] \).

Then \( \langle H, V[G], k', \delta' \rangle \) models \( L^* \), where \( \sigma = \sigma' \mu \). \( \text{GED (4.1)} \)

Now let \( N^* = \langle H', W', N, \sigma, \lambda_1, \ldots, \lambda_m, \lambda, IP, \ldots \rangle \) where \( \delta > 0\) in \( W'. \) Let \( p \) permute to \( N^* \). Set:

\[ \bar{\mathcal{N}}^* = \mathcal{N}^*(p^*N^*) = \langle H', W', N, \sigma', \lambda_1, \ldots, \lambda_m, \lambda', IP, \ldots \rangle \]

Let \( \bar{E}^* \) be defined in \( \bar{\mathcal{N}}^* \) like \( E^* \) in \( N^* \). Set 
\( \delta^* = \delta' \mu \). Set \( \bar{C} = c \bar{E}, c' = \sigma^* \delta^* c'. \) Since 
\( \sigma^* \tilde{\alpha} : \tilde{\alpha} < Q' \) cofinally (where \( Q' = Q_p \) is defined in \( \bar{\mathcal{N}}^* \) like \( Q \) in \( N^* \)), we have:

\( q' \in IP \) where \( q' \) is defined by:\
\( \varphi_0 = \langle M, c' \rangle, \quad \varphi_1 = \rho \).

We show that this \( \varphi \) satisfies the claim. Let \( G \geq \varphi \) be IP-generic. Note that, since

\[ \langle N', \sigma^* \rangle \in \text{the lift} \text{ of } \langle N, \sigma^* \mid N \rangle \]

and \( \sigma^* \upharpoonright \bar{Q} \subseteq \bar{Q}' \) with \( \sigma^* \upharpoonright \bar{C} = c' = \varphi_1 \), we have:

\[ \langle \bar{M}, \bar{C} \rangle \leq \varphi_0 = \langle \rho, c' \rangle \]

with

\[ \langle \bar{M}, \bar{C} \rangle \leq \varphi_0 \leq \langle M, c' \rangle \]

and \( \bar{M} \subseteq M \). But \( \varphi_0 \leq \langle M, c \rangle \) with

\[ \langle \bar{M}, \bar{C} \rangle \leq \langle M, c \rangle \]

Now let \( \pi^* \subseteq \varphi \) be \( \pi^* \) \text{ s.t. } \bar{N} \subseteq N^* \).

Set \( \sigma_0 = \pi^* \sigma^* \).

Then (a) - (c) are readily established. We show:

(d) \( \sigma_0 \upharpoonright \bar{G} \subseteq \pi^* \).

Let \( \pi \in \bar{G}, \pi = \sigma_0 \tau \).

Then \( \pi_0 = \sigma_0 \).

But then \( \pi_0 \leq \langle \bar{M}, \bar{C} \rangle \) and \( \pi_0 \upharpoonright \langle \bar{M}, \bar{C} \rangle = \pi_0 \).

Hence \( \pi_0 \leq \langle M, c \rangle \) and \( \pi_0 \upharpoonright \langle M, c \rangle = \sigma_0 \upharpoonright \pi_0 \).

And since \( \langle \bar{M}, \bar{C} \rangle, \langle M, c \rangle = \pi \upharpoonright \bar{G} \).

At remains only to show:

Claim: \( \sigma_0 \upharpoonright \pi_0 \upharpoonright \langle M, \bar{a} \rangle \leq \langle M, a \rangle \)

When ever \( \langle a, \bar{a} \rangle \in F^\pi \).

Let \( \langle a, \bar{a} \rangle = \sigma_0 \langle a', \bar{a}' \rangle \) where \( \langle a', \bar{a}' \rangle \in F^{\pi_0} \).

Then \( \pi \upharpoonright \pi_0 \upharpoonright \langle M, \bar{a} \rangle \leq \langle M, a' \rangle \). But

\[ \sigma_0 \upharpoonright \bar{M}; \langle \bar{M}, \bar{a}' \rangle \leq \langle M, a \rangle \]

And since \( \sigma_0 \langle a', \bar{a}' \rangle = \langle M, a \rangle \).

\[ QED \ (\text{Lemma 4}) \]
Note that $\bar{\mathcal{P}}(\mathcal{P}_B) \leq 2^{\beta \mu+1}$, since either
\[\beta = \beta \mu+1 \text{ if } (\beta) = \omega_1 \] and $\mathcal{P}(\mathcal{P}_B)$ has a
dense subset $\mathcal{P}^*$ of $\beta$, by Sublemma 3.15.1, o r we have $\beta = \beta \mu+1$, $\mathcal{P}_B = 2^\beta = 2^{\beta \mu+1}$ since then
GCH holds below $\kappa$, Now let $\mathcal{P}_B = \mathcal{P}_B$, $\mathcal{B}$ being the canonical generic name.
We then form $\mathcal{P}_B \ast \mathcal{B}$, which also has
Cardinality $\leq 2^{\beta \mu+1}$, since $\mathcal{P}_B$ has
Cardinality $\leq 2^{\beta \mu} \leq \beta \mu+1$, and since $2^{\beta \mu+1} = \beta \mu+1$.
Let $k : \mathcal{P}_B \rightarrow \mathcal{P}_B \ast \mathcal{B}$ be the natural
injection. Choose $\mathcal{P}_B \ast \mathcal{B} \sim \mathcal{B}$ s. t. there
is a isomorphism $\mathcal{B} \cong \mathcal{B} \ast \mathcal{B}$ with:
\[
\begin{array}{ccc}
\mathcal{P}_B & \xleftarrow{k} & \mathcal{P}_B \ast \mathcal{B} \\
\mathcal{P}_B & \cong & \mathcal{P}_B \ast \mathcal{B}
\end{array}
\]
We ensure that $\mathcal{P}_B \subset \mathcal{H}_+$. By Lemma 4,
we know that $\mathcal{B} \ast \mu+1$ is subcomplete.
However, we have found it necessary to
dovice another representation of $\mathcal{B} \ast \mu+1$ in
order to elicit its closer properties. Working
in $\mathcal{T}$ define as before:
\[\mathcal{Q} = \mathcal{H}_\omega, \mathcal{M} = \mathcal{L}(\mathcal{B} \ast \mu), \mathcal{N} = \langle \mathcal{H}_\beta, \mathcal{M}, <, \cdots \rangle.
\]
We then define a class $\Pi'$ of triples
as follows:
Def. \( \Gamma' \) is the set of \( u = \langle R_b, B, c \rangle \) s.t.

- \( R = L_{\beta}^{A, B} \) model ZFC - or Zermelo
- \( B \in R \) a complete BA in \( R \) and \( B \upharpoonright B - \) generic over \( R \).
- \( \omega = x^u = \omega_b \in \omega \subseteq R_b, Q = Q_b = H_y, \)
  \( Q^B = Q_u = H_{R_b}, u^* = \langle R_b, c \rangle \)
- \( \langle R_b, c \rangle \in \Gamma \) with \( \langle Q^B, c \rangle \in \Gamma^* \)
- \( B \subseteq Q \) satisfies \( \forall - cc \) in \( R \).

Def. Let \( u = \langle R_u, B_u, c_u \rangle \), \( v = \langle R_v, B_v, c_v \rangle \in \Gamma' \)

- \( \pi : R_u < R_v \) s.t. \( \pi^* B_u < B_v \)
- \( \pi \upharpoonright Q_u : Q_u < Q_v \) cofinally
- Let \( \pi : R_u \rightarrow R_{u^*} \) cofinally. Then
  \( \langle R_u, v, \pi \rangle \) is the lift up of \( \langle M_u, \pi \upharpoonright Q_u \rangle \)
- Let \( \pi^* \upharpoonright \pi \) be the unique extension of
  \( \pi \) s.t. \( \pi^* : R_u^{B_u} < R_v^{B_v} \). Then \( \pi^* c_u = c_v \).

We then get:

Lemma 5.1 Let \( \pi : u \triangleleft v^* \), then \( \pi^* : u^* \triangleleft v^* \).

The proof is straightforward.

Lemma 5.2 Let \( \pi : u \triangleleft v^* \) where \( v \in \Gamma' \).

There is a unique pair \( \langle \pi^*, \tilde{u} \rangle \) s.t.

- \( \tilde{u} \in \Gamma' \) and \( \tilde{\pi} : \tilde{u} \triangleleft v^* \)
- \( u = \tilde{u}^* \), \( \pi = \tilde{\pi}^* \).
proof of 5.2

Let \( \bar{u} = \langle M^u, B^u, C^u \rangle \), \( u = \langle M_u, C_u \rangle \).

Let \( M_u = L^{A_u \cap B_u} B_u \). Set \( \tilde{\bar{u}} \)

\[ M^u = L^{A_u \cap B_u}, B^u = B_u, C^u = C_u. \]

It follows easily that \( \bar{u} \in \Pi \).

Moreover, \( Q^u_a = H^\gamma M^u \), \( Q_u = Q^B^u \), where \( \gamma = \gamma^u = \omega_1^M \).

Clearly, \( M_u = M^u \).

Set \( \tilde{\bar{u}} = \pi \cup M^u \). We verify

Claim \( \tilde{\bar{u}} : \bar{u} \leq \bar{u} \).

\( \tilde{\bar{u}} : M_u \leq M^u \) and \( \tilde{\bar{u}} \cup B^u \leq B_u \) trivially,

\( \bar{u} : \tilde{\bar{u}} \cup Q^u_a \leq Q^u \) cofinally,

Now let \( \tilde{\bar{u}} : M^u \to M_u \) cofinally.

Claim \( \langle \tilde{\bar{u}}, \tilde{\bar{u}} \rangle \) in the lift up of \( \langle Q^u_a, \tilde{\bar{u}} \cup Q^u_a \rangle \).

proof:

Let \( x \in \tilde{\bar{u}} \). Then \( x \in \pi (a) \) for an \( a \in M_u \)

i.e. \( x < \delta \) in \( M_u \). Let \( f \in M_u \) s.t.

\( f : \delta \to a \) for a \( \delta \leq x \). We may suppose \( w, b, \delta \), e.g.,

such that \( f = f^B \) where \( f \in M^u \)

and \( \tilde{f} = f \cup \delta \) a function defined on \( \delta \).

Arguing in \( M^u \) choose for each \( u < \delta \) a maximal antichain \( A^u \) in

in the set \( \{ b \in B_u \mid \forall \chi > b \uparrow f (u) = x \} \).
For each $b \in A$, let $x_b$ be the
s.t. $b \models f(x_b) = \bar{x}$. Set $X = \{x_b : x < \bar{c} \land b \in A\}$.
Clearly $\bar{x} \in \pi(X)$, since $\bar{A}$
satisifies the $x_{-}$ chain condition and
$\bar{x}$ is regular in $\bar{M}_\omega$. But $x \in \bar{x}(X)$
QED (Claim)

Finally we note that, if $\bar{x}^* \in \pi^*$ is the
unique extension of $\bar{x}$ s.t. $\bar{x}^* : M_\omega^{\bar{B}} < \bar{M}_\omega^{B^\omega}$,
then $\bar{x}^* = \bar{x}$, since $\bar{x}$ has its defining
properties. In particular, then
$\bar{x}^* = C_{\bar{A}} = C_{\bar{A}}$
The uniqueness of the pair $<\bar{A}, \bar{x}>$
is evident. QED (Lemma 5.2)

Finally we prove:

Lemma 5.3 Let $\pi : u \in \mathcal{V}$ where
$u = \langle Mu, B^u, C^u \rangle$, $v = \langle M_v, B^v, C^v \rangle \in \mathcal{V}$.
Let $\pi : <Mu, \bar{A}> \subseteq <Mu, A>$, where
$<Mu, A>$ models ZFC - or Zermelo.
Then $\pi^*: <Mu, \bar{A}> \subseteq <M_v^{B^v}, A>$.

Proof:

$B^u$ is $IB^u$-generic over $<M_v, A>$, $B^u$ is
$IB^u$-generic over $<Mu, \bar{A}>$, and $\bar{A} \in B^u$.

Hence there is a unique $\pi^* : \pi \subset \pi$ s.t.
\[ \Pi^+; \{M_k^{B_u}, A\} \leq \{M_k^{B_v}, A\}, \text{ But then } \Pi^+ = \Pi^* = \text{the unique } \Pi^* \cap A \text{ s.t. } \Pi^* \leq M_k^{B_u} \leq M_k^{B_v}. \]

**Theorem 5.4** Let \( u = (M_k, B_u, C_u), v = (M_k, B_v, C_v) \in \Gamma' \). Let \( \Pi', \{Q_u, C_u\} \leq \{Q_v, C_v\} \) s.t. \( \Pi'(B_u x) = B_v \Pi'(x) \) for \( x \in Q_u \). Let \( \Pi: M_u \leq M_v \) s.t. \( \Pi^* Q_u = \Pi^* Q_u \) and \( \{M_u, \Pi^* Q_u\} \) is a lift of \( \{M_u, Q_u\} \), where \( \Pi: M_u \rightarrow M_v \) is essentially unique. Then \( \Pi: \{Q_u, B_u, C_u\} \leq \{Q_v, B_v, C_v\} \). Moreover, \( \Pi^* Q_u B_u = \Pi^* Q_v B_v \), where \( \Pi^* \cap A \text{ s.t. } \Pi^* M_k^{B_u} \leq M_k^{B_v} \).

**Proof.**

It suffices to show that \( \Pi^* Q_u B_u = \Pi^* Q_v B_v \).

For each \( x \in Q_u B_u \), there is a pair \( \langle d, r \rangle \in Q_u \) s.t. \( r \in d \subseteq \mathcal{A}^2 \) and \( \langle d, r \rangle \supseteq \langle C(\{x\}), \varepsilon \rangle \).

It suffices to show \( \Pi^* \langle r \rangle = \Pi^* \langle r \rangle \)

for \( r \in Q_u B_u \) s.t. \( r \in \mathcal{A}^2 \). Let \( r = i \varphi \), where \( i \in \mathcal{A}^2 \), \( \varphi \in \mathcal{A}_u \).

For each \( z \in d \subseteq \mathcal{A}^2 \), let \( A_z \) be a maximal antichain in \( \{b \in B_u \mid b \ni \varphi \} \). Then \( A_z \in \mathcal{Q}_u \) and \( A = \{\langle b, z \rangle \mid z \in \mathcal{A}^2 \wedge b \in A_z \} \in \mathcal{Q} \).

Since \( Q = H_{\mathcal{A}_u} B_u \) and \( B_u \) satisfies \( \mathcal{A}_u - \mathcal{C} \) in \( M_u \), set \( a = a_y \cup \mathcal{Q} \).

Then \( a \in Q \) and \( \Pi^* (a \cap B_u) = \Pi^* (a \cap B_v) = \Pi^* (a) \cap B_v \).
But \( \pi^*(x) = \bar{\pi}^*(\{z \mid Vb \in anB \land b \in z \in A \}) = \{z \mid \forall b \in \pi(a) \land b^\varphi \land b \in A \} = \pi^*(x) \).

QED (5.4)

Working in \( V \), define \( Q = H \beta, M = L_{\beta}^{A, B} \) and \( N = \langle H_{\beta^+}, N, <, \ldots > \rangle \) as before. We shall define an infinitary language \( L' \) on \( N \).

Using \( L' \) we shall then define an \( L' \) - forcing \( \text{IP}' = \text{IP}_L \) s.t. \( \text{BA}(\text{IP}) \) is isomorphic to \( B \times \beta \), hence to \( B_{\beta+1} \).

\( L' \) is the infinitary language on \( N \) with:

**Predicate e**, **Constant x (x \in N), B, C**

**Axiom**: \( \exists C \quad \forall \mu (\mu \in x \implies \forall \nu \in C \land \nu = z \) for \( x \in N \), \( \beta \in \varepsilon \)

\( H_\omega = H_{\omega_1}, \langle N, B, C > \in L' \) and

\( \forall x \in M \forall u \in H_\omega \forall \pi (\pi : \mu \in \langle N, B, C > \land \langle x \in \eta \rangle \land u \in \Psi(\pi)) \) (where:

\( \Psi(\pi) = \cup \eta(u) \in M \) if \( \mu \) is regular, \( \pi = \pi^* \) otherwise.)

We now pause to introduce formally a convention which we have already employed tacitly. Let \( L \) be an infinitary language on an admissible set \( N \). All of our languages have what we shall call the "special constant" \( x (x \in N) \) and the axioms include:

\( \forall u \ (u \in x \iff \forall \nu \in \varepsilon) \) for \( x \in N \)
Now let $\mathcal{M}$ be a solid model s.t. $N \subset$ wfcore ($\mathcal{M}$) and $\mathcal{M}$ interprets the predictor and non-special constants of $L$. We say "$\mathcal{M}$ models $L$" to mean that $\mathcal{M}$ becomes a model of the axiom $L_1$ if we enhance it by giving the special constants the interpretation $\nu \mathcal{M} = \nu$.

This convention was often used tacitly in [LF] and was employed here in the formulation of Cor. 3.10, where we wrote: "$\langle H_\Theta \cap c \rangle$ models $L(p)$".

We now prove:

Lemma 6.1 $L'$ is consistent.

Proof:
Let $B$ be $IB$-generic and let $\mathcal{M}=\langle H_{IV}, c_{\mathcal{M}} \rangle$ be a model of $LB$. Then $\mathcal{M} = \langle H_{\mathcal{M}}, B, c_{\mathcal{M}} \rangle$ models $L'$ by Lemma 5.1. QED (6.11)

We also note:

Lemma 6.2 Let $\mathcal{M} = \langle H_{\mathcal{M}}, B_{\mathcal{M}}, c_{\mathcal{M}} \rangle$ model $L'$. Then $B_{\mathcal{M}}$ is $IB$-generic over $\mathcal{M}$, $NB \subset$ wfcore ($\mathcal{M}$) and $\langle H_{\mathcal{M}}, C, c_{\mathcal{M}} \rangle$ models $LB$. 

We obviously have the analogue of Lemma 2:

**Lemma 6.3** Let $M$ be a model of $L'$. Let $B = B^M$, $C = C^M$. Let $\langle A_n | n < \omega \rangle \subseteq M$ s.t. $A_n \subseteq MB^n$ for $n < \omega$. Then the conclusion of Lemma 2 holds.

We now define $P' = P_{L'}^P$.

**Def** $P'$: the set of $p = \langle P_0, P_n \rangle$ s.t.

- $P_0 = \langle M, B^P, C^P \rangle \in P \cap \text{Hw}$
- $P_n = F^P$ in an at most countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $\bar{a} \subseteq M_P, a \subseteq M$.

**Def** $F'$: For $p \in P'$ let $\varphi_p$ be the conjunction of

- $\varphi_p \triangleq \langle M, B, C \rangle$ s.t. $\varphi_p = \forall \bar{x}, < M, B, C >$.
- $\varnothing' : \langle M, \bar{a} \rangle \prec \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in F^P$.
- $\varnothing' : M_P \rightarrow M$ cofinally if $\beta$ is regular.

**Def** $L'(p) = L' + \varphi_p$

$L'(p) = \{ p \mid L'(p) \text{ is consistent} \}$

Set $\mathcal{P} = \text{sats}(L'), \mathcal{P} = \text{dom}(F^P)$.

**Def** $F'$: For $p, q \in \mathcal{P}$ s.t.

- $p \leq q$ implies $\langle q_0, \bar{p}_0 \rangle \wedge R^q \subseteq R^p$
- $\varnothing' : \langle M, \bar{a} \rangle \prec \langle M, a \rangle$
- $\varnothing' : M_\bar{p}_0 \rightarrow M_\bar{p}_0$

where $\forall < a, \bar{a} > \in F^q, < a, \bar{a} > \in F^p$.
As before we get:
\[
P \leq q \iff (R^p \subseteq R^q, L'(p) \vdash (L'(q) \land
\land \text{rng}(R^p) \subseteq \text{rng}(R^q)).
\]

We set:
\[\text{Def } \pi q^p = \pi q^q p_0 \quad \text{for } p \leq q,\]

For \( p \in IP' \) define:
\[\text{Def } p^* = \langle p_0^*, p_1^* \rangle \quad \text{where}\]
\[p_0^* = \langle M_p^B, c^B \rangle, \quad p_1^* = p_1,\]

Using Lemma 5.1 - 5.3 we easily get:

**Lemma 6.4** Let \( p \in IP' \)

(a) \( \text{Def } M' = \langle 1\omega_1, B, C \rangle \) is a solid model of \( L'(p) \), then \( M' = \langle 1\omega_1, C \rangle \) is a solid model of \( L_B(p^*) \).

(b) \( \text{Def } M = \langle 1\omega_1, B, C \rangle \) is a solid model of \( L_B(p^*) \), then \( M = \langle 1\omega_1, B, C \rangle \) is a solid model of \( L'(p) \).

The proof of Lemma 3.2 goes through as before, as do the proofs of the extension lemmas 3.4 - 3.8. In particular:

**Lemma 6.5** \( p \) is compatible with \( q \) in \( IP' \) if \( L'(p) \cap L'(q) \) is consistent.
Using the extension lemmas we conclude just as before:

**Lemma 6.6** Let $G$ be $\mathcal{IP}'$-generic. Then

(a) $\left< p_0 \mid p_0 \in G \right>, \left< \overrightarrow{p} \mid p \in G \right>$ is a directed system with the direct limit:

$$\left< M, B^G, c^G \right>, \left< \pi^G_p \mid p \in G \right>$$

(Moreover, $\pi^G_p = \bigcup \{ \pi \mid \pi_0 = p_0 \land \pi \in G \}$)

(b) $\pi^G_p : p_0 \triangleleft \left< M, B^G, c^G \right>$ for $p \in G$

(c) $\pi^G_p : \left< \langle M_p, a \rangle \right> \triangleleft \left< \langle M, a \rangle \right>$ for $a, \overrightarrow{a} \in E F^p$.

---

**Def:** $B' = BA' (\mathcal{IP}')$

We define an embedding $k' : IB \rightarrow B'$

by $k'(b) = [\overrightarrow{b} \in B^G]$, where $\mathcal{G}$ is the canonical $\mathcal{IP}'$-generic name.

**Lemma 6.7** $k'$ is an injective homomorphism.

$k'$ is clearly a complete homomorphism.

ex. $k' (\bigcap b_i) = \bigvee \bigcap b_i \in B^G = \bigcap \{ b_i \in B^G \} = \bigcap k'(b_i)$. Injectivity follows from:

Claim $k'(b) = 0 \rightarrow b = 0$.

Suppose not. Let $b \in B$ where $B \nvdash IB$.

- generic. Let $\mathcal{U} = \left< \mathcal{U}, c \right>$ be a solid model of $\mathcal{L}'$. Then $\mathcal{U}' = \left< \mathcal{U}, B, c \right>$ is a solid model of $\mathcal{L}'$. By lemma there is $p$ s.t., $\mathcal{U}' \nvdash \mathcal{L}'(p)$ and

$b \notin \pi^G_p$. Let $\pi^G_p(b) = b$. Then

$b \in B^G$. At follows that $p \nvdash b \in B^G$,

Hence $0 \neq [(p) c[\overrightarrow{b} \in B^G]] = k'(b) = 0$.

*QED* (6.17)
We now consider the factor algebras \( IB'/B \), where \( B \in B_a - \text{generic} \). For greater perspicuity we write \( IB'/B \) for \( IB'/k'B \) and \( b/B \) for \( b/k'B \) when \( b \in IB' \). Remembering the definition of \( p^* (p \in \mathcal{P}') \) we prove:

**Lemma 6.8** Let \( B \) be \( IB - \text{generic} \). Let \( p, q \in \mathcal{P}' \), \( [p]/B, [q]/B \) are compatible in \( IB'/B \) iff \( p^* \) and \( q^* \) are compatible in \( IB' \).

**Proof**

(\( \leftarrow \)) Suppose not. Then \( [p] \cap [q] \cap B = 0 \).

Hence \( [p] \cap [q] \cap k'(b) = 0 \) for a \( b \in B \).

Let \( M' \) be a solid model of \( L_B (p^*) \cup L_B (q^*) \).

Then \( M' = \langle M', B, C^M \rangle \) is a solid model of \( L_B (p^*) \cup L_B (q^*) \) by Lemma 6.4.

Hence by Lemma 6.3 there is \( r \leq p, q \) s.t. \( M \models L_B (r) \) and \( b \in \text{rng}(C^M) \). Let \( C^M (b) = b \). Then \( b \in B_r \). Thus \( r \leq p, q \) and \( r \models b \in B_r \). Hence \( [r] \models [b] \models [b^* B_r] = k'(b) = 0 \). Contd!

(\( \rightarrow \)) Suppose not. Then \( L_B (p^*) \cup L_B (q^*) \) is inconsistent. Since \( B \in IB_a - \text{generic} \) there is \( b \in B \) s.t.

(1) \( b \models L_B (p^*) \cup L_B (q^*) \) is inconsistent.
Now let $\tilde{G}$ be $IB'/B$ - generic r.t. $\{p\}/B$, $\{q\}/B \leq \tilde{G}$, Set:

$$G = \{ p \in IP' \mid [p]/B \leq \tilde{G} \}$$

Then $G$ is $IB'$ - generic, $p, q \in G$, and $B = B^G$. By genericity, there is $r \leq p, q$ in $G$ r.t. $b \in \text{rg}(\pi^n)$. Let $\pi^n(b^-) = b$.

Then $b \in B^\pi$ since $b \in B$. Now let $M' = \langle IB', B', C \rangle$ be a solid model of $L'(\pi^\pi)$. Then $b \in B'$ since $\pi^n(b^-) = b$ and $\tilde{b} \in B^\pi$. But then $M = \langle IB', C \rangle$ is a solid model of $L(B, (p^*) \cup L(B, (q^*) ))$, where $b \in B'$ and $B'$ is $IB'$ - generic.

\underline{Contradiction!} (by (11), QED (6.8))

\underline{Lemma 6.9} Let $B$ be $IB'$ - generic. Then $[p^*]_{IB'} \not\equiv 0$ in $IB'/B$ \iff dense in $IB_B = BA(IP_B)$.

\underline{Proof:}

We first note that $\{ [p^*]_{IB'} \not\equiv 0 \} \subseteq \text{dense in } IB'$.

where $\hat{IP}_B$ is the set of $p \in IP_B$ r.t. $F^p \subseteq U$.

To see this, let $p = \langle p_0, p_1 \rangle \in \hat{IP}_B$ and $p_0 = \langle \tilde{M}_p, C_p \rangle$ with $\tilde{M}_p = L_{A_p}^{p}\langle B, B, \rangle$. Set $p^\prime = \langle M^p, B_p, C_p \rangle$ with $M^p = L_{A_p}^{p}\langle B, B, \rangle$.

$p^\prime = p^\prime$. Then if $M'$ is a solid model
of \( \mathcal{L}_B(p) \), it follows that \( \mathcal{M}' = \langle \mathcal{M}_B, B, C \rangle \) modulo \( \mathcal{L}(p') \), where \( \mathcal{M}_B = \langle \mathcal{M}_B, B, C \rangle \) (using Lemma 5.2). Hence \( p' \in \mathcal{I}_{B} \), \( p = p' \) and \( [p']/B \not\equiv 0 \) by Lemma 6.8 (taking \( p' = p \) in the statement of Lemma 6.8).

Hence it suffices to show:

**Claim** Let \( q \in \mathcal{I}_{B} \). There is \( p \in \mathcal{N}_{B} \) s.t. \( [p] \not\equiv [q] \).

**Proof.**

Let \( A = \langle a_1, \ldots, a_n, r \rangle \in \mathcal{V}[B] \) enumerate \( R \).

Let \( D \subseteq \beta \) s.t. \( A \in (\mathcal{M}_B, D) \) - definable, where \( D \subseteq \mathcal{V}[B] \). Let \( D = \beta \) and set:

\[
E = \{ \langle v, b \rangle \mid b \in B, v = \beta \wedge b \models \chi \wedge v \in \mathcal{D}^B \}
\]

Then \( D = \{ v \mid v \in \mathcal{D}^B \} \). Hence \( A \in (\mathcal{M}_B, E) \) - definable. Set:

\[
\mathcal{N}^* = \langle H_\theta, N_B, M_B, \langle B, E, A \rangle \rangle \in \mathcal{V}[B]
\]

where \( \theta > (2^{\aleph_0})^+ \) is a cardinal. Let \( p \leq q \)

in \( \mathcal{I}_{B} \) s.t. \( p \) conforms to \( \mathcal{N}^* \). Set:

\[
\mathcal{N} = \mathcal{N}^*(p, \mathcal{N}^*) = \langle H, \overline{N}, \overline{M}, \langle B, E, \overline{A} \rangle \rangle.
\]

Then \( \overline{M} = M_p = (L_B, \beta_B, \beta_B, \beta_B, \beta_B) \), \( \overline{B} = B^p \), and \( \overline{A} \in (\overline{M}, E) \) - definable by the same definition. Now form \( p' \) by:

\[
p_0' = p, \quad p'_1 = \{ \langle a, \overline{a} \rangle \in \mathcal{E}^B \mid a \in R \}
\]

\[
p_0'' = p, \quad p''_1 = \{ \langle E, \overline{E} \rangle \} \text{ where } (E, \overline{E}) \in \mathcal{E}.
\]

Then \( p' \leq q \) in \( \mathcal{I}_{B} \) and \( p'' \leq \mathcal{N}_{B} \). We show:
Claim \( [p''] \subset [p'] \subset BA(I P_B) \).

Let \( G \ni p'' \) be \( I P_B \) - generic. We show:

Claim: \( p' \in G \).

Since \( p_0 = p'_0 \) we have:

\[ \varpi : p_0 \triangleleft <M_B, C^G> \text{ where } \varpi = \varpi_{p''}. \]

At remain to show:

\[ \varpi : <M_{p'}, a> \triangleleft <M_B, a> \text{ for } <a, \bar{a}> \in \mathcal{F}_p'. \]

\( a = A(i, l) \) in \( <M_B, E> \) - definable and \( \bar{a} = \bar{A}(i) \) in \( <M_{p'}, E> \) definable by the same definition. But \( \varpi : <M_{p'}, E> \triangleleft <M_B, E>. \)

QED (6.4)

Set: \( IB_B = BA(I P_B) \). Set:

\[ A' = \{ [p]/B \mid p \in I P' \land [p]/B \neq 0 \}. \]

Then:

\( A' \) is dense in \( IB'/B \). But:

\[ A = \{ [p^*]/B \mid [p]/B \in A' \} \triangleleft \] is dense in \( IB_B \).

By Lemma 6.8 we have:

\[ [p]/B \land [q]/B = 0 \implies [p^*] \cap [q^*] = 0 \text{ in } IB'/B \text{ in } IB_B \]

But for \( a, b \in A \) we have \((\text{in } IB_B)\):

\[ a \leq b \iff \forall c \in A \text{ (cnb} = c \to \text{cn}a = c) \]

since \( A \) is dense in \( IB_B \). Similarly:

for \( A', IB'/B \). Hence:
\[ [p] / B \leq [q] / B \text{ in } IB / B \quad \iff \quad [p^*] \leq [q^*] \text{ in } IB_B. \quad \text{Hence:} \]

Cor 6.4.1: There is \( \sigma_B : IB / B \xrightarrow{\sim} IB_B \) uniquely defined by: \( \sigma_B([p] / B) = [p^*] \).

But \( \sigma_B = \sigma_B^* \) where:

\[
\vdash_{IB_B} \sigma^* : IB / B \xrightarrow{\sim} IB_B
\]

and \( \vdash_{IB_B} IB = BA (IB_B^*) \).

\( \sigma(a) = \text{that } a' \text{ s.t. } \vdash_{IB} a' = \sigma^*(a / B) \).

Then \( \sigma : IB / B \xrightarrow{\sim} IB_B \times IB_B \) is an injective homomorphism. But \( \sigma \) is onto since \( \vdash_{IB} a \in IB \times IB \) and \( \vdash_{IB} t = \sigma^{-1}(a) \). Then

\( \vdash_{IB} t \in IB / B \) and hence there is a unique \( b \) s.t. \( \vdash_{IB} b / B = t \). Hence

\( \vdash_{IB} a = \sigma(b / B) \).

We note finally that \( \sigma^{-1} = k \), where

\( k : IB_B \xrightarrow{\sim} IB_B \times IB_B \) is the natural injection.

Let \( c = k'(b) = \llbracket b \in B / B \rrbracket_{IP} \). We then have!
Therefore, having established that $IB'$ is a representation of $IB^{m+1}$, we examine its properties more closely. We note, of course, that by the last result, $IB'$ adds no reals and is, in fact, a complete. Hence, if $G \in IP'$ - generic, we know that $\langle H, IB', C, G \rangle$ models $L'$ whenever $\theta > 2^\alpha$ is regular.

(Th e only problematical axiom was $H_{\omega_1} = H_{\omega_1}$, which is now established.)
Let $B$ be $\mathcal{P}^\kappa$-generic and $C < \mathcal{Q}^B$
be countable.

$G^{B, C} = \{ p \in \mathcal{P}^\kappa \mid \text{there is } \pi \text{ s.t.}
\]
\[
\pi : p_0 \leq \langle M, B, C \rangle
\]
\[
\pi : \langle M, \bar{a} \rangle < \langle M, a \rangle \text{ whenever } \langle a, \bar{a} \rangle \in F^p.
\]

**Lemma 6.11** Let $G$ be $\mathcal{P}^\kappa$-generic. Let


**Proof**

Suppose not. Then there is $p \in G^{B, C} \setminus G$.
But then there is $q \in G$ s.t. $p, q$ are
incompatible. Hence $p, q \in G^{B, C} +

\text{ hence } \langle H, B, C \rangle = L(p) \cup L(q)

\text{ for } G = 2^{B^+}$. Contr. QED (6.11)

The following lemma is sometimes
useful in dealing with the case
that $\kappa \notin A_ \mathcal{C}.

Set $\bar{B}_\lambda = \bigcup \{ B \in \mathcal{P}^\kappa : \exists \delta \exists \lambda (B \cap \delta \in \mathcal{C}). \text{ for } \lambda \leq \mu, \].
Hence $\hat{\mathcal{B}}_\lambda$ is dense in $\mathcal{B}_\lambda$ whenever $\lambda \neq A \subseteq o$, $cf(\lambda) = \omega_1$, since $\mathcal{B}_\lambda$ is then the direct limit of $\langle \mathcal{B}_i : i < \lambda \rangle$.

Lemma 6.12 Let $\mu \in \text{Acc}$. (Hence $\mu = \kappa$ is strongly inaccessible in $V_\kappa$.) Let $\mathcal{B} \subseteq \mathcal{B}$, let $\mathcal{B} \cap \mathcal{B}_\kappa$ be generic for $\kappa < \mu$. Then $\mathcal{B}$ is generic.

Proof:
$\mathcal{B} = \bigcap_{\kappa < \mu} \mathcal{B}_\kappa$ is dense in $\mathcal{B}$ and $\mathcal{B}_\mu$ satisfies $\mu = \text{cc}$, hence $\mathcal{B}_\mu = \mathcal{B}_\mu$. Let $\Delta$ be dense in $\mathcal{B}_\mu$. Then $\exists \lambda < \mu \setminus \Delta \cap \hat{\mathcal{B}}_\lambda$ is dense in $\hat{\mathcal{B}}_\lambda$. Then $\exists \lambda < \mu \setminus \Delta \cap \hat{\mathcal{B}}_\lambda$ is dense in $\hat{\mathcal{B}}_\lambda$. Then $\exists \lambda < \mu \setminus \Delta \cap \hat{\mathcal{B}}_\lambda$ is dense in $\hat{\mathcal{B}}_\lambda$. Hence $\Delta \cap \hat{\mathcal{B}}_\lambda$ is dense in $\mathcal{B}_\lambda$ for a $\lambda < \mu$. Hence $\Delta \cap \mathcal{B} \cap \mathcal{B}_\lambda \neq 0$ by genericity. Q.E.D. (6.12)

$\mathcal{P}'$ satisfies a rather strong form of reproductivity.
Lemma 6.13 Let $p \in IP$. Let $B', C'$ be set.

$B'$ is $B^p$-generic over $M_p$.

$\Omega_p = \Omega_B^p$.

$C' < \Omega_p$ cofinally.

Then $p' \in IP$, where $p' = (M_p, B', C')$, $p'_1 = p_1$.

Proof Let $M'$ be a notched model of $L'(p)$. Let $\pi = \pi_p^M$ and $\pi^* = \pi^* \sigma$, s.t. $\pi^*: M_p^{B'} < M_{B'}$.

Set $B = \bigcup_{x \in \Omega_p} (x \cap B')$, $C = \pi^* C'$.

Let $M'$ be the result of replacing $B'$, $C'$ with $B$, $C$.

Claim $M' \models L(p')$.

(1) $B$ is $B^p$-generic over $V$.

Proof

Case 1 $\mu \in A_C$

Then $B^p_\mu$ has a dense set of $\lambda \in d (B^p_\mu) < \delta^+$, $\mu$ since $d (B^p_\mu) = \omega_1$ in $V[\beta^\kappa]$. Let $P_\mu \in B^p$ - generic over $V$, since it is $B^p_\mu$-generic over $N = H^V_{\beta^+}$. But then $B^p$ has a dense set of $\lambda \in d (B^p_\mu) < \delta^+$ in $M_p$. Let $P_\mu \in \Omega_p$ be dense in $IB^p$. Then $B \cap P_\mu$ is $P_\mu$-generic over $Q_p$.

But $\pi(P_\mu) = D$ where $D$ is dense in $ IB^p$ and $\pi^* (B \cap P_\mu) = B \cap D$; when $\pi_* Q_p^{B^p} < \Omega_B^{B^p}$. Since $B \cap P_\mu$ is $P_\mu$-

- generic over $Q_p$, $B \cap D$ is $D$-generic.
\[ Q = H_\eta \] hence over \( V \), Hence \( B \cup B_\mu \) -
generic over \( V \), QED (Case 1)

Case 2 \( \mu \neq \aleph_\alpha \).

Then \( \mu = \kappa \) is strongly inaccessible.

At follows that the set \( D \) of \( \lambda < \mu \) such
\( \forall \eta \in V_\eta \cup \mathcal{B}_\mu \) is predeces in \( B_\mu \) \( \mathcal{H} \) if it
it is predeces in \( B \cup V_\eta \), \( \mathcal{H} \) = club in \( \mu \). \( D \) \( \mathcal{H} \) \( \mathcal{H} \) definable. Hence there are

arbitrarily large \( \delta \in \mathcal{D}(\mathcal{V}_\delta) \).

Let \( A \) be dense in \( B_\mu \) and let \( A \subset D \)
be a max. antichain. Then \( A \cup V_\delta \) for all \( \delta \in D \)
\( \delta \) \( \delta \) definable. Then

\[ \mathcal{P}^*(B_\mu \cap V_\delta) = B \cap V_\delta \] and

\[ \mathcal{M}_\delta \] = Every \( \delta \in V_\delta \) is predeces in \( B_\mu \) \( \mathcal{H} \) in \( V_\delta \cap \mathcal{B}_\mu \)

Since \( B' \subset B_\mu \) - generic it follows that

\[ \mathcal{M}_\delta \models (\forall \delta)(B_\mu \cap V_\delta) \neq \emptyset \] whenever \( \delta \) is

\( \mu \) genericity. But then the same

holds of \( B_\mu \cap V_\delta \). Hence \( B_\mu \cap V_\delta \) \( \neq \emptyset \).

Hence \( B \cap A \neq \emptyset \), QED (1)

(1) \( Q^B = Q^{B_\mu} \)
Set \( Q(3) = L(3) [A, B, B] \) for \( 3 \leq 8 \).

Similarly for \( Q(8) \).

Let \( Q_p(3) \), \( Q_p(8) \) have the same definition in \( A, B \) for \( 3 \leq 8 \). Then
\[
\pi^*(Q_p(3)) = Q(B^{8-3}), \quad \pi^*(Q_p(8)) = Q(B^{8-8}),
\]

(1) Let \( 3 < 8 \), \( \bar{3} = \pi(3) \). Then
\[
\bar{3} > 3 \text{ and } Q_p(3) \in Q(B^8), \text{ and } Q_p(8) = Q(B^8), \text{ Hence } Q(B(\bar{3})) \in Q(B^{8-8}).
\]

Where \( \pi(\bar{3}) = 5 \).

(2) is entirely similar. QED (2)

But then \( \pi^* R Q_p(8) \). Let \( Q_p(8) \approx Q(B) \) and \( \pi^* C = C \). Hence:

(3) \( \pi^* Q_p(8) \); \( Q_p(8) \approx Q(B) \) and \( \pi^* C = C \). Hence:

By the definition of \( B \) we have:

(1) \( \pi^*(B^* x) = B \cap \pi(x) \) for \( x \in Q_p(8) \).

Since \( \langle \bar{M}, \pi \rangle \) is the lift up of \( \langle M_p, \pi \rangle \), where \( \pi : M \rightarrow \bar{M} \) naturally, we conclude by Lemma 5.4 that:

(5) \( \pi^* \langle M_p, B, C \rangle \approx \langle M, B, C \rangle \),

where \( \pi^* = \pi^* M_p \).

Since \( p^* = p \), we, of course, have:

\( \ldots \)
(6) \( \overline{\pi}; \langle M, a \rangle < \langle M', a \rangle \) whenever \( \langle a, \overline{a} \rangle \in F' \).

Thus we have shown:
(7) \( M' \models \varphi' \).

All remain only to show:

Claim: \( M' \models L' \).

Let \( x \in M \). By Lemma 6.13,

Lem \( u = \langle \overline{M}, \overline{B}, \overline{C} \rangle \in H_{\omega_1} \) and \( \pi \in \mathcal{M} \) set \( \pi: u \not\models \mathcal{A}' \langle M, B^\omega, C^\omega \rangle \),
and let \( y \in \operatorname{rng}(\overline{\pi}) \), and let \( \pi : \langle \overline{M}, \overline{B}, \overline{C} \rangle < \langle \overline{M}, \overline{B'}, \overline{C'} \rangle < \langle M, B'_\overline{y}, C'_\overline{y}, B, C \rangle \).

Using Lemma 5.4 it follows easily that:

\( \pi : \langle \overline{M}, \overline{B'}, \overline{C'} \rangle \models \mathcal{A}' \langle M, B, C \rangle \),
where \( x \in \operatorname{rng}(\pi) \). QED (Lemma 6.13)

As a corollary of the proof:

Lemma 6.14: Let \( p, B', C', p' \) be as above.

Let \( q \leq p \), Set \( B = \bigwedge_{x \in p} \pi^*_p (x \cap B') \),

\( C = \pi^*_p " C' \). Then

(a) \( q_\pi^B = q_\pi^B \) and \( B \in \mathcal{P}^q \) generic over \( M_q \)

(hence \( q' \in \mathcal{P} \) where \( q' = \langle M_q, B, C \rangle \), \( B' \), \( C' \))

(b) \( \pi' \models q' \).

Proof (sketch). Repeat the proof of (1)-(5) mutatis

mutandis. QED (6.14)
Lemma 7 Let $G$ be $\mathbf{P}'$-generic, $B = B^G$, $C = C^G$. Let $B'$ be $B$-generic but, $Q^B' = Q^B$. Let $C' \subseteq Q^B$ be countable and cofinal in $Q^B$. Assume that $B', C'$ lie in a generic extension of $V[G]$ which adds no real. Set: $G' = G \setminus (B \cap C)$. Then $G'$ is $\mathbf{P}'$-generic and $V[G'] = V[G]$.

Proof:
There is a $p \in G$ s.t. $C' \subseteq \text{rng}(\pi^*)$ and $B' \cap x \in \text{rng}(\pi^*)$ for all $x \in C$, where
\[ \pi = \pi^G \quad \text{and} \quad \pi^* \in \text{the unique } \pi^* \in \mathcal{M}^G \]
\[ \text{s.t. } \pi^* \cap M^B \subseteq M^B. \]

Set:
\[ B^p' = \bigcup_{x \in C} \pi^* \setminus (B' \cap x), \quad C^p' = \pi^* \setminus \pi^* C'. \]

Then:
1. $B', C' \in V[G]$ into
   \[ B' = \bigcup_{x \in C^p} \pi^*(B' \cap x), \quad C' = \pi^* \setminus C^p \]
2. $\pi^*(B^p' \cap x) = B' \cap \pi^* x$ for $x \in M^p$
3. $B^p' \text{ is } B^p\text{-generic over } M^p$

Proof:
Let $\Delta \subseteq M^p$ be dense in $B^p$. Then $\pi(\Delta)$ is dense in $B^p$. Hence $\pi(\Delta) \cap B' \cap \pi^* \Delta \neq \emptyset$ for some $x \in C^p$, i.e., $B' \cap \pi(\Delta) \neq \emptyset$. Hence $\Delta \cap B^p \cap x \neq \emptyset$. QED (3)
Proof:
Set \( \varphi_p(\zeta) = \mathcal{L}_3^* [A_p, B_p, B_p^*], \) for \( 3 < \delta_p, \)
\[ \varphi_p(3) = \mathcal{L}_3 [A_p, B_p, B_p^*] \]

Then \( \varphi^*(\varphi_p(\zeta)) = \varphi((\varphi(\zeta))) = \mathcal{L} [A_p, B_p^*, B_p^*] \)
and \( \varphi^*(\varphi_p(\zeta)) = \varphi((\varphi(\zeta))) = \mathcal{L} [A_p, B_p, B_p] \).

Since \( A_p = B_p, \) there is \( 3 < \delta_p, \)

Let \( \varphi((\varphi(\zeta))) \in \varphi((\varphi(\zeta))). \) Hence \( \varphi_p \subset \varphi_p^* \)

Similarly \( \varphi_p^* \subset \varphi_p^* \), \( \aleph \in \mathcal{D} \)

By Lemma 6.13 we conclude:

(5) \( p' \in \mathcal{P}' \) where \( p' = \langle M_p, B_p^*, C_p^* \rangle, \)

Now let \( q \leq p. \) Set:
\[ B^q = \bigcup_{x \in \varphi_p} \varphi^* (B^p \wedge x), \ C^q = \varphi^* \cup C_p^* \]
\[ q_p' = \langle M_q, B^q, B^q \rangle, \ q'_p = \varphi_p' \]

By Lemma 6.14 we have:

(6) \( ca \) \( B^q \) in \( B^q, \) generic over \( M_q \), and
\[ \varphi_q = \varphi_q \]

(b) \( q' \in \mathcal{P}' \)

(c) \( \varphi_p' q_p' = \varphi_p' q_p' \)
At \( q \leq r \leq p \), we could, of course, define \( q' \) from \( r' \) the way we defined it from \( p' \). At it easily seen that we get the same thing: Hence:

\[ 71 \quad q \leq r \leq p \rightarrow q' \leq r' \leq p'. \]

Set \( \Delta_0 = \{ q | q \leq p \} \), \( \Delta_1 = \{ q | q \leq p' \} \).

Set \( \sigma(q) = q' \) for \( q \in \Delta_0 \). Then:

\[ \sigma^{\prime} \Delta_0 \subseteq \Delta_1 \quad \text{and} \quad r \leq q \rightarrow \sigma^{\prime}(r) \leq \sigma(q) \]

for \( r, q \in \Delta_0 \).

Now let \( q \leq p' \). We can reverse the above operation \( \sigma^{\prime} \) by letting:

\[ B^{\sigma^{\prime}} = \bigcup_{k \leq q' \leq p'} (B \otimes k'), \quad C^{\sigma^{\prime}} = \bigcap_{k' \leq q} (C \otimes k'), \]

\[ \mu^{\sigma^{\prime}} = \sigma^{-1}(q'), \quad \tilde{\sigma}^{\sigma^{\prime}}(q) = \sigma^{-1}(q'), \]

\[ \sigma^{\sigma^{-1}}(q) \in \Delta_1, \quad r \leq q \rightarrow \sigma^{-1}(r) \leq \sigma^{-1}(q). \]

Moreover \( \sigma^{-1}(q) = q \) for \( q \in \Delta_0 \) and \( \sigma \sigma^{-1}(q) = q \) for \( q \in \Delta_1 \).

Hence:

\[ 81 \quad \sigma : \langle \Delta_0 \rangle \leq \langle \Delta_1 \rangle. \]
At \( q \in \Delta_0 \cap G' \) it then follows easily that \( \sigma(q) \in G' = G_B^{*} \cap C' \). Similarly, \( q \in \Delta_1 \cap G' \rightarrow \sigma^{-1}(q) \in G = G_B^{*} \cap C \). Hence:

(9) \( \sigma^{-1}(\Delta_0 \cap G) = \Delta_1 \cap G' \).

Hence

(10) \( V[G] = V[G'] \), since

\[
G' = \left\{ x \mid \forall q \in \Delta_0 \cap G, \sigma(q) \leq x^2 \right\}
\]

\[
G = \left\{ x \mid \forall q \in \Delta_1 \cap G', \sigma^{-1}(q) \leq x^2 \right\}
\]

Finally:

(11) \( G' \cap \Pi' \) is generic.

**Proof**

Let \( \Delta \) be dense in \( \Pi' \). Set

\[
\Delta' = \{ q \in \Delta_0 \mid \sigma(q) \in \Delta^2 \}.
\]

Then \( \Delta' \) is dense above \( p \) in \( \Pi' \).

Hence \( G \cap \Delta' \neq \emptyset \). Hence

\[
G' \cap \Delta \cap \Delta_1 = \sigma^{-1}(G \cap \Delta') \neq \emptyset.
\]

QED (Lemma 7)
Setting \( |B| = BA(\{p\}) \), we note that
\( F \upharpoonright |B| \) is generic if and only if
\[ G = \{ p \mid \[ p \] \in F \} \]

is \( |B| \) generic and
\[ F = F_G = \{ b \in |B| \mid G \upharpoonright b \neq \emptyset \} \].

The proof of Lemma 7 actually gives:

**Corollary 1.** Let \( G, B, C, B', C', G' \) be as above.

There is \( \sigma^* \in \mathcal{V} \) s.t. \( \sigma^* : |B'| \looparrowright |B| \) and
\[ F_G' = \sigma^* \circ F_G \].

**Proof.** (Assume w.l.o.g. \( G' \neq G \))

Set \( \Delta_2 = \{ x \mid \text{incompatible with } p \text{ and } p' \} \). Then \( \Delta_0 \cup \Delta_1 \cup \Delta_2 \) is dense in \( |B| \).

Since \( G \neq G' \), we have \( B \not\equiv B' \) or \( C \not\equiv C' \).

Hence \( B \parallel p \not\equiv B' \parallel p' \) or \( C \parallel p \not\equiv C' \parallel p' \). Thus \( p, p' \) are incompatible and \( \Delta_0, \Delta_1, \Delta_2 \) are mutually disjoint.

Set \( \Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \). We define:

\[ \sigma' : \langle \Delta, \leq \rangle \looparrowright \langle \Delta, \leq \rangle \]

by:

\[ \sigma'(q) = \begin{cases} \sigma(q) & \text{if } q \in \Delta_0 \\ \sigma^{-1}(q) & \text{if } q \in \Delta_1 \\ q & \text{otherwise} \end{cases} \]

Since \( f : |B| \looparrowright BA(\langle \Delta, \leq \rangle) \) where
\[ f(b) = b \cap \Delta, \]
we have:
σ' induces \( \sigma^* : IB' \leftrightarrow IB' \) defined
by: \( \sigma^*([p]) = [\sigma'(p)] \) for \( p \in A' \).
Q.E.D. (7.1)

At we run the proof with \( C' = C^G \),
we get:

Cor 7.2 Let \( G \) be \( IB_{\mu+1} \) - generic and
let \( B = G \cap IB_\mu \). Let \( B' \) be \( IB_\mu -
\) generic s.t. \( H_y[B'] = H_y[B] \),
where \( y = y_{\mu+1} \). Then there is \( \pi \in V
\) s.t. \( \pi \) is an automorphism of \( IB_{\mu+1}
\) and \( \pi''B = B' \) (Hence \( C' = \pi''G \cap
IB_{\mu+1} \) - generic and \( V[C'] = V[C] \).)
Lemma 8 \( \mathfrak{B} \) \( \mu + 1 \) is symmetrically preserved over \( \mathfrak{B} \) whenever \( \exists \leq \mu \) s.t. \( \exists \in \mathcal{A} \mathfrak{C} \),

proof

This reduces to:

Main Claim Let \( \theta > 2^\omega \) be big enough to verify the preservation of \( \mathfrak{B}_{\xi} \) for all \( i \leq \lambda \) s.t. \( i \in \mathcal{A} \mathfrak{C} \). Let \( G \) be \( \mathbb{P}' \)-generic and let \( \pi \in \mathbb{V}[G] \) s.t. \( G \vdash \pi \)-conforming and \( \pi: \mathbb{W} < \mathbb{W} = H_\theta \), where \( \mathbb{W} \) is countable and transitive. Set \( \beta = \mathbb{B}^G \cap \mathfrak{B}_\xi \). Let:

\[
\pi(\bar{\xi}, \mathbb{P}', \langle \mathfrak{B}_i \mid i \leq \mu \rangle) = \bar{\xi}, \mathbb{P}', \langle \mathfrak{B}_i \mid i \leq \lambda \rangle.
\]

Suppose that \( \mathfrak{B}' \) is \( \mathbb{B}_{\bar{\xi}} \)-generic over \( \mathbb{W} \) and \( \mathfrak{B}' \) is \( \mathbb{B}_{\bar{\xi}} \)-generic s.t.

\[
\mathbb{V}[\mathfrak{B}'] = \mathbb{V}[\mathfrak{B}] \text{ and } \pi'' \mathfrak{B}' \subset \mathfrak{B}'.
\]

Let \( \bar{G}' \) be \( \mathbb{P}' \)-generic over \( \mathbb{W} \) s.t. \( \bar{G}' = \mathbb{B}^G_{\bar{\xi}} \). There is \( \bar{G}' \) s.t.

- \( \bar{G}' \) is \( \mathbb{P}' \)-generic
- \( \mathfrak{B}' = \mathbb{B}_{\bar{\xi}} \cap \mathbb{B}_{\bar{\xi}}^G \)
- \( \pi'' \bar{G}' \subset \bar{G}' \)
- There is \( \sigma: \mathfrak{B}' \prec \mathfrak{B}' \) s.t.
  \[
  \sigma'' \mathfrak{F}_\sigma = \mathfrak{F}_{\bar{G}}.
  \]

(\( \sigma \) here \( \mathfrak{B}' = \mathcal{B} \mathcal{A} (\mathbb{P}') \))
proof.

Case 1: \( \mu \in A^c \)

We can assume w.l.o.g. that \( \bar{z} = \mu \).

Since \( V[B'] = V[B] \) we conclude:

\( Q B' = Q B \), using the fact that \( B \)

satisfies \( \mathcal{A} - C \). Set \( \bar{c}' = c \bar{c}', \bar{c}' = \bar{\pi}^* \bar{c}' \)

where \( \bar{\pi}^* : \bar{a} \mapsto \bar{a}, \bar{\pi}^*: W[B'] \rightarrow W[B] \)

and \( \bar{\pi}^*(\bar{B}') = B' \). Set \( G' = GB', C' \).

Then by Lemma 7.1 we have:

1. \( G' \) is \( \mathcal{P}' \) generic and \( B' = BG' \)

2. There is \( \sigma : IB' \rightarrow IB' \) s.t. \( \sigma^* \bar{F}_G = \bar{F}_G' \)

We need only prove:

3. \( \bar{\pi}' \bar{G}' \subseteq G' \)

proof.

Let \( \bar{\pi} \in \bar{G}' \), \( \bar{p} = \bar{\pi}(\bar{a}) \). Since

\( \bar{\pi}' \bar{G}' : \bar{p}_0 \triangleleft \langle \bar{M}, B', \bar{c}' \rangle \) and \( \bar{\pi}(\bar{p}_0) = \bar{p}_0 \).

\( \bar{\pi}^*(\langle \bar{M}, B', \bar{c}' \rangle) = \langle M, B', C' \rangle \), we have:

\( \bar{\pi} : \bar{p}_0 \triangleleft \langle M, B', C' \rangle \), where

\( \bar{\pi} = \bar{\pi}' \bar{\pi}^* \bar{G}' \). At least we remain

only to show:

\( \bar{\pi} : \langle M, \bar{a} \rangle \leq \langle M, a \rangle \) whenever

\( \langle a, \bar{a} \rangle \in \bar{\pi}(\bar{p}) = \bar{\pi}(\bar{F}_G) \).

Let \( \bar{\pi}(\bar{a}') = a \). Then \( \langle a', \bar{a} \rangle \in \bar{F}_G \)

and \( \bar{\pi}' \bar{G}' : \langle M, \bar{a} \rangle \leq \langle \bar{M}, a \rangle \).
Hence $\hat{\tilde{\omega}} = \hat{\omega} \circ \tilde{\pi} \hat{\tilde{\omega}}'; \langle \tilde{\omega}, \hat{\tilde{\omega}}' \rangle < \langle \tilde{\omega}, \hat{\tilde{\omega}} \rangle$.

Since $\pi(\langle \tilde{\omega}, \hat{\tilde{\omega}}' \rangle) = \langle \tilde{\omega}, \hat{\tilde{\omega}} \rangle$.

Q.E.D. (Case 1)

**Case 2** Case 1 fails.

Then $\mu = \chi$ is strongly inaccessible. Let $\langle \chi, \zeta < \omega \rangle \subseteq V[G]$ be monotone and cofinal in $\Theta$ w.r.t. $\chi^\chi = \omega$ and $\chi^\chi \subseteq C^G$ for $i < \omega$. Let $\pi(\tilde{\omega}_i) = \chi_i$. We may also assume w.l.o.g. that $\chi_i \subseteq C^G$.

Hence $B_{\chi_i+1}$ is proper over $P_{\chi_i}$ for $i < \omega$. Set:

$B = B^G$, $B_i' = B \cap B_{\chi_i}$, $C = C^G$

$B'' = B^G$, $B_{i'}'' = B'' \cap B_{\chi_i}$, $C'' = C^G$.

(Hence $\tilde{B}_0 = B$, $\tilde{B}_0'' = B$). By $P$-induction we may successively choose $B''_i (i < \omega)$ w.r.t. $B''_0 = B'$, $B''_{i+1} \supseteq B''_i$ and $B''_i \in B_{\chi_i}$ - generic, $\pi'' B''_i \subseteq B''_i$ and $V[B''_i] = V[B_i']$. Set:

$B'' = \{ a \in \Pi \mid \forall \zeta < \omega \ B''_\zeta \text{ b.c.a.} \}$.

Then $B'' \cap B_{\chi_i} \subseteq B''_i$ - generic for $i < \omega$.

Hence by Lemma 6.12:

(4) $B'' \cap P_{\chi_i}$ - generic.
We then set \( G' = G^{13''}, C'' \), where \( C'' = \pi^{*} \widehat{C}'' \). The rest of the proof is exactly as in Case 1.

Q.E.D (Lemma 8)

Finally, we prove:

**Lemma 9**: \( IB_{\kappa+1} \) is symmetrical over \( IB_{\kappa} \).

**Proof.**

This reduces to:

**Main Claim**: Let \( \sigma; IB_{\kappa} \prec \sim IB_{\kappa} \). There exists \( \sigma^{*}; IB' \prec \sim IB' \), \( \sigma^{*}k' = k'\sigma \).

**Proof.**

Let \( N^* = \langle H_8, N, \sigma, <, IB_{\kappa} \rangle \), \( \theta > 2^B \).

Set \( \Delta = \{ p | p \text{ conforms to } N^* \} \).

Then \( \Delta \) is dense in \( IB' \). For \( p \in \Delta \), define \( p' = \overline{\sigma}(p) \) by:

Set \( \overline{N}^* = \overline{N}^*(p, N^*) = \langle \overline{H}, \overline{N}, \overline{\sigma}, <, IB' \rangle \).

Hence \( IB^p \cup \overline{IB} = IB^p \)-generic over \( \overline{M} = M_{p^*} \) and \( \overline{\sigma}; \overline{IB} \prec \sim \overline{IB} \).

Hence \( QIB^p = Q\overline{\sigma}IB^p \), and

\( \overline{N}(IB^p) = \overline{N}(\overline{\sigma}IB^p) \) and
\[ Q^{B^p} = H \tilde{N}^{[B^p]} \omega \]

Set:

\[ p'_0 = \langle M_p, \tilde{\sigma}^{-1} B^p, C^p \rangle, \quad p'_1 = p_1 \]

Then \( p'_1 \in IP_1 \). It follows easily by earlier lemmas that

\[ p \leq q \iff \tilde{\sigma}(p) \leq \tilde{\sigma}(q) \]

and \( \tilde{\pi} p q = \tilde{\pi} \tilde{\sigma}(p), \tilde{\sigma}(q) \) for \( p, q \in \Delta \).

Moreover, if we set \( \tilde{\sigma}^{-1}(p) = p'' \) where

\[ p'' = \langle M_p, \tilde{\sigma}^{-1} B^p, C^p \rangle, \quad p''_1 = p_1 \]

Then \( \tilde{\sigma}^{-1} \sigma(p) = \sigma \tilde{\sigma}^{-1}(p) = p \).

Thus \( \tilde{\sigma} : \langle \Delta, \leq \rangle \longrightarrow \langle \Delta, \leq \rangle \). Hence \( \tilde{\sigma} \) induces \( \sigma^* : IB' \simeq IB' \) as \( \sigma^*(p) = \tilde{\sigma}(p) \), \( \Delta \) remains only to show:

Claim \( \sigma^* \bar{k} = \bar{k} \tilde{\sigma} \).

Let \( b \in IB_n \).

Set:

\[ \Delta^* = \{ p \in \Delta \mid b, \tilde{\sigma}(b) \in \text{any} \ (\bar{\pi} p) \} \]

Then \( \Delta^* \) is dense in \( \Delta \) and \( \tilde{\sigma}^* \Delta^* = \Delta^* \) since \( \bar{\pi} p \) depends only on \( p_1 \), hence \( \bar{\pi} p = \bar{\pi} \tilde{\sigma}(p) \). Then for \( p \in \Delta^* \), we have \( \bar{\pi} p (\bar{\sigma}(b)) = b \), (hence \( \bar{\pi} p (\bar{\sigma}(b)) = \sigma (b) ) \);
\[ \sigma^*([p]) \subseteq \sigma^* k'(b) \iff [p] \subseteq k'(b) = \bigcup \{ \tilde{b} \in B \mid \tilde{b} \subseteq \sigma(b) \} \]
\[ \iff \tilde{b} \in B^p \iff \tilde{\varphi}(\tilde{b}) \in \varphi^{-1}(B^p) = B^{\varphi(p)} \iff \]
\[ \iff \sigma^*([p]) \subseteq k' / \varphi(b) \]
\[ \Rightarrow \sigma^*([p]) \subseteq k'(\sigma(b)) \]

Since \( \sigma^*([p]) \mid p \in \Delta^* \subseteq \sigma \) is dense in \( B' \),
we conclude: \( \sigma^* k'(b) = k'(\sigma(b)) \)

\( \text{Q.E.D. (Lemma 9)} \)
Lemma 10. \( \langle \mathcal{B}_i \mid i \leq \mu + 1 \rangle \) satisfies (a1)-(h) of §2.3.

Proof.

(a1) is straightforward.

We prove (a1) for \( i = \mu + 1 \). Let \( \mathcal{B} \) be \( \mathcal{B}_\mu \)-generic and \( G \) be \( \mathcal{B}_\mu \)-generic.

At \( \gamma = \gamma_\mu \leq \gamma < \beta_{\mu+1} \), and \( \gamma \) is regular in \( V_\gamma \).

Then \( \gamma \) remains regular in \( V[\mathcal{B}] \), since \( \gamma = \omega_2 \) in \( V[\mathcal{B}] \) and \( \mathcal{B} \subset H_\gamma \). But then \( \text{cf}(\gamma) = \omega \) in \( V[\mathcal{B}][G] \) by Lemma 3.13.

At \( \mathcal{B} = \mathcal{B}_{\mu+1} \), then \( \text{cf}(\beta) = \omega_1 \) in \( V \), hence in \( V[\mathcal{B}][G] \), since no new scales are added. But then \( \bar{\beta} = \omega_1 \) in \( V[\mathcal{B}][G] \) by Lemma 3.12 and \( \beta^+ = \omega_1 \) in \( V[\mathcal{B}][G] \) by Lemma 3.15.

Otherwise \( \beta_{\mu+1} = \beta^+ \in A_0 \), where \( 2^\beta = \beta^+ \).

Hence \( \bar{\beta}_{\mu+1} = \text{cf}(\beta_{\mu+1}) = \omega_1 \) and \( \beta^+_{\mu+1} = \omega_2 \) in \( V[\mathcal{B}][G] \) by Lemma 3.14.1 and the remark following it, \( G \in U(a1) \).

(b) follows for \( i = \mu + 1 \) by Lemma 4.
(c), (d) are vacuous for \( i = \mu + 1 \).

(e) holds by Lemma 8 and (f) by Lemma 9.

It remains to prove (g). We imitate the proof of (g) in the first successor case.

Let \( h \leq \mu \) and set \( \hat{B}_i = B_i / B \) for \( h \leq i \leq \mu + 1 \),

where \( B \in \hat{B}_h \) - generic. We know:

\[ \hat{B}_{\mu + 1} / B = \hat{B}_\mu = \text{BA}(\hat{B}_\mu / B) \] whenever \( B' \in \hat{B}_h \) - generic. Hence if \( \hat{B} \in \hat{B}_\mu \) - generic we have:

\[ \hat{B}_{\mu + 1} / \hat{B} = (\hat{B} / B) / \hat{B} = \hat{B} / B' = \text{BA}(\hat{B}_\mu / B) \]

where \( B' = B \star \hat{B} = \{ b \in \hat{B}_\mu \mid B \upharpoonright B \in \hat{B}_\mu \} \) \( \in \hat{B}_\mu \) - generic. Hence:

\[ \hat{B}_{\mu + 1} / B = \text{BA}(\hat{B}_\mu / B) \] , \( B' \) being the canonical generic name, exactly as before we construct in \( V[B] \) a

\[ \sigma : \hat{B}_\mu \times \hat{B} \to \hat{B}_{\mu + 1} / \hat{B} \] , where

\[ \hat{B} = \text{BA}(\hat{B}_\mu / B) \] , and observe

That \( \sigma(b) = \hat{b}'(b) \) for \( b \in \hat{B}_\mu \), where \( \hat{b}' : \hat{B}_\mu \to \hat{B} \times \hat{B} \) is the natural projection. Hence \( \langle \hat{B}_i, i \leq \mu + 1 - h \rangle \) has the required properties of \( \langle \hat{B}_i, i \leq \mu + 1 \rangle \) and we can repeat all of our proofs in \( V[B] \),

QED