§4 The limit case

§4.1 The 1st limit case

Suppose that \( \lambda \) is an \( \omega \)-point of the iteration, \( \mathcal{V} = \mathcal{V}^\lambda \), and we wish to give \( \mathcal{V}^+ \) the cofinality \( \omega \). Let \( \beta \) be the supremum of the cardinals \( \beta_i < \beta^\lambda \). Then either \( \beta = \beta^\lambda \), \( 2^{\beta^\lambda} = \beta^+ \), and \( \text{cf}(\beta) = \omega_1 \), or else we are done by the second construction (hence \( 2^{\beta^\lambda} = \beta^+ \) by CC(H) and \( \beta^\lambda = \beta^+ \in \mathcal{A} \), where \( \beta^\lambda < \gamma(\omega_1) \). We must find a completion \( \mathcal{V}^\beta \) of \( \mathcal{V}^\lambda \) which gives every regular \( \mathcal{V}^\lambda \) the cofinality \( \omega \). If \( \text{cf}(\beta) = \omega_1 \), we want \( \beta^+ \) to remain a cardinal in \( \mathcal{V}^\beta \), otherwise we want \( \beta^+ \) to have cofinality \( \omega_1 \) in \( \mathcal{V}^\beta \). We know, of course, that \( 2^{\mathcal{V}^\lambda} = \mathcal{V}^\lambda \), that every \( \gamma < \mathcal{V}^\lambda \) is collapsed to \( \omega_1 \) by some \( \mathcal{V}^\lambda \)-complete \( \mathcal{V}^\gamma \). Thus \( \mathcal{V}^{\beta^+} \) gives a \( \omega \)-complete \( \mathcal{V}^{\beta^+} \) in \( \mathcal{V}^{\beta^+} \).

Set \( M = L_{\beta^+}^{\mathcal{A}} \), where \( L_{\gamma}^{\mathcal{A}}[\mathcal{A}] = H_\gamma \), whenever \( \gamma \leq \beta \) and \( 2^{\gamma} = \gamma \). Set \( Q = \langle L_{\beta^+}^{\mathcal{A}}, \langle B_i(i \in \Lambda) \rangle \rangle \), \( N = \langle H_{\beta^+}, Q : M, <_m \rangle \rangle \) where \( < \) well-ordenes \( H_{\beta^+} \).
Amitating the definitions in the second succesor case, we set:

**Def** \( \Gamma_* = \text{the set of } \langle Q, B, C \rangle \in T, \)

- \( Q = \langle L_A, IB \rangle \) models Zermelo
- \( IB = \langle IB_i : i < \delta_i \rangle \) where \( \delta_i \leq \delta \) is a limit ordinal; \( IB_i \in Q \) is a complete BA in \( Q \), \( IB_i \) is forced over \( IB_i \) in \( Q \) whenever \( i < 1 < \delta_i \) and \( i \notin A \) in \( Q \), \( \sup d_i = \gamma \) where \( i < \delta_0 \)
- \( d_i = d IB_i \) is \( \omega \) for \( 0 < i < \delta_0 \) and \( IB_0 = \emptyset \)
- \( BC \cup IB_i \in T, B_i = B \cap IB_i \) is generic over \( Q \) for \( i < \delta_0 \)
- \( C \in \delta_0 \), \( \sup C = \delta_0 \), \( c^T (C) = \omega \)

*(Set \( c^T \) the \( n \)-th element of \( C \)).

**Def** For \( u = \langle Q_u, B_u, C_u \rangle \), \( \sigma = \langle Q_\sigma, B_\sigma, C_\sigma \rangle \)

\( u \in T \iff \langle u, \pi \rangle \in T \iff \pi : u \leq \pi^u \iff (\pi \vdash \pi^u \land \pi^u B_u \subseteq B_\sigma \land \pi^u C_u \subseteq C_\sigma) \)

**Def** \( u \equiv \pi \iff \forall \pi \pi : u \equiv \pi \)

**Fact 1** Let \( \sigma = \langle Q_\sigma, B_\sigma, C_\sigma \rangle \in T_* \), \( d = \vdash \omega Q_\sigma \). There is at most one pair \( \langle u, \pi \rangle \in T, \pi : u \leq \pi^u \) and \( d = d u \).
Proof of Fact 1.
Set \( Q^B = Q[B] = \bigcup_{i \leq \omega} Q[B_i] \) for \( \langle Q_i[B, C] \rangle \in \Gamma_x^R \). Let \( \pi : u <_x v \) s.t.
\[ i = c_u. \]
Then, since \( \pi_i >^u B_i^u \subset B_i^v \) for \( i < \omega^c u \), \( \pi \) extends to a unique
\[ \pi^*: \bigcup_{i \leq \omega} Q_i[B_i^u] \leftarrow Q_i[B_i^v] \] s.t.
\[ \pi^*_i(B_i^u) = B_i^v. \]
Set \( \pi^* = \bigcup \pi^*_i. \)
Then \( \pi^*: Q^u[B^u] \rightarrow Q^v[B^v] \)

Finally, Set \( f^u_c = \text{the } Q_i[B_i^u] \)-least
\[ f \text{ r.t. } f \text{ maps } x = \omega_i Q_i^u \text{ onto } c_d. \]
Let \( f_i^v \) be defined similarly for \( i < \omega^c u \).
Then \( \pi^*(f^u_c) = f_i^v \) and \( \pi^*(f_i^u(x)) = \pi(f_i^u(x)) = f_i^v(x) \) for \( i < \omega, \ z < \).

Hence, \( \text{rng}(\pi) = \{ f_i^v(\bar{z}) \mid i < \omega, \ z < \} \).
QED (Fact 1)

Hence:
Fact 2. Let \( u, v \in \Gamma_x^R \). There is at most
one \( \pi \) s.t. \( \pi : u <_x v \).

Def. Let \( u <_x v \). \( \pi_{u,v} = \text{that } \pi \text{ r.t. } \)
\[ \pi : u <_x v. \]
We easily get:

**Fact 3** Let \( u \triangleleft_\ast v \triangleleft_\ast w \). Then \( u \triangleleft_\ast w \) and

\[
\overline{\nu}_w = \overline{\nu}_w \cdot \overline{\nu}_v
\]

(i.e., \( \langle \overline{\nu}_v, \overline{\nu}_w \rangle \) is a commutative system where \( \overline{\nu}_w = \langle \overline{\nu}_u, \overline{\nu}_v \rangle \).

**Fact 4** \( \langle \overline{\nu}_v, \overline{\nu}_w \rangle \) is continuous i.e.,

if \( I \subseteq \overline{\nu}_v \), \( R \subseteq I^2 \) s.t. \( I \) is directed

(i.e., \( \forall u, v \in I \exists w \in I \ u \sim v \sim w \)) and

\( u \triangleleft_\ast v \triangleleft_\ast w \) whenever \( v \sim w \), then

there is a unique \( w \) s.t., \( u \triangleleft_\ast w \triangleleft_\ast w \)

for \( u \in I \) and \( \langle \overline{\nu}_w, \langle \overline{\nu}_u, \overline{\nu}_v \rangle \rangle \) is

the direct limit of \( \langle I, \langle \overline{\nu}_u, \overline{\nu}_v \rangle \rangle \).

(Note \( \overline{\nu}_w \) follows that \( \overline{\nu}_w = \overline{\nu}_w \cdot \overline{\nu}_v \)

for \( u \in I \).)

Hence:

**Fact 5** \( \{ \nu \in \nu \nmid u \triangleleft_\ast \nu \} \) is closed in \( d_{\nu + 1} \)
Def. \( M \) is a smooth model iff:

- \( M = L^A_\beta \) models \( \text{ZFC}^* \) or Zermelo
- \( 2^{\omega} = \omega_1 \) in \( M \) (in particular, \( \omega_1^M \) exists)

Def. \( Q \) is a smooth segment of \( M \) iff

\[
Q = \langle L^A_{\gamma}, R_1, \ldots, R_n \rangle \quad \text{where}
\]

- \( M = L^A_\beta \) is a smooth model
- \( \gamma > \omega \) is a limit cardinal in \( M \) and \( L_\gamma[A] = H^M_\gamma \)
- \( R_1, \ldots, R_n \in Q \)

Def. \( \Gamma \) is the set of \( \langle Q, M, B \rangle \) s.t.

- \( Q \) is a smooth segment of \( M \)
- \( \langle Q, B, C \rangle \in \Gamma \) for a \( C \) s.t.,
  \[ C \in M[B_i] \quad \text{for some} \ i < \gamma^A_\gamma. \]

(Nota. Each \( B_i \) in \( M[B_i] \) - generic over \( M \),
where \( Q = \langle L^A_\gamma, B \rangle \).

Def. For \( u, v \in \Gamma \), \( u = \langle Q_u, M_u, B^u \rangle \), \( v = \langle Q_v, M_v, B^v \rangle \), \( \pi : u < v \) iff

- \( \pi \upharpoonright Q_u : \langle Q_u, C, B^u \rangle \prec \langle Q_v, \overline{\pi}(C), B^v \rangle \)
  for a \( C \in M_u[B^u_i] \) for \( i < \omega \).
- There is \( M_{uv} \) s.t. \( \langle M_{uv}, \pi \rangle = \) the lift up of \( \langle M_u, \pi \upharpoonright Q_u \rangle \).
**Fact 6** Let \( u, v \in R \). There is at most one \( \pi \) s.t. \( \pi \cdot u < v \).

**Proof:**
Let \( \pi \cdot u < v \). We show that \( \pi \) is unique.
Let \( i \) be least s.t. \( v^u \in \omega \)-cofinal in \( M_u \). Let \( C = \text{the least } C \in M_u[B_i^u] \) s.t.
\( C \subseteq v^u = \sup C \) and \( Ap(C) = \omega \). Let \( i' = \pi(i) \).
Then \( \pi^u B_i^u \subseteq B_i^{v'} \), where \( \pi^u B_i^u = 1B_i^{v'} \). Hence
there is \( \pi' \) s.t. \( \pi' \cdot M_u[B_i^u] \subseteq M_v[B_i^{v'}] \)
and \( \pi^u B_i^u = 1B_i^{v'} \). Let \( C' = \pi'(C) \). Then
\( C', i' \) are defined from \( \langle Qu, M_u, B_i^u \rangle \) as
\( C, i \) were defined from \( \langle Qu, M_u, B_i^u \rangle \). Set:
\( u = \langle Qu, B_i^u, C \rangle \), \( v' = \langle Qu, B_i^{v'}, C' \rangle \). Then
\( u \) is defined from \( u + v \). \( v \) is defined from \( v' \) by the same def. But clearly
\( \pi \cdot Qu \cdot u \triangleq v' \); hence \( \pi \cdot Qu = \pi \cdot v' \).
Thus \( \pi \cdot Qu \) depends only on the pair \( u, v \).
But then \( u < v \) does \( \pi \), since \( \pi \) depends only on \( \pi \cdot Qu \). QED (Fact 6)

**Def** \( u < v \iff \forall \pi \quad \pi \cdot u \triangleq v \)

For \( u < v \) we set:
\( \pi uv = \text{that } \pi \cdot u \triangleq v \)

We easily get:
Fact 7. \(<\Gamma, \Pi> \) is a continuous commutative system, where \(\Pi = \{ \overline{a}u \mid u \leq v \} \).

Hence:

Fact 8. \(\{ \alpha u \mid u \leq v \} \) is closed in \(\alpha u \).

Fact 9. Let \(u, v \leq w\), \(\text{rng}(\overline{a}uw) \subset \text{rng}(\overline{a}vw)\).

Then \(u \leq v\) and \(\overline{a}uw \overline{a}vw = \overline{a}uw\).

Let \(Q, M, N\) be as defined above. Let \(IB = \{B_i \mid i < \gamma \}\) where \(\gamma = \lambda\) be the iteration up to \(\lambda\). Suppose \(B \subseteq \bigcup_i B_i\).

Then \(B_i = B \cap B_i\) \(\cap iB_i\) \(\leq \gamma\) generic for \(i < \lambda\).

Then \( < Q, M, B > \in \Gamma \).

As a preliminary to defining \(IB_\lambda\) we define:

Let $L$ be the language on $N$ with:

**Predicate $\in$, Constants $x, (x \in N), \beta$**

**Axiom** $\neg\text{FC}$, $\lambda\sigma (\nu \in \xi \leftrightarrow \forall \nu \sigma = \nu)$,

$H_\omega = H_\omega_1, \quad \langle \sigma, \mu, \beta \rangle \in \Gamma \quad \text{and; }$

(*) For each $\xi < \beta$ there are $u, \nu \in \Gamma, u \in H_\omega_1, \pi : u \downarrow \langle \sigma, \mu, \beta \rangle \land \xi \in \text{rng} (\pi) \lor \psi$

where $\psi = \{ \sup \pi \beta = \beta \text{ if } \beta \text{ is regular, }$

$u = u \text{ if not} \}$

Note $\sup \pi \beta = \beta$ means the same as

$\mu \downarrow \langle \sigma, \mu, \beta \rangle = \mu$ recalling

$\langle \mu, \pi \rangle = \text{ the lift up of } \langle \mu, \pi \rangle \langle \sigma_0 \rangle$

for $u \downarrow \varphi$.

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$\vdash B \in IB_{\xi_i}$ - generic for some $i < \xi_0$, we can,

in $V[B]$, define: $IB/B = \langle IB_{i+1}^B, B \mid j < \xi_0 - i \rangle$

$Q^B = \langle Q[B], B \rangle = \langle L^A, IB/B \rangle$

$M^B = \langle M[B], B \rangle = L^A_{\beta}$

$N^B = \langle N[B], B \rangle = \langle H_{\beta}, V[B], \in, Q^B, M^B, \prec, \ldots \rangle$

where $\prec$ well order $N^B$. 

We can then define a language $L_B$ over $N_B$ as before in $V[B]$ with $Q^B, M^B$ in place of $Q, M$.

If $B = \bigcup B_i$, s.t. $B_i = B \cap IB_i \cup IB_i$, -

- generic for $i < \delta_0$, we set:

$B/B_i^c = \{ b/B_i^c | b \in B \}$, where $b \to b/B_i^c$ is the canonical projection of $\bigcup B_i$ onto $\bigcup B_i^c$.

(We shall follow the convention in §3 of taking that $M = \langle 1M_1, e, 2^{\infty} \rangle$ is a solid model of $L$: $M$ is solid, $N \subseteq \omega$-core $(M)$, and $M$ becomes a model of $L$ if we interpret $\exists \tilde{x}$ by $\exists \tilde{x}$ for $\tilde{x} \in N$.)

We then get:

\underline{Lemma 0}

(a) If $M = \langle 1M_1, e, 2^{\infty} \rangle$ is a solid model of $L$, then $B = \bigcup B_i$ and $B_i = B \cap IB_i \cup IB_i$ - generic over $V$ for $i < \delta_0$.

(b) Let $B = \bigcup B_i$, s.t. $B_i = B \cap IB_i \cup IB_i$ - generic for $i < \delta_0$. Let $\delta_0 < \delta_0$. Then $M = \langle 1M_1, e, 2^{\infty} \rangle$ is a solid model of $L$ if $M_0 = \langle 1M_1, e, 2^{\infty}, B/B_i^c \rangle$ is a solid model of $L_{B_i}$.

The proof is straightforward.
Lemma 1: \( L \) is consistent.

**Proof.**

Since \( \delta \), is an \( \omega \)-point, there is an \( i < \delta \) s.t. \( L \models \delta \). By Lemma 0 it suffices to show that if \( B \) is \( 1 \)-generic, then \( L[B] \) is consistent (this being a statement in \( V[B] \)). Hence we may assume \( \delta \), i.e., that \( \delta \) is \( \omega \)-cofinal in \( V \). Let \( C = \{ c_i : i < \omega \} \)
where \( \langle c_i : i < \omega \rangle \) is monotone and cofinal in \( \delta \). Set \( B' = \) the inverse limit of \( \langle B^\alpha \rangle_{\alpha < \omega} \).

Then \( B' \) is subcomplete by the \( \omega \)-one of the iteration lemma for subcomplete algebras. Let \( B \) be \( B' \)-generic. Set \( B_{\delta} = B \cap B' \) for \( \delta < \delta \). In \( V[B] \) let \( \sigma: \bar{N}[B] \rightarrow \bar{N}[B] \), \( \sigma(B) = B' \), \( \sigma(C) = C \),
where \( \bar{N} \) is countable and transitive.

Let \( \sigma(B) = B \), Clearly we have \( B \) and \( \sigma(B) = B \).

Let \( \sigma(B) = B \), Clearly we have \( B \) and \( \sigma(B) = B \).

1. \( \sigma(\bar{Q}) : \langle \bar{Q}, B, \bar{C} \rangle \triangleleft \langle Q, B', C \rangle \)

Set \( \langle \bar{N}, \bar{\sigma} \rangle = \) the lift up of \( \langle \bar{N}, \sigma(B) \rangle \)

Then \( \bar{\sigma}: \bar{N} \rightarrow \bar{N} \) cofinally and there is a unique \( \bar{k}: \bar{N} \rightarrow \bar{N} \) s.t. \( \bar{k}\bar{\sigma} = \bar{id} \) and \( \bar{k}\bar{\sigma} = \bar{id} \). Let \( \bar{Z} \) be defined.
over $\tilde{\mathcal{L}}$ the way $\mathcal{L}$ was defined over $\mathcal{N}$ 
(with $\tilde{\mathcal{M}} = k^{-1}(M)$ taking the place of $M$). 
Then $k : \langle \tilde{\mathcal{N}}, \tilde{\mathcal{L}} \rangle \prec \langle \mathcal{N}, \mathcal{L} \rangle$ and it suffices to prove: 
**Claim:** $\tilde{\mathcal{L}}$ is consistent.

We show, in fact, that $\langle H_{\omega_2}, B \rangle$ 
models $\tilde{\mathcal{L}}$. The proof is a virtual repetition of the corresponding step 
in the proof of §3 Lemma 1. The details are left to the reader.

Q.E.D (Lemma 1)

The same proof obviously yields:

**Cor 1.1** Let $B$ be $13e$-generic ($i < \delta_0$). 
Then $\mathcal{L}_B$ is consistent.

Exactly as in §3 we get:

**Lemma 2** Let $\mathcal{M}$ be a solid model of $\mathcal{L}$. 
Let $\langle A_n \mid n < \omega \rangle \in \mathcal{M}$ and $A_n \subseteq M$ for $n < \omega$. 
There is $\nu = \langle \delta_n, M_n, B_n \rangle \in \mathcal{M} \cap H_{\omega_1}$ such that 

- $\nu \triangleleft \langle Q, M, B \rangle$
- $\bar{\pi} : \langle M_n, \bar{A}_m \rangle \prec \langle M, A_n \rangle$ for $m < \omega$

where $\bar{\pi} = \bar{u}, \langle Q, M, B \rangle$, $\bar{A}_m = \bar{\pi}^{-1} \ pledged A_n$. 


\[ \sup \mathbb{P}_p \mathcal{O}_{mp} = \beta \text{ if } \beta \text{ is regular.} \]

\[ \text{Let } \mathbb{B} \text{ be } \mathbb{B}_1 \text{ in generic } (\triangleleft \delta). \]

\[ \text{We now define the conditions } \mathbb{P} = \mathbb{P}_{\mathcal{B}}. \]

\[ \pi_{\mathcal{P}} = \langle \pi_{p}, \mathbb{M}, \mathbb{B} \rangle \in \mathcal{P} \cap \mathcal{N}. \]

\[ \mathbb{P} = \mathbb{P}_{\mathcal{B}} \text{ is a countable set of pairs } \langle \alpha, \gamma \rangle. \]

\[ \mathbb{P} = \langle \mathcal{M}_{\mathcal{B}}, \mathbb{B} \rangle \in \mathcal{P} \cap \mathcal{N}. \]
Exactly as before:

**Lemma 3.1** Let \( p, q \in IP \). Then \( p \leq q \) if

- \( R^q \subseteq R^p \)
- \( L(p) \models \forall \overline{q} (\overline{r} \in \text{rng}(\overline{q}) \rightarrow \exists \overline{r} (\overline{p} \in \text{rng}(\overline{r}))) \)

Define \( \overline{p} \models \overline{q} \) for \( p \leq q \).

**Lemma 3.2** Let \( p \in IP \). Then

- \( (FP)^{-1} \) is a function
- \( H \) \( R^p \) is closed under set difference, then \( F^p : D^p \leftrightarrow R^p \)
- \( \overline{p} \models \overline{q} \) if \( F^p \models \overline{M} \) is injective into \( M \).

**Lemma 3.3** \( IP \neq \emptyset \). Moreover, \( \forall q \models \text{if } p \in IP \) and \( L(p) \cup L(q) \) is consistent, there is \( r \leq p, q \).

Moreover, for any countable \( RC \subseteq (M) \) we can choose \( r \) so that \( RC \subseteq R^r \).

**Corollary 3.4** \( p, q \) are compatible in \( IP \) if and only if \( L(p) \cup L(q) \) is consistent.

**Corollary 3.5** Let \( p \in IP \). Let \( RC \subseteq (M) \) be countable. There is \( q \leq p \) so that \( RC \subseteq R^q \).

**Corollary 3.6** Let \( p \in IP \). Let \( u \subseteq M \) be countable. There is \( q \leq p \) so that \( u \subseteq \text{rng}(\overline{p}) \).
Lemma 3.7 Let \( p \in \text{IP}, u \leq M_p, u \text{ finite.} \)
There \( u \leq p \) s.t. \( q = p, v \in 
\text{any} (\Pi G). \)

Lemma 3.8 Let \( G \) be \( \text{IP} \)-generic. Then
\( (a) \left\langle \left\langle Q_p, M_p, 1_{B_p} \right\rangle \mid p \in G \right\rangle, \left\langle \pi_p^G \mid q \leq p \text{ in } G \right\rangle \)
in a directed system with limit:
\( \langle Q, M, B^G \rangle, \left\langle \pi_p^G \mid p \in G \right\rangle. \)
Moreover: \( \pi_p^G = \bigcup \{ \pi_q^G \mid q \leq p, \pi_q^G = 1_{B_q}, q \in G \} \)
\( (b) p \in G, \langle Q, M, B^G \rangle \text{ with } \pi_p^G = \pi_p^G, \langle Q, M, B^G \rangle \)
\( (c) \pi_p^G : \langle M_p, a \rangle < \langle M, a \rangle \text{ for } \langle a, \bar{a} \rangle \in E_p. \)

The proofs are exactly as in \( \S \overline{3}, \)
Let $p \in \text{IP}$. Let $\overline{B}$ be s.t.
- $\overline{B} \subseteq_{\pi < \delta_0^p} B_i^p$ and $\overline{B}_i = \overline{B} \cap B_i^p \cap B_i^p$ — generic over $\mathbb{Q}_p$ for $i < \delta_0^p$
- $\mathbb{Q}_p[\overline{B}] = \mathbb{Q}_p[B_i^p]$

Then $p' \in \text{IP}$ where $p' = 〈\mathbb{Q}_p, M_p, \overline{B} ›$, $p'_1 = p'_1$.

**Proof (sketch)**

Let $M' = 〈M'_1, B' ›$ be a solvible model of $L(p)$.

Set $B = \bigcup_{i < \delta_0^p} \pi_i^p(\overline{B})$. Then $\mathbb{Q}[B] = \mathbb{Q}[B''']$ and $B \subseteq_{\pi < \delta_0} B_i$ s.t. $B_i = B \cap B_i^p \cap B_i^p$ — generic over $\mathbb{Q}$ (hence over $V$) for $i < \delta_0$.

Set $M' = 〈M'_1, B ›$.

**Claim**: $M'$ models $L(p')$.

We first show that $M'$ models $L$. All axioms are trivial except (4). We verify (4),

Let $i < \delta_0^p$ s.t. $\pi_i^p \delta_0^p$ is $\omega$-cofinal.

Then, since $\pi_i^p(\overline{B}_i^p) \subseteq B_i$, $\pi_i^p$ extend to $\pi_i^*: M_i^p(\overline{B}_i^p) \subseteq M_i^p(\overline{B}_i^p)$, and extend $\pi_i^*|B_i^p = \pi_i^*|B_i^p$. Let $C \in M_i^p \overline{B}_i^p$, s.t.

$C \subseteq \delta_0^p = \pi_i^*(C)$, $\pi_i^*(C) = \omega$. Let $\pi_i^*(C) = C$. At follows exactly that

$\pi_i^*(\mathbb{Q}_p) 〈\mathbb{Q}_p, \overline{B}, C › \Delta_x 〈\mathbb{Q}, B, C ›$.
But then \( \pi_p : \langle Q_p, M_p, B \rangle \preceq \langle Q_1, M_1, B \rangle \).

Now let \( \pi_1 \). By Lemma 2 there is \( \pi_1 : u \preceq \langle Q_1, M_1, B' \rangle \) s.t. \( u \in H_{\pi_1} \), \( \pi_1 \in M \) and \( \text{rng}(\pi) \cup \{3\} \subset \text{rng}(\pi_1) \).

Set \( \pi_0 = \pi_1 \cdot \pi_p \). At follow that

\[
\pi_0 : p_0 \preceq u, \quad \pi_1 : u \preceq \langle Q_1, M_1, B' \rangle
\]

and \( \pi_0 = \pi_1 \cdot \pi_0 \).

At we set \( B' = \bigcup_{i \leq \delta_0} \pi_0(B_i) = \bigcup_{i \leq \delta_0} \pi_1^{-1}(B_i) \), we have \( \pi_0 : p_0 \preceq u' \wedge \pi_1 : u' \preceq \langle Q_1, M_1, B' \rangle \), where \( u' = \langle Q_1, M_1, B' \rangle \). (Recall that \( p_0 : \langle Q_p, M_p, B \rangle \). But \( 3 \in \text{rng}(\pi_1) \), which proves (1). Hence \( M' = L \). Since \( \pi_0 : p_0 \preceq \langle Q_1, M, B \rangle \) and \( p_1 = p_0 \), we see that \( M \equiv \alpha \models \mathcal{L}(p') \).

Q.E.D (4.1)

At turn out, however, that the

\[ Q[B] = Q[B'] \]

is in restrictive for our purposes. We formulate an other rem

viability principle which evades the restriction.
Lemma 4.2 Let $\theta$ be big enough to verify

The product of $1B_i$ over $1B_i$ for all $i' < i < \delta_0$

s.t. $i', i \in Ac$. Suppose moreover that

$\theta > 2^{\beta}$. Let $N^* = \langle H_\theta, N, <, 1P, 1B, m \rangle$.

Let $p$ conform to $N^*$. Let $N^{*'} = \langle H^*, N^*, <, 1P, 1B, m \rangle$.

(1+en 1B = 1B^p.) Let $B' \subseteq _{i'} 1B_0$ s.t. $B'_0 = \overline{B}_0 \cap 1B_0$ is generic over $N$ for

$i < \delta_0^p$. Then $p' \in P'$ where

$p'_0 = \langle \bar{q}_p, M_p, \overline{B}' \rangle$ and $p' = p'$.

Proof

Let $M_T = \langle H_T, B^M \rangle$ be a notid model of

$L(p)$. Set $1B = B^M$. Then $\overline{\tau} = \overline{\tau}^i$ extends

to $\overline{\tau}^* : N^* \to N^*$ s.t. $\overline{\tau}^* \cap F^p = \overline{\tau}^*$. We

assume that $M_T$ (and hence $\overline{\tau}^*$) live in a generic

extension of $T$. Note that $B'_0$ is 1B$_0$ - generic over $\varphi$, hence

over $T$ for $i < \delta_0^p$. Let $\langle \delta^i \mid i < \omega \rangle$ be

a monotone cofinal sequence in $\delta_0^p = \delta_0^0$ s.t. $\delta^i \in Ac$ in $N^*$ for $i < \omega$.

Set $\delta^i = \overline{\tau}^i(\delta^i)$. Then $\langle \delta^i \mid i < \omega \rangle$ is

monotone and cofinal in $\delta_0$ and

$\delta^i \notin Ac$ for $i < \omega$. 
But then \( 1B_{\delta_i} \) is proved over \( 1B_{\delta_h} \) for \( h < i < \omega \), where \( B_{\delta_i} \) is \( \pi^* \)-conforming.

Thus we can successively form 
\( B_{\delta_i} \) \( i.t. \) \( B_{\delta_i} \) is \( 1B_{\delta_i} \) - generic,
\( \pi^* " B_{\delta_i}^i \subset B_{\delta_i}^i \), and \( \mathcal{U}[B_{\delta_i}^i] = \mathcal{U}[B_{\delta_i}^i] \).

Set \( B' = \bigcup B_{\delta_i}^i \). Then \( B_3' = B_3' \cap B_3' \)
is \( B_3' \) - generic for \( 3 < \delta_0 \). Let
\( \bar{C} \in M_{P} [B_{\delta_0}^i] \) \( i.t. \) \( \bar{C} \subset \delta_0 = \sup \bar{C} \)
and \( \sup \bar{C} = \omega \), \( \pi^* \) extends uniquely to 
\( \tilde{\pi} : \mathcal{N}^* [B_{\delta_0}^i] \subset \mathcal{N}^* [B_{\pi(i_0)}^i] \).
\( \tilde{\pi}(B_{\delta_0}^i) = B_{\pi(i_0)}^i \), so \( \pi^* " B_{\delta_0}^i \subset B_{\pi(i_0)}^i \).

Let \( \tilde{\pi}(\bar{C}) = C \). Then \( \pi^* \bar{C} = C \).

At follows easily that:
(1) \( \pi \upharpoonright \bar{Q} : \langle Q, \bar{B}, \bar{C} \rangle \triangleleft \mathcal{K} \langle Q, B, C \rangle \),
Hence:
(2) \( \pi : \langle Q, M_p, \bar{B} \rangle \triangleleft \langle Q, M, B \rangle \),
where \( p_0' = \langle Q, M_p, \bar{B} \rangle \).

Now set \( \mathcal{L} = \langle \mathcal{L}, B \rangle \). At
follow as in Lemma 4.1 that \( \mathcal{L} \models ZF(p') \). QED (4.2)
By a virtual repetition of the proof of §3 Lemma 3.9 using 4.2, we get:

**Lemma 4.3**  IP adds no reals.

But then, as before:

**Lemma 4.4**  Let $\theta \geq 2^\lambda$ be regular. If $G \in \text{IP} \text{- generic}$ and $B = B^G$, then $\langle H^\theta, V[G], B^G \rangle$ is also $\lambda$-minimal.

**Def**  Let $B = \bigcup_{0 < \delta_0} B_i$, s.t. $B_i = B \cap B^G$ is $\text{IP} \text{- generic}$ for $i < \delta_0$.

$G^B = \text{the set of } p \in \text{IP such that}$
\[ p \not\in \langle \theta, \lambda, B \rangle \text{ and, letting } \theta^i = \theta^i, \text{ we have:} \]
\[ \Pi^i : \langle M_p, \alpha^i \rangle \subseteq \langle M, \alpha \rangle \text{ for all } \langle \alpha, \alpha^i \rangle \in F_p. \]

Exactly as in §3 Lemma 3.11 we have:

**Lemma 4.5**  Let $G$ be IP - generic.

Then $G = G^B$ where $B = B^G$.

Hence as in §3 Lemma 3.12:

**Lemma 4.6**  Let $G$ be IP - generic.

Then $B \subseteq \omega_1$ in $V[G]$. 
Def. We define a homomorphism of \( \mathcal{C}_{\mathcal{B}} \) into \( \mathcal{B}(\mathbb{P}) \) by:

\[
k(b) = \left[ \left[ b' \in \mathcal{B}^\mathbb{P} \right] \right]_{\mathbb{P}}
\]

**Lemma 4.7** \( k \) is injective

**Proof.**

Suppose not. Then \( k(b) = 0 \) for a \( b \in \mathcal{C}_{\mathcal{B}} \), s.t. \( b \neq 0 \). But \( \mathcal{L} + b \in \mathcal{B} \) is consistent. (To see this choose \( B \) s.t. \( b \in B \) in the proof of Lemma 1.) Let \( M \) be a solid model of \( \mathcal{L} + b \in \mathcal{B} \). Then \( b \in \mathcal{B}^M \). By Lemma 2 there

\[ u \in \Gamma \setminus \text{Homo}_u \text{ s.t. } u \triangleleft \langle \mathfrak{m}, b \rangle \]

and \( \tilde{v}(b) = b \) for some \( \tilde{v} \) where \( \tilde{v} = \pi u, \langle \mathfrak{m}, b \rangle \). Define \( p \) by:

\[ p = u, \quad p = \left\{ \langle b, \tilde{v} \rangle \right\} \]

Then \( M \) models \( \mathcal{L}(p) \). Hence \( p \in \mathbb{P} \). But if \( G \triangleright p \triangleright \mathbb{P} \)-generic, then \( b \in \mathcal{B}^G \), hence \( c \notin [\mathbb{P}]/G \subset \left[ \left[ b \in \mathcal{B}^G \right] \right] = 0(b) \).

**Contr.** QED (4.7)
We now show that $BA(P)$ is, in fact, generated by $\{k(b) \mid b \in \omega_s \} \in \mathcal{B}_c^R$.

As a preliminary we prove:

**Lemma 4.8** Let $G$ be $P$-generic, $B = B^G$.

Then $G \in V[B]$.

**Proof.**

Let $u \in H_{\omega_1} \cap R$, $\pi : u \leq \langle Q, M, B \rangle$.

Then $u \in V[B]$.

**Proof.**

Let $\xi$ be least s.t., $|\mathcal{B}_E^R|^{(\leq \xi)} = \omega$.

Thus $\exists \xi \mathcal{B}_E^R(\xi)$.

$\exists u \in M^u \cap R$, $\pi : M^u \leq M$, $\pi(\mathcal{B}_E^R) = B^u$.

Let $\pi(u) = \bar{u}$. Let $\bar{u} \in M^u$, $\bar{u} \in M^u = M^R(\bar{c})$.

$\text{otp}(\bar{c}) = \omega$. $\pi$ extends uniquely to

$\pi^* : M^u[B^u] \to M^u[B^u]$, s.t., $\pi^*(B^u) = B^u$.

Thus $\pi^* : B^u \to B^u$. Let $C = \pi^*(\bar{c})$.

Then $C = \pi^*(\bar{c})$ and

$\pi^*(\langle Q, u, B^u, \bar{c} \rangle) \leq \langle Q, B, C \rangle$,

where $\langle Q, B, C \rangle \in V[B]$. Hence $\text{rng}(\pi^*) = \text{the smallest } X \leq \langle Q, B \rangle$.

s.t. $\omega_1^{Q_u} \subseteq C \subseteq X$.

Hence $\pi^*(\langle Q_u \rangle) \subseteq V[B]$. But then
\[ \pi \in V[B], \text{ since, letting } \tilde{\pi}; M_\mu < \tilde{\pi} \text{ cofinally we have:} \]

\[ \langle \tilde{\pi}, \pi \rangle = \text{the lift up of } \langle M_\mu, \pi \setminus \lambda_\mu \rangle. \quad \text{QED (4.1)} \]

But then by Lemma 4.5 we have

\[ p \in G \iff V[B] \models p \in G^B. \]

Hence \( G \in V[B] \). \quad \text{QED (4.8)}

As a corollary of the proof:

\[ \text{Lemma 4.9} \quad \text{BA}(IP) \text{ is generated by } \sum k(b) | b \in (\omega, IB_3^B). \]

**Proof.**

Let \( \dot{B} \in V^{IP} \), i.e. \( \models IP \)

Standard methods show:

\[ \left[ \phi(\check{x}_1, \ldots, \check{x}_n, \check{B}) \right]_{IP} \in IB^*, \]

for all \( \check{x}_1, \ldots, \check{x}_n \in V \) and all \( \exists F \in V \) - formulae \( \phi \), where \( IB^* \) is the complete subalgebra of \( \text{BA}(IP) \) generated by \( \{ k(b) | b \in (\omega, IB_3^B) \} \).

By the proof of Lemma 4.8, however:
\[ \text{Lemma 4.10} \quad \text{BA}(\mathbb{P}) \leq \beta_\lambda^+ . \]

proof:

At $\beta \neq \beta_\lambda$, then $\beta_\lambda = \beta^+ = 2^\beta$, since GCH then holds below $\lambda$. But $\mathbb{P} \leq 2^\beta$ and hence the result follows. Now let $\beta = \beta_\lambda$. Then $cf(\beta) = \omega_1$. At suffices to show:

Sublemma 4.10.1 BA(\mathbb{P}) has a dense subset of size $\beta$.

proof (sketch)

The proof is virtually identical to that of §3 Sublemma 3.15.1. Set $H = H_{\beta^+}$. Then $\langle H[G], B \rangle$ models $L$ whenever $G \in \mathbb{P}$ - generic and $B = B^G$. We can give every $L$ -sentence $\Psi$ an interpretation $[\Psi]_G \in \text{BA}(\mathbb{P})$ in $H[\mathbb{P}]$, interpreting $\dot{B}$ by $\dot{B}$, where $\dot{B} \in H[\mathbb{P}]$, $H[\mathbb{P}] \models \dot{B} = B^G$, $G$ being the canonical generic name, $\dot{X}$ $\dot{\omega}$
interpret def by \( x \). It then suffices to show:

**Claim:** For each \( p \in \mathbb{I}^* \) there is an \( L \)-statement \( \exists y \) with \( \| y \| \neq 0 \) and \( \| y \| \leq [p] \) in \( \text{BA}(\mathbb{I}^*) \).

The proof is an almost literal repetition of the corresponding step in the proof of §3 Sublemma 3.15, 1. We leave the details to the reader. \( \Box \) (4.10)

**Lemma 4.11** Let \( G \) be \( \mathbb{I}^* \)-generic. Then \( c(\beta) = \omega \) in \( V[G] \) whenever \( \beta \in [\delta^*_\lambda, \beta_\lambda] \) is regular in \( V \).

**Proof:**
Let \( \bar{\beta} = \Pi^\mathbb{I}^*_{p} (\bar{\beta}) \). Then \( \sup \Pi^\mathbb{I}^*_{p} (\bar{\beta}) = \bar{\beta} \), since \( \Pi^\mathbb{I}^*_{p} : \mathbb{I}^*_{p} \rightarrow \check{Z} \) is \( \check{\alpha} \)-cofinal, where \( \Pi^\mathbb{I}^*_{p} (\bar{\beta}) = \delta^*_\lambda = \delta^*_\lambda \). \( \Box \) (4.11)

**Lemma 4.12** Let \( G \) be \( \mathbb{I}^* \)-generic. Then

(a) \( \beta^*_\lambda = \omega_1 = c^+ (\beta_\lambda) \) in \( V[G] \)
(b) \( \beta^*_\lambda = \omega_2 \) in \( V[G] \)

**Proof:**
\( \bar{\beta} \leq \omega_1 \) since \( \beta \subseteq \bigcup \mathbb{I}^* (\Pi^\mathbb{I}^*_p) \), where \( \Pi^\mathbb{I}^*_p = \Pi^\mathbb{I}^*_p (\check{\omega}_1, \check{\mathbb{I}^*}, \check{\beta}^*_\lambda) \) depends only on \( p_0 \in \mathbb{I}^* \).
If $\beta = \beta_\lambda$, then $\text{cf}(\beta) = \omega_1$ in $V$, hence in $V[G]$. Moreover, $\beta^+$ remains a cardinal in $V[G]$, since $\text{BA}(\kappa)$ has a dense subset of size $\beta$.

Now let $\beta_\lambda^+ = \beta^+$. Then $\text{cf}(\beta) = \omega_1$ in $V[G]$, since either $\beta$ is regular or $\text{cf}(\beta) = \omega$ in $V$.

We also know $\beta^+ = 2^\beta$, since GCH holds below $\kappa$. By [LF] §4 Lemma 4.1 (Fact 11 of §3 in this paper) we conclude that $\beta_\lambda^+ = \omega_1$ in $V[G]$. But $\beta_\lambda^+$ remains a cardinal in $V[G]$ since $\overline{\beta}^+ \leq \beta^+ = \beta_\lambda^+$.

Hence $\text{cf}(\beta_\lambda^+) = \omega_1$ in $V[G]$, since otherwise another application of §3 Fact 11 would give $\beta_\lambda^+ \leq \omega_1$ in $V[G]$.

QED (Lemma 4.12)
We choose $\sigma$, $IB_{\lambda}$ (recall $\lambda = \lambda_{0}$) set $\nu$.

$\sigma : BA(\Gamma) \leftrightarrow IB_{\lambda}$

$\Gamma \uparrow$

$\bigcup_{i} IB_{i}$

and $IB_{\lambda} \in H_{\beta+}$. $IB_{\lambda}$ is the completion of $\bigcup_{i} IB_{i}$ which we sought. However, we must still verify many of its properties.

Now let $\xi_{\lambda} < \delta_{0}$ + let $B$ be $IB_{\xi_{\lambda}}$ - generically.

Set: $IB^{+} = IB^{\bigcup IB_{\lambda}} = <IB_{\lambda} | \nu \leq \lambda >$.

We shall have to prove that the properties we have shown to hold of $IB^{+}$ also hold of $IB/B = <B_{\xi_{\lambda}} | B | \nu \leq \delta_{0} - \xi_{\lambda} >$ in $V[B]$. We know by the induction hypothesis that the salient properties of $IB$ hold of $IB/B = <B_{\xi_{\lambda}} | B | \nu < \delta_{0} - \xi_{\lambda} >$ in
But in $V[B]$ we can use the language $L_B$ over $N_B$ to construct conditions $IP_B$ exactly as $IP$ was constructed from $L$. At we have established properties of $IB, IP$, we can repeat our proofs to verify the same properties of $IB_B, IP_B$ in $V[B]$. Now let

\[ k_B : (I^n - IB, IP) \rightarrow BA(IP_B) \]

be defined in $V[B]$ as $k$ was defined in $V$. We then define $\sigma^B, IB_B^B, \lambda^{-i_0}$ in $V[B]$ like $\sigma, IB, \lambda$ in $V$. We have:

\[ BA(IP_B) \xrightarrow{\sigma_B} IB_B^B \]

At suffice to prove

Lemma 5.1: There is $\mu : BA(IP)/B \rightleftharpoons IB_B^B$

such $\mu(b/B) = b/B$ for $b \in \omega BI_B$.

Proof

This is equivalent to...
Claim: There is $\delta : BA(IP)/k(B) \xrightarrow{\sim} BA(1P_B)$
\[ \text{s.t. } \delta(k(b)/k(B)) = k_B(B/B) \text{ for } b \in \bigcup i_B B_i, \]
where $k(B) = k^B(B)$. 
We can then set $\mu = \sigma^B \circ \delta$.

\[ BA(IP)/k(B) \xrightarrow{\delta} BA(1P_B) \xrightarrow{\sigma^B} B^B_{\lambda,\mu} \]

The proof stretches over several lemmas and closely follows that of
\[ \S 3 \text{ Cor. 6.9.1.} \] $IP_B$ plays the same role as
the $IP_B$ of that paper. The role of $IP'$ is
played by:
\[ IP' = \{ p \in IP \mid \exists i_0 \in \pi_1 IP \} \]

Notice that $IP'$ is dense in $IP$.

As before, we assign to each $p \in IP'$ a
potential element $p^* \circ \pi B_B$.
\[ p^* = \langle Q_p, \beta^p, B_p^\beta \rangle, \quad p^* = p_4, \]
where $\pi IP(p) = \pi$ and
\[ B_p^\beta /B = \{ b/B \mid b \in B_p^\beta \} \text{ whenever } B \in B^p_{\lambda,\mu} \text{ generic over } M_p. \]
Lemma 5.1.1 Let $B$ be $\mathcal{B}_0$-generic. Let $p, q \in \mathcal{P}$, $[p] / k(B)$, $[q] / k(B)$ are compatible in $\mathcal{B}A(\mathcal{P}) / k(B)$, i.e. $p^*, q^*$ are compatible in $\mathcal{I}P_B$.

Proof:

(\rightarrow) Suppose not. Then $[p] \cap [q] / k(B) = 0$. Hence $[p] \cap [q] \cap k(b) = 0$ for $a, b \in B$. Let $\mathcal{M} = \langle \mathcal{W}^*, B^* \rangle$ be a solid model of $L_B(p^*) \cup L_B(q^*)$. Then $\mathcal{M} = \langle \mathcal{W}^*, B' \rangle$, a solid model of $L(p) \cup L(q)$ where $B' = B^* B^* = \{b \in \mathcal{B}_0^* \mid b/B \in B^*, \mathcal{L}_B(p) \cup \mathcal{L}_B(q) \}$, by Lemma 0. Hence there is $\pi \leq \rho$, $\pi \neq A$.

Then $\mathcal{M} = \mathcal{L}(\pi)$ and $b \in \operatorname{rng}(\pi^2)$. Let $\pi^2(b) = b$. Then $b \in \mathcal{B}^2$. Hence $\pi \parallel b \in B^*$. Hence $[\pi] \cap \{b \in B^* \} = k(b)$, where $[\pi] \cap k(b) = 0$.

(\leftarrow) Suppose not. Then $L_B(p^*) \cup L_B(q^*)$ is inconsistent. Hence there is $b \in B$ s.t.

(1) $b \models (L_B(p^*) \cup L_B(q^*)$ is inconsistent).

Let $\mathcal{G}$ be $\mathcal{B}A(\mathcal{P}) / k(B)$-generic s.t. $[p] / k(B)$, $[q] / k(B) \in \mathcal{G}$. Set $G = \{p \in \mathcal{P} \mid [p] / k(B) \in \mathcal{G}\}$ Then $G$ is $\mathcal{I}P$-generic with $p, q \in G$ and $B = B_0$. Let $\pi \in G$ s.t. $\pi \leq p, q$ and $b \in \operatorname{rng}(\pi^2)$. Let $\pi^2(b) = b$. 


Then $\overline{b} \in B^x$. Let $M' = \langle 10^1, B' \rangle$ be a solid model of $L(\beta)$. Then $M' = \langle 10^1, B'/B \rangle$ is a solid model of $L_B(p^*) \cup L_B(q^*)$, where $b \in B'$ and $B' \in B_0$ -- generic, contradicting (1). Contradiction! QED (5.1.1)

As a corollary:

Corollary 5.1.2 \( [p^*] / k(B) \neq 0 \iff p^* \in IP_B \)

for \( p \in IP \) and \( B_0 -- \text{generic} B_0 \).

Lemma 5.1.3 \( \{ [p^*] \mid [p] / k(B) \neq 0 \} \)

is dense in $BA(IP_B)$.

Proof: (We follow the proof of 5.3 Lemma 6.9.) By the same argument as before, using Lemma 1.2, $\exists p^* : [p^*] / k(B) \neq 0$. \( \exists p^* : [p^*] / k(B) \neq 0 \) \( \exists p^* : [p^*] / k(B) \neq 0 \) in the same sense as the set $IP$ of $p \in IP_B$ such that $p \models \forall x \exists y (\pi p)$ and $E \subseteq V$. We show:

Claim: Let $q \in IP_B$. Then if $p \in IP \cap q^*$, \( [p] \subseteq [q^*] \).

Let $A = \langle a_i : i < \omega \rangle \in V[B]$ an unnameable. Let $\Delta \subseteq \beta \cap A \in P_B$ as before. Let $D \subseteq \beta \cap A$ in $P_B$ as before. Let $D = D_B$ and set, as before,

\[
E = \{ \langle \nu, b \rangle : b \in B_0, \nu \in \beta \land b \models V \}
\]

...
Then $A \in \langle M^B, E \rangle$ - definable as before, in $\mathcal{V}[B]$ define:

$N^* = \langle H_\theta, N^B, M^B, O^B, B, E, A, \ldots \rangle$, where $\theta > (2^\kappa)^+$

is a cardinal and $< \text{ well ordered } N^*$. Let $P \leq \mathcal{Q}$ s.t. $P$ conforms to $N^*$. Again set $\bar{N}^* = \bar{N}^*(P, N^*) = \langle \bar{A}, \bar{N}, \bar{M}, \bar{O}, \bar{B}, \bar{E}, \bar{A}, \ldots \rangle$.

Then $\bar{M} = M^P$, $\bar{O} = Q^P$, $\bar{B} = B^P$ and $\bar{A} \in \langle \bar{M}, \bar{E} \rangle$ - definable by the same definition.

Form $P', P''$ by:

$P'_0 = P_0 \upharpoonright P'_1 = \{ \langle a, \bar{a} \rangle \in E^P \mid a \in R^{=\kappa} \}$

$P''_0 = P_0 \upharpoonright P''_1 = \{ \langle E, \bar{E} \rangle \}$, where $\langle E, \bar{E} \rangle \in E^P$.

Then $P' \leq \mathcal{Q}$ in $\mathbb{P}_B$ and $P'' \in \mathbb{P}_B$. We show:

Claim $[P''] \subset [P']$ in $\mathcal{B}(\mathbb{P}_B)$.

Let $G \in \mathbb{P}''$ be $\mathbb{P}_B$ - generic. It suffices to show:

Claim $P' \in G$.

Since $P' = P''$ we have:

$\overline{\sigma} : P'_0 \not\in \langle Q^B, M^B, B^G \rangle$ while $\overline{\sigma = \pi P''}$.\!

We need only show:

$\overline{\pi} : \langle M^P, \bar{a} \rangle \not\in \langle M^B, a \rangle$ for $\langle a, \bar{a} \rangle \in E^P$,

as before, $a = A^{\langle i \rangle} \in \langle M^B, E \rangle$ - definable and $\bar{a} = \bar{A}^{\langle i \rangle} \in \langle M^P, \bar{E} \rangle$ - definable by the same definition, where $\overline{\pi} : \langle M^P, \bar{E} \rangle \not\subset \langle M^B, E \rangle$.

Q.E.D (5.1.3)
Lemma 5.14 \([p]/k(B) \leq [q]/k(B)\) in \(BA(IP)/B\)

\[\begin{align*}
\forall p, q \in IP',
&\quad \bigcup [p^*] \subseteq \bigcup [q^*] \text{ in } BA(IP^*_B) \\
&\quad \text{for } p, q \in IP'.
\end{align*}\]

Proof:

\[
S = A = \{ [p]/k(B) \mid p \in IP' \} \setminus \{ 0 \} \\
A' = \{ [p^*] \mid p \in A \} = \{ [p^*] \mid p \in IP' \} \setminus \{ 0 \}
\]

Since \(A\) is dense in \(BA(IP)\) we have:

\[
a < b \iff \forall c \in A \ (c b = 0 \implies c a = 0)
\]

for \(a, b \in BA(IP)/k(B)\), in particular, this holds for \(a, b \in \{ [p]/k(B) \mid p \in IP' \}\).

Similarly:

\[
a < b \iff \forall c \in A' \ (c a = 0 \implies c b = 0)
\]

for \(a, b \in BA(IP^*_A)\) (in particular for \(a, b \in \{ [p^*] \mid p \in IP' \}\)).

But:

\[
[p]/k(B) \setminus [q]/k(B) = 0 \iff [p^*] \setminus [q^*] = 0
\]

for \(p, q \in IP'\) by Lemma 5.1.1.

The conclusion is immediate.

QED (5.14)

Cor 5.15 There is a unique

\[
\delta : BA(IP)/k(B) \leftrightarrow BA(IP^*_B) \quad \text{such that}
\]

\[
\delta([p]/k(B)) = [p^*] \quad \text{for } p \in IP'.
\]
We complete the proof of 5.1 by showing:

**Lemma 5.1.6** \( \delta \left( \frac{k(b)}{k(B)} \right) = \frac{\mathcal{B}}{B} (b/B) \) for \( b \in \mathcal{B}_\mathcal{C} \).

**Proof.**

It suffices to show that if \( F \) is a \( \mathcal{B}A(P)/k(B) \) \( \delta \)-generic filter and \( F^* = \delta^-F \), then

\[
\frac{k(b)}{k(B)} \in F \iff \frac{\mathcal{B}}{B}(b/B) \in F^*
\]

Set \( G = \{ p \in IP \mid [p]/k(B) \in F^* \} \). Then \( G \) is \( IP \)-generic. Hence, there is \( p \in IP \setminus G \) not being \( \pi(p) \). Let \( \pi(p)(b) = b \). Then \( b \in B^p \iff b \in B^G \iff \frac{k(b)}{k(B)} = \left[ \left[ b \in B^G \right] /k(B) \right] \in F^* \), since \( F = \{ a/k(B) \mid a \in \mathcal{B}A(P) \land a \not\in \mathcal{G} \} \).

But \( \bar{b} \in B^p \iff \frac{\bar{b}}{B^p} \in B^p/B^p_{\mathcal{C}} = B_{\mathcal{C}}^p \),

Set \( G^* = \{ p \in IP \mid [p] \in F^* \} \). Then \( G^* \) is \( IP \) \( \delta \)-generic. But \( \pi_{G^*}(B^p) = B^p \), hence \( \pi_{G^*} : M_p \to M_B \). Hence \( \pi_{G^*}(\frac{k(b)}{B^p_{\mathcal{C}}}) = \frac{k(b)}{B} \), and

\[
\frac{k(b)}{k(B)} \in F \iff \bar{b} \in B^p \quad \text{and} \quad \frac{\bar{b}}{B^p_{\mathcal{C}}} \in B_{\mathcal{C}}^p \quad \text{Hence} \quad \frac{b}{B} = \frac{\mathcal{B}}{B} (b/B) \in \mathcal{B} \in \mathcal{G}^*. \quad \text{Hence} \quad \frac{k_B(b/B)}{= \left[ \left[ b/B \in B \right] /B \right] \in F^*},
\]

Similarly, \( \vdash \frac{k(b)}{k(B)} \notin F \), then \( \frac{\mathcal{B}}{B} (b/B) \notin F^* \). \( \square \) (Lemma 5.1.1)
Lemma 6 \( \mathcal{IB}_\lambda \) is subcomplete.

Proof.
There is \( i_0 < \lambda \) s.t. \( \mathcal{IB}_{i_0} \models \lambda(\check{\lambda}) = \omega \). Since
\[
\mathcal{IB}_\lambda \cong \mathcal{IB}_{i_0} \times \mathcal{IB}_0 \text{ where } \mathcal{IB}_0 \models \mathcal{IB} = \mathcal{B}_\lambda / \mathcal{B}_0
\]
(\( \mathcal{B}_i \) being the canonical generic name),
and \( \mathcal{IB}_{i_0} \) is subcomplete, it suffices to show that \( \mathcal{IB}_\lambda \) is subcomplete in \( V[\mathcal{B}] \) whenever \( \mathcal{B} \models \mathcal{B}_0 \)-generic.

But \( \mathcal{IB}_\lambda \cong \mathcal{BA}(\mathcal{IB}_0) \), so it suffices to show that \( \mathcal{IB}_0 \) is subcomplete in \( V[\mathcal{B}] \). But then it suffices to prove the subcompleteness of \( \mathcal{IB}_0 \) under the assumption \( i_0 = 0 \), since the same proof would apply to \( \mathcal{IB}_0 \) in \( V[\mathcal{B}] \).

Let \( W = L^\beta \) where \( 2^\beta < \Theta < \omega_1 \). \( \Theta \) is regular,
\( H_\Theta \subset W \), and \( W \) verifies the pronoun \( \sigma \) of \( \mathcal{IB}_\lambda \) over \( \mathcal{IB}_0 \) for \( \delta < \Theta < \delta_0 \). Let \( \sigma : \check{W} \to W \)
whenever \( \check{W} \) is countable and full. Let
\[
\sigma(\check{\theta}, \check{\mathcal{P}}, \check{Q}, \check{M}, \check{N}, \check{Z}, \check{X}_i) = \theta, \check{\mathcal{P}}, \check{Q}, \check{M}, \check{N}, \check{Z}, \check{X}_i
\]
\( i = 1, m, m \) where \( \check{\mathcal{P}} \models \mathcal{H}_\lambda \). (hence \( \mathcal{N} \models \mathcal{H}_\lambda \)) and \( \delta_i < \Theta \) for \( i = 1, m, m \). Let \( \Theta \) be \( \check{\mathcal{P}} - \)generic over \( \check{W} \).
Claim: There is $g \in IP$ s.t. whenever $G \ni g$

$IP$-generic, there is $\bar{\sigma}_0 \in V[G]$ with:

(a) $\bar{\sigma}_0 : \bar{W} \leq W$

(b) $\bar{\sigma}_0 (\bar{e}, \bar{P}, \bar{Q}, \bar{M}, \bar{N}, \bar{\xi}, \bar{\xi}_i) = \bar{\sigma}_0, \bar{P}, \bar{Q}, \bar{M}_i, \bar{N}_i, \bar{\xi}_i, \bar{\lambda}_i$

($i = 0, \ldots, m$)

(c) $\sup \bar{\sigma}_0 \bar{\xi}_i = \sup \bar{\sigma}_{\bar{\sigma}} \bar{\xi}_i$ ($i = 0, \ldots, m$)

where $\bar{\xi}_0 = 0 \in \bar{W}$

(d) $\bar{\sigma}_0 \bar{G} \subset G$, monotone and

Let $\langle e_i, i < \omega \rangle \in \bar{M}$ be cofinal in $\bar{\xi}_0 = \sigma^{-1}(\bar{\xi}_0)$.

Set $c_i = \sigma(e_i)$. Then $\langle c_i, i < \omega \rangle$ is monotonous and cofinal in $\bar{\xi}_0$. Let $\sigma(\bar{M}) = M$, where $M = \langle M_i, i < \omega \rangle$. Then $\sigma(\bar{M}) = \bar{M}$ for $n < \omega$.

Let $\bar{M} \circ \bar{M}_n$ be the inverse limit of $\langle \bar{M}_n, n < \omega \rangle$. Modifying slightly the proof of the $\omega$-case of the iteration theorem for $\omega$-bicompleteness, we get:

Fact: Let $\bar{B} \subset \bigcup_{n < \omega} \bar{M}_n$ r.t. $\bar{M}_n = \bar{M} \cap \bar{M}_n$ is $\bar{M}_n$-generic over $\bar{W}$ for $n < \omega$. There is a $b \in \bar{B} \setminus \{0\}$ r.t. whenever $\bar{B} \ni b$ is $\bar{B}$-generic,

then there is $\bar{\sigma} \in V[\bar{B}]$ r.t.

(a) $\bar{\sigma} : \bar{W} \leq W$

(b) $\bar{\sigma} (\bar{e}, \ldots, \bar{\xi}_i) = 0, \ldots, \bar{\xi}_i$ ($i = 1, \ldots, n$)

(c) $\sup \bar{\sigma} \bar{\xi}_i = \sup \bar{\sigma}_{\bar{\sigma}} \bar{\xi}_i$ ($i = 0, \ldots, n$)

(d) $\bar{\sigma} \circ \bar{B}_n \subset \bar{B}_n = \bar{B} \cap \bar{B}_{\omega}$ for $n < \omega$. 

Making use of this fact we get:

**Sublemma 6.1** Let \( \sigma^* \) be least \( \omega \)-admissible. The following language \( L^* \) on \( L_\omega(W) \) is consistent:

**Predicate**: \( \in \), \( \text{Constante} \rightarrow (x \in L_\omega(W)), \sigma^*, \mathcal{B} \)

**Axioms**: \( \mathbb{ZFC}^- \), \( \forall \nu (\nu \in x \leftrightarrow \forall \mu \in x \), \( \sigma^* \downarrow \mathcal{M} : \langle \overline{a}, \overline{m}, \overline{b} \rangle \triangleleft \langle \overline{a}, \overline{m}, \overline{b} \rangle \), \( \sigma^*(\overline{a}, \overline{p}, \overline{a}, \overline{m}, \overline{x}, \overline{z}, \overline{c}) = \overline{a}, \overline{p}, \overline{a}, \overline{m}, \overline{x}, \overline{z}, \overline{c} \) \( (i = 0, \ldots, m) \), \( \lambda \lim_p \sigma^*(\overline{x}_i) = \lambda \lim_p \sigma^*(\overline{x}_i) \) \( (i = 0, \ldots, m) \)

[Note: These axioms do not say that \( \omega = \omega_1 \).]

**Proof (sketch)** We follow the proof of §3 Lemma 4.1.

We first let \( L_0 \) be like \( L^* \) except that the axiom \( \lambda \lim_p \sigma^*(\overline{x}_i) = \lambda \lim_p \sigma^*(\overline{x}_i) \) is replaced by \( \lambda \lim_p \sigma^*(\overline{x}_i) = \lambda \lim_p \sigma^*(\overline{x}_i) \), where \( \lambda_0 = \overline{e} \).

Let \( \sigma^*, \mathcal{B} \) be as in the above fact. Let \( H^\infty = \overline{H^\infty} \), where \( \sigma(\overline{a}) = \overline{a} \). Let \( \overline{a} \uparrow \overline{H} \triangleleft \overline{H} \) up to finally, and let \( \langle W', \sigma' \rangle \) be the lift up of \( \langle W, \sigma \rangle \). Note that

\( \sigma' \uparrow \overline{a} : \langle \overline{a}, \overline{b}, \overline{c} \rangle \triangleleft \langle \overline{a}, \overline{b}, \overline{c} \rangle \),

where \( \overline{c} = \{ \overline{c}_i \mid i < \omega \} \), \( C = \{ c_i \mid c_i < \omega \} \).

Hence \( \sigma' \uparrow \overline{M} : \langle \overline{a}, \overline{m}, \overline{b} \rangle \triangleleft \langle \overline{q}, \overline{m}, \overline{b} \rangle \),

where \( \sigma'(\overline{M}) = \overline{M} \). There is a canonical \( k : W' \leq W \) s.t. \( k\sigma' = \sigma \) and \( k|H = 1|H \).

Let \( L_0' \) be defined on \( L_\omega(W) \) the way \( L_0 \) was defined on \( L_\omega(W) \), where
\(5\) is least s.t. \(L_{\bar{\beta}}(W')\) is admissible, (More precisely, \(L_0\) is defined in \(W\) and parameters \(\bar{\beta} \in W\); \(L_0\) has the same def., in \(W'\) and \(k^{-1}(\bar{\beta})\).) Then \(\langle H_{W_2}, \sigma' \rangle\) models \(L_0'.\) Hence \(L_0'\) is consistent. Assume \(\pi, \lambda_0 > \lambda_1 > \ldots > \lambda_m\) and let:\n\[\sigma \in H_{\bar{\omega}}: H_{\bar{\omega}} \leq H'\] finally, \(\sigma\).

Let \(\sigma'': \bar{\omega} \leq W''\) be the lift of \(\bar{\omega}\) by \(\sigma \in H_{\bar{\omega}}\). Let \(k'': W'' \leq W\) s.t. \(k''H'' = \text{id}\) and \(k'' \sigma'' = \sigma'.\) Then there is \(k': W' \leq W''\) s.t. \(k' \sigma' = \sigma''\) and \(k' H' = \text{id}\).

Moreover \(k = k'k'\). Let \(\delta''\) be least s.t. \(L_{\delta''}((W''))\) is admissible and let \(L_0''\) be defined over \(L_{\delta''}((W''))\) as \(L_0\) was defined over \(L_{\delta}(W)\). The statement that \(L_0'\) is consistent is \(T_{\lambda}(L_{\delta'}((W'))\) in parameters \(\bar{\beta}' = k^{-1}(\bar{\beta})\) and \(W'\). Hence the statement that \(L_0''\) is consistent in \(T_{\lambda}(L_{\delta''}((W''))\) in \(W''\) and \(\bar{\beta}'' = k''^{-1}(\bar{\beta}) = k'(\bar{\beta}')\). Hence \(L_0''\) is consistent by the full line of \(\bar{\omega}\). Note that \(k''(M) = M\). Let \(M = \langle |M|, \sigma'', B'' \rangle\) be a valid model of \(L_0''\) lying in some generic extension \(V[\delta']\) of \(V[\bar{\beta}]\).
Then \( \langle H^V[\Omega], k''', \sigma'''', B'''' \rangle \) models \( L^*_\kappa \).

QED (Lemma 6.1)

Now set: \( N^* = \langle H^*_\kappa, W, \Omega, M, \Lambda, B, x, \sigma, \lambda_1, \ldots, \lambda_n \rangle \),

where \( \Omega > \kappa \) is a cardinal. Let \( P \) conform to \( N^* \). Set \( \bar{N}^* = \bar{N}^* (P, N^*) = \langle H^*, W^*, x^*, B^*, \sigma^*, \lambda^*_1, \ldots, \lambda^*_n \rangle \).

Let \( L^* \) be defined in \( \bar{N}^* \) like \( L^* \) in \( N^* \). Let \( \mathcal{M} \in H^*_\kappa \) be a solid model of \( L^* \).

Set: \( \sigma^* = \sigma^*_\mathcal{M}, B^* = B^*_\mathcal{M} \). Define \( \mathcal{G} \in \mathbb{P} \) by:

\[ \mathcal{G}^* = \langle \mathcal{G}_p, M_p, B' \rangle, \mathcal{G} = P_1 \mathcal{G} \]

(Then \( \mathcal{G} \in \mathbb{P} \) by Lemma 4.2.) We show that this \( \mathcal{G} \) satisfies the Claim.

Let \( G \in \mathcal{G}^* \) be \( \mathcal{P} - \)generic. Then

\[ \langle \bar{Q}, \bar{M}, B' \rangle \cup \mathcal{G}^* \] with

\[ \overline{\langle \bar{Q}, \bar{M}, B' \rangle}, \mathcal{G}^* \]

where \( B = B^G \). Moreover \( \mathcal{G}^*, \langle \bar{Q}, \bar{M}, B' \rangle \]

Hence \( \langle \bar{Q}, \bar{M}, B' \rangle \cup \langle \bar{Q}, \bar{M}, B' \rangle \] with:

\[ \overline{\langle \bar{Q}, \bar{M}, B' \rangle}, \langle \bar{Q}, \bar{M}, B' \rangle \] = \[ \mathcal{G}^* \sigma^*_\mathcal{M} \]

Let \( \pi^* \) be the unique \( \pi^* \in \overline{\mathcal{G}^*} \) with

\[ \pi^* \bar{N}^* < \bar{N}^* \]

Set: \( \sigma^*_0 = \pi^* \sigma^*_\mathcal{M} \).

Then \( \overline{\langle \bar{Q}, \bar{M}, B' \rangle}, \langle \bar{Q}, \bar{M}, B' \rangle \] = \[ \mathcal{G}^* \mathcal{M} \]

Then \( \sigma^*_0 \) witnesses the Claim.

(a) - (c) are straightforward,
We verify (d): \( \sigma_0 \tilde{G} \subseteq G \).

Let \( \tilde{x} \in \tilde{G} \), \( x = \sigma_0 (\tilde{x}) \). Then \( \tilde{x} \sigma_0 = \tilde{x} \sigma_0 \) since \( \sigma_0 \tilde{H}^\tilde{M}_{\omega_1} = \text{id} \). Thus \( \tilde{x} \sigma_0 \subseteq \langle \tilde{Q}, \tilde{M}, \tilde{B} \rangle \) with
\[
\tilde{x} \sigma_0, \langle \tilde{Q}, \tilde{M}, \tilde{B} \rangle = \tilde{x} \sigma_0 \tilde{G}.
\]
Set \( \tilde{u} = \sigma_0 \tilde{x} \sigma_0 \).

Claim: \( \tilde{x} \tilde{G} \subseteq G \) with \( \tilde{G} \tilde{G} = \tilde{G} \).

We know that \( G = G \tilde{B} \), so it is enough to show:
\[
\tilde{x} : \langle M, a \rangle < \langle M, a \rangle \text{ whenever } \langle a, \tilde{a} \rangle \in F^x.
\]

Let \( \langle a, \tilde{a} \rangle \in F^x \). Then \( a = \sigma_0 a' \) where \( \langle a', \tilde{a} \rangle \in F_{\tilde{x}} \). Thus \( \tilde{x} \sigma_0 : \langle M, a \rangle < \langle M, a' \rangle \).

and \( \sigma_0 \tilde{M} : \langle M, a' \rangle < \langle M, a \rangle \).

Thus \( \sigma_0 (\langle M, a' \rangle) = \langle M, a \rangle \). Q.E.D. (Lemma 6)
Lemma 7.1. Let \( G \) be IP-generic, \( B = B^G \). Let \( B' \subseteq \{ 1, 2 \} \) a.t. \( B' \cap B \quad \) is \( B \)-generic for \( i < \z_0 \). Let \( Q [ B ] = Q [ B' ] \). Assume moreover that \( B' \) lies in a generic extension of \( V [ G ] \)
which adds no reals. Then
(a) \( G' = G \cup B' \) is IP-generic.
(b) \( V [ G ] = V [ G' ] \)

Proof.
We imitate the proof of §3 Lemma 7. Let \( i_0 < \z_0 \) a.t. \( \vdash \chi ( x^*) = \omega \). Let \( C < M [ B_0 ] \)
\( \vdash \psi ( C ) = \omega \). Let \( \chi ( c_i ) = i < \omega > \) enumerate \( C \). There is \( p \in \mathbb{Q} \) a.t.
\( \psi ( B_{c_i} ) = \psi ( \pi^* ( p ) ) \) for \( i < \omega \), where \( \pi^* \) is the
unique \( \pi^* > \pi_0 \) a.t.
\( \pi^* : Q [ B ] \rightarrow Q [ B ] \)
and \( \pi^* ( B^p_{c_i} ) = B^p_{c_i} \) for \( i < \omega \). So \( ; \)
\( B^p_i = \bigcup_{i < \omega} \pi^* ( B^p_{c_i} ) \). Then:
(1) \( B' \subseteq V [ G ] \) since \( B' = \bigcup_{i < \omega} \pi^* ( B^p_{c_i} ) \)
(2) \( \pi^* ( B^p_{c_i} ) = B^p_\z ( \z ) \) for \( \z < \z_0^p \)
(3) \( B^p_\z \cup 1B^p \quad \) generic over \( M_\z \) for \( \z < \z_0^p \)
proof. (Let straightforward)
(4) \( Q_\z [ B^p_\z ] = Q_\z [ B^p ] \)
\( \vdash \) straightforward
Hence by the 1st reversibility principle.
(5) \( p' \subseteq IP \) where \( p' = \langle Q_\z, M_\z, B^p_\z, \rangle, \quad p'_1 = p_1 \).
Now let \( q \leq p \). Let \( \pi_q^* \) be the unique extension of \( \pi_p \cap Q \) s.t. \( \pi_q^* : Q_q[B^p] \leq Q_q[B^q] \) and \( \pi_q^* (B^p) = B^q \). Let \( \pi_q^* (B^q) \) for \( i \leq p \). Set:

\[ B^q := \bigcup_{i \leq p} \pi_q^* (B^p) \cap \mathcal{B}_i, \quad q_0 = \langle Q_q, Q_q[B^p], B^q \rangle, \quad i_q^* = \pi_q^* \]

We easily get:

(6) (a) \( B_i^q \cap B_i^p \) - generic over \( M_q \) for \( i \leq q_0 \).

(b) \( q' \leq p' \).

(c) \( \overline{\pi_q^*} = \pi_{q'}^* \).

Proof straightforward, using the fact that \( \pi_q \) extends to \( \pi_q^* : Q_p[B^p] \leq Q_q[B^q] \) s.t. \( \pi_q^* (B^p) = B^q \) for \( i \leq q_0 \), QED (6).

At \( q \leq p \), we could define \( q' \) from \( p' \) the way we defined it from \( p' \). Hence:

(7) \( q \leq p \rightarrow q' \leq p' \);

moreover, \( \pi_{q'}^* \neq \pi_{p'}^* \).

Set:

\[ \Delta_0 := \{ q : q \leq p \cap \mathcal{B} \}, \quad \Delta_1 := \{ q : q \leq p' \} \]

Set \( \sigma (q) = q' \). Arguing exactly as in \( \S 3 \) Lemma 7 we get:

(8) \( \sigma : \langle \Delta_0 \rangle \rightarrow \langle \Delta_1 \rangle \),

But then \( q \in \Delta_0 \cap G \rightarrow \sigma (q) \in G' = G \) and

\[ q \in \Delta_1 \cap G' \rightarrow \sigma^{-1} q \in G = G \].

Hence:

(9) \( \sigma^{-1} (\Delta_0 \cap G) = \Delta_1 \cap G' \),

As before we conclude:
(10) \[ V[G] = V[G'] \].

(11) \( G' \in IP \) - generic, since if \( \Delta \) is dense in \( IP \), then \( \Delta' = \{ q \in L_0 \mid \sigma(q) \subseteq \Delta \} \) is dense above \( p \) in \( IP \).
Hence \( G \cap \Delta' \neq \emptyset \). Hence \( G' \in G \cap \Delta' \).
\[ Q.E.D. \](Lemma 7.1)

Carrying this proof a step further, we get:

Cor 7.2 Let \( G, B, G', B' \) be as above. There \( \in \sigma' \in V \rightarrow \cdot \cdot \cdot \), \( \sigma' : BA(\Delta) \sim \rightarrow BA(\Delta') \) and \( F' = \sigma''(\mathcal{F}) \) (where \( F = \{ b \in BA(\Delta) \mid b \cap G \neq \emptyset \} \) is the generic ultrafilter given by \( G \)).

proof (assume w.l.o.g. \( G \neq G' \)).

As in §3 Cor 7.1 set \( \Delta_2 = \{ x \mid x \) is incompatible with \( p \) and \( p' \} \), then \( B \neq B' \) since \( G \neq G' \). Hence \( B \cap B' \).

Hence \( \Delta_0, \Delta_1, \Delta_2 \) are mutually disjoint.
Set \( \Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \), Define \( \sigma' : \langle \Delta, \leq \rangle \sim \rightarrow \langle \Delta, \leq \rangle \).

\[ \sigma'(q) = \begin{cases} \sigma(q) & \text{if } q \in \Delta_0 \\ \sigma'(q) & \text{if } q \in \Delta_1 \\ q & \text{otherwise} \end{cases} \]

Since \( \Delta \) is dense in \( IP \), \( \sigma' \) induces a unique \( \sigma' : BA(\Delta) \sim \rightarrow BA(\Delta') \) s.t. \( \sigma'(p \cap G) \)
for \( p \in \Delta \). \[ Q.E.D. \](7.2)
Cor. 7.3 Let $G$ be $IB_\lambda$-generic and $\mathcal{B} = G \cap \bigcup_{i<\lambda} B_i$. Let $B' = \bigcup_{i<\lambda} B_i$. Let $B'_i = B_i \cap B'_i$ be $IB_\lambda$-generic for $i<\lambda$ and $H^*_\lambda[B'] = H^*_\lambda[B]$. There is an automorphism $\pi \in U$ of $IB_\lambda$ such that $\pi''B = B'$. Hence $G' = \pi''G$ is $IB_\lambda$-generic and $V[G'] = V[G]$ with $B' = G \cap \bigcup_{i<\lambda} B_i$.}
Lemma 7.4 \[ IB_\lambda \text{ is symmetrically proud over } IB_\delta \text{ whenever } \delta \leq \lambda \cap Ac. \]

**Proof.**

This follows obviously by:

**Main Claim.** Let \( \theta > 2^\lambda \) be big enough to verify the proudnest of \( IB_\delta \) for all \( \delta \leq \lambda \cap Ac. \) Let \( G \) be \( P \)-generic and \( P \)-commuting, where \( \bar{P} : \bar{W} < W = H_\theta, \) and \( \bar{W} \) is countable and transitive.

Set \( \bar{B} = B^\bar{G}. \) Let \( \bar{\delta} \in \lambda \cap Ac \) and:

\[
\bar{P} \left( \bar{\delta}, \bar{P}, \langle IB_\bar{\delta}, i : \bar{\lambda} \rangle \right) = \bar{\delta}, \bar{P}, \langle IB_\bar{\delta}, i : \bar{\lambda} \rangle.
\]

Suppose that \( \bar{B}'' \) is \( IB_\bar{\delta} \)-generic over \( \bar{W} \) and \( B'' \) is \( IB_\delta \)-generic r.t.

\[ V''[B''] = V''[IB_\delta] \] (where \( B'_\delta = B \cap IB_\delta) \)

and \( \pi'' \bar{B}'' \subset B''. \) Let \( \bar{G}' \) be \( \bar{P} \)-generic over \( \bar{W} \) r.t. \( \bar{B}'' = \bar{B} \cap \bar{IB}_\delta \).

Set \( \bar{B} = \bar{B}^\bar{G}' \). There \( \bar{G} \) r.t.:

- \( G' \) is \( P \)-generic
- \( B' \supset B'' \), where \( B' = B^G' \)
- \( \pi'' \bar{G}' \subset G' \)
- There is \( \sigma : IB(A(P)) \rightleftarrows BA(A(P)) \) r.t.

\[
\sigma'' \bar{F}_G = \bar{F}'_G.
\]
proof.
Since \( cf(\lambda) = \omega \) in \( V[G] \) and \( \pi \) takes \( \lambda \)
cofinally to \( \lambda \), we can pick \( \langle \xi_i : i < \omega \rangle \)
monotone and cofinal in \( \lambda \). Set \( c_i = \pi(\xi_i) \). Then \( \langle c_i : i < \omega \rangle \) is monotone
and cofinal in \( \lambda \). We may suppose
without loss of generality that \( c_0 = \emptyset \) and \( c_i \subseteq \mathcal{A}_c \)
for \( i < \omega \). Thus \( \overline{B}_i \) is proud over \( \overline{B}_h \)
for \( h < i < \omega \). Set \( \overline{B}_i' = \overline{B}_i \cap \overline{B}_c \) for \( i < \omega \). Using proudness, we successively pick \( B'_i \) \( (i < \omega) \) set,
\[
\begin{align*}
* \quad & B'_0 = B'' \\
* \quad & B'_i \supset B'_{i+1} \text{ in } B'_{i+1} \\
* \quad & \forall B'_i = \forall B' \text{ where } B_i = B \cap \overline{B}_c \\
* \quad & \forall \overline{B}_c' \subseteq B'_i \\
\end{align*}
\]
Set \( B' = \bigcup B'_i \). Since \( Q = H_y \) in \( V \)
and \( \overline{B}_c \in Q \), we have \( Q[B'_i] = Q[B_c] \).
Hence \( Q[B'] = \bigcup Q[B'_i] = \bigcup Q[B_c] = Q[B] \).
By Lemma 7.1 and 7.2.
The conclusion then holds for
\( G' = G[B'] \). All verifications
are trivial except for...
\[ \pi' \bar{G} \subset G' \]

which we now prove. We first note:

1. \( \pi' \bar{G} : \langle \bar{Q}, \bar{B}, \bar{C} \rangle \triangleleft \langle Q, B, C \rangle \)

for any \( \bar{C} \in \bar{M} \) such that \( \bar{C} \subset \bar{C}_0 = \sup \bar{C} \) and \( \operatorname{otp}(\bar{C}) = \omega \).

(Here \( \pi'(\bar{Q}, \bar{M}) = Q, M \))

But then:

2. \( \pi' \bar{M} : \langle \bar{Q'}, \bar{M}, \bar{B'} \rangle \triangleleft \langle Q, M, B \rangle \)

proof:

Let \( \pi' \bar{M} : \bar{M} \to \bar{F} \) cofinally. We must show that \( \langle \bar{M}, \pi' \bar{M} \rangle \) is the lift up of \( \langle \bar{M}, \pi' \bar{Q} \rangle \). In other words, we must show that \( \pi' \bar{M} : M \to \bar{M} \) is \( \bar{F} \) - cofinal. Let \( \bar{G} = \pi'(\bar{G}) \). Then \( \bar{G} \) is \( \bar{F} \) - generic over \( \bar{M} \). Hence \( \pi' \) extends uniquely to \( \pi' : \bar{M} \to \bar{F}[\bar{G}] \subseteq \bar{F}[G] \) and \( \pi'((\bar{G})) = G \). But then, for any \( x \in \bar{M} \) there is \( \exists y \in \bar{M} \) such that \( x \in \pi(\bar{A}) \) (where \( \bar{M} = \bar{L}_{\beta}^{\bar{A}} \)). Hence there is \( \exists z \in \bar{G} \) such that \( z \in \exists y \in \pi(\bar{A}) \).

But \( \pi' \bar{G} : \langle \bar{z}, \bar{G} \rangle \triangleleft \langle \bar{Q}, \bar{M}, \bar{B} \rangle \), where \( \bar{B} = B \bar{G} \). Let \( \pi_{\bar{G}} : M \to \bar{M} \) cofinally. Then the map \( \bar{z} \in \bar{G} \) - cofinal - i.e.,

\[ \forall z \in \bar{Z} \exists \bar{V} \in \bar{M} \quad (\bar{z} \leq \bar{z} \in \bar{M} \quad \land \bar{z} \in \bar{G}(\bar{V})) \]
Let $\tilde{\mathcal{M}} = \tilde{\pi}(\tilde{\mathcal{M}}_1)$. Then $\tilde{\mathcal{M}}_1$ is a segment of $\mathcal{M} = \tilde{\pi}(\tilde{\mathcal{M}})$ and since $\tilde{\pi}(\tilde{\mathcal{M}}_1) = \mathcal{M}_1$. Hence $\mathcal{A} \subset \tilde{\mathcal{M}}_1 \subset \mathcal{M}_1$. But:

Let $\tilde{z} \in \tilde{\mathcal{M}}_1$ with $z \in \mathcal{M}_1$ ($\tilde{\mathcal{A}} \subset \mathcal{A}$.)

Hence $\mathcal{A} \subset \tilde{\mathcal{A}} \subset \mathcal{A}$.

Now let $\tilde{\mathcal{A}} \subset \tilde{\mathcal{G}}$.

Claim $\mathcal{A} \subset \tilde{\mathcal{G}}$.

We again have $\tilde{\mathcal{A}} = \tilde{\pi}(\tilde{\mathcal{A}})$, hence:

$\mathcal{A} \subset \tilde{\mathcal{A}} \subset \mathcal{A}$.

Set $\tilde{\pi} = \tilde{\pi} \circ \tilde{\pi}$. We claim that $\tilde{\pi} \circ \tilde{\mathcal{G}}$.

Clearly $\mathcal{A} \subset \tilde{\mathcal{G}}$, since $\tilde{\mathcal{G}} \subset \tilde{\mathcal{G}}$. Since $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}$, it is sufficient to show:

$\tilde{\pi} : \langle \tilde{\mathcal{M}}_1, \tilde{\mathcal{A}} \rangle \subset \langle \mathcal{M}, \mathcal{A} \rangle$

whenever $\langle \mathcal{A}, \mathcal{A} \rangle \subset \tilde{\mathcal{G}}$. 

Then $F^2 = \mathbb{F} (F^2)$ and $a = \pi (a')$, where $\langle a', a \rangle \in F^2$. Hence

$$\pi \bar{G} : \langle M, a \rangle \leq \langle \bar{M}, a' \rangle,$$

and

$$\pi \bar{M} : \langle M, a' \rangle \leq \langle M, a \rangle.$$

Since $\pi (\langle \bar{M}, a' \rangle) = \langle M, a \rangle$, 

QED (Lemma 7.4)

**Lemma 8** $IB_\lambda$ is symmetrical over $IB = \langle IB_i | i < \lambda \rangle$

**Proof.**

This will follow from:

**Claim** Let $\sigma : \cup IB_i \leftrightarrow \cup IB_i$. Let

$\sigma : IB_i \leftrightarrow IB_i$ for sufficiently large $i$.

There is $\sigma' : BA(IP) \leftrightarrow BA(IP)$. Let

$\sigma' \sigma = \sigma$

**Proof.**

Set $N^* = \langle H_N, N, \sigma, <, \cdots \rangle$ where $\theta > Z^B$,

$\Delta = \{ p \in IP | p \text{ conforms to } N^* \}$.

Then $\Delta$ is dense in $IP$. For $p \in \Delta$ set:

$N^*_p = \langle H_N, N, \sigma, \langle p \rangle, \cdots \rangle$

where $N_p = \langle H_p, N_p, \sigma_p, <, \cdots \rangle$.

Then $p = \langle S_p, N_p, IB_p \rangle$. For $p \in \Delta$
define \( p' \in \text{IP} \) by:
\[
p_\sigma = \langle \sigma, M_p, \sigma' B_p \rangle, \quad p'_\sigma = p
\]
Theorem \( B_p = \sigma P \) and \( p' \in \text{IP} \) by the first reversibility lemma. But then \( p' \in \Delta \) since \( p' = p \). It follows easily that \( p \leq q \iff p' \leq q' \) for \( p, q \in \Delta \).
Moreover, if \( p^* \) was defined as \( p' \) was defined with \( P_\sigma \) in place of \( P \), then \( p \leq \Delta \rightarrow p^* \in \Delta \), \( p \leq q \iff p^* \leq q^* \) for \( p, q \in \Delta \), \( p^* = p \) and \( p' = p \) for \( p \in \Delta \). Hence \( p \rightarrow p' \) is an automorphism of \( \langle \Delta, \leq \rangle \). Hence there is a unique automorphism \( \sigma \) of \( \text{BA}(\text{IP}) \) such:
\[
\sigma([p]) = [p'] \quad \text{for } p \in \text{IP}. \quad \text{We must show:}
\]
\textbf{Claim} \( \sigma' k = k \sigma \)
Recalling that \( k(b) = [\{ b' \in B \} \] for \( b \in \cup_{i} B_i \), this becomes:
\[
\sigma'([\{ b' \in B \} \] = [\{ \sigma(b) \in B \] \]
Claim. Let $G$ be $IP$-generic. Then

$$G \cap \sigma'(\{b \in B^{\infty}_c\}) \neq \emptyset \iff \sigma(b) \in B^{\infty}_c.$$ 

Proof.

($\Longleftarrow$) $D = \{p \in \Delta \mid b \in \text{rng}(\pi^g) \} \cap \Delta_{\infty}$ is dense in $\{b \in B^{\infty}_c\}$. Hence $p^* \in \Delta$ for a $p \in D$. Then $\sigma_{p^*} = \sigma_p$, $M_{p^*} = M_p$, $F^g = F^p$, and $B^{\infty}_p = \sigma_p \cdot B^g$. Hence $\pi^{p^*} = \pi^p$ and $N^{\infty}_{p^*} = N^{\infty}_p$. Hence $\sigma_{p^*} = \sigma_p$.

Let $\pi^p(b^-) = b$. Then $\sigma_{p^*}(b^-) = \sigma_p(b^-) \in B^{\infty}_p$.

But then $\pi^{p^*}(b^-) = b$ and $\pi^{p^*}(\sigma_p(b^-)) = \sigma_{p^*} \cdot \pi^{p^*}(b^-) = \sigma_p(b) \in B^{\infty}_c$, hence $\pi^{p^*}$ extends to $\pi^g$; $N^{\infty}_{p^*} < N^{\infty}$ (hence $\pi^g(\sigma_{p^*}) = \sigma$).

QED ($\Rightarrow$)

($\Longleftarrow$) Let $\sigma(b) \in B^{\infty}_c$. Then there is $p \in G \Delta_n$ s.t. $\sigma(b) \in \text{rng}(\pi^p)$. Note that

$$\tau_p : \mathcal{B}_c^p \leftrightarrow \mathcal{B}_c^p ,$$

and $\pi^g(\tau_p) = \sigma$,

where $\pi^g$ extends $\pi^p$, $\pi^g \cdot N^{\infty}_p < N^{\infty}_c$.

Hence there is $b \in \mathcal{B}_c^p$ s.t.

$$\pi^g(\sigma_p(b^-)) = \sigma(b) ,$$

hence $\sigma_p(b^-) \in B^p_c$. Then

$$\tau_p(\sigma_p(b^-)) = \sigma_p(b) \in \mathcal{B}_c^p ,$$

and $p \models \chi'( \{b \in B^{\infty}_c\} ) \wedge G$.

QED (Lemma 8).
In order to satify the condition (f) of §2.3 we need:

**Lemma** Let $c(\lambda) = \omega$ in $V$. Let $\langle c_i \mid i < \omega \rangle$ be cofinal and monotone in $\lambda$. Let $\langle b_i \mid i < \omega \rangle$ be a thread in $\langle IB, c_i \mid i < \omega \rangle$ (i.e., $b_{c_i} c_{i+1} \supset b_i, c_i$). Then $\bigcap b_i \neq \emptyset$ in $\mathcal{IB}_\lambda$.

**Proof.**
This reduces to:

**Claim** Let $\langle c_i \mid i < \omega \rangle, \langle b_i \mid i < \omega \rangle$ be as above. Then $\bigcap b_i \neq \emptyset$ in $\mathcal{IB}(IP)$.

But $b_i = \{b_i : b_i \in \mathcal{B}^* \}$. We modify the proof that $\mathcal{L}$ is consistent to find a $p \in IP$ s.t. $p \in \bigcap \{b_i : b_i \in \mathcal{B}^* \}$.

We first note that $\mathcal{L}' = \mathcal{L} + \langle \dot{b}_i : b_i \in \mathcal{B} \rangle$ is consistent. To see this we re-run the consistent proof of $\mathcal{L}$, taking a $\mathcal{IB}'$-generic $\mathcal{B}'$ s.t. $\bigcap b_i \in \mathcal{B}'$, where $\mathcal{B}' = \bigcup \mathcal{IB}_i$ is the inverse limit.

Now let $M'$ be a solid model of $\mathcal{L}'$.

By Lemma 2 there is $u = \langle au, M, \mathcal{B}_u \rangle \in \mathcal{P} \cap H_{\omega_1}$ s.t. $u \notin \langle \mathcal{Q}, M, \mathcal{B} \rangle$ and $\pi(b_i) = b_i$ for $i < \omega$, where $\pi = \overline{u}, \langle \mathcal{Q}, M, \mathcal{B} \rangle$.

Set $p_0 = u, p_n = \langle b_{c_n}, b_m \rangle$ for $m < \omega$. Then...
Then \( p \in P \) since \( \varphi \models L(p) \). But then \( b_i \in B^p \), since \( b_i \in B^\varphi \). Hence, if \( G \models \iota \) \( p \in P \) -
- generic, we have \( b_i \in B^G \) for \( i < \omega \). Hence \( G \cap \bigcap k(b_i) = G \cap \bigcap \{ b_i \in B^G \} \neq \emptyset \). This shows that \( [p] \cap \bigcap k(b_i) \). QED (Lemma 9)

Note Using Lemma 5.1 it follows easily that Lemma 9 holds of \( \langle B_i, B \rangle \models \lambda - \iota_0 \rangle \) in \( T \) \( B \) whenever \( \iota_0 < \lambda \) and \( B \models \langle B_i, \lambda \) -
- generic.

We are now ready to show:

**Lemma 10** \( \langle B_i, i \leq \lambda \rangle \) satisfies the condition (a) - (h) of § 2.3.

**Proof**

(h) is trivial. (a) follows by Lemma 4.11 and Lemma 4.17. (b) follows by Lemma 6. (c) holds vacuously. (d) holds by Lemma 9. (e) holds by Lemma 7.3. (f) follows by Lemma 8. (g) follows by Lemma 5.1

QED

This completes the first limit case.
§ 4.2 The 2nd limit case

We now consider the case that $\lambda$ is an $\omega$-point, $\delta = \delta_\lambda$, and $\delta^+ \in A_\omega$. Hence $\delta^+$ must acquire cardinality $\omega_1$. We have $\beta_\lambda = \delta^+$; hence we want: $\delta^{++} = \omega_2$ in $V[G]$ if $G$ is $\mathcal{P}_\lambda$-generic. Our construction closely parallels § 4.1. Indeed, most of the proof can be repeated almost verbatim. We shall therefore confine ourselves to detailing the differences in the two constructions. From now on let $\beta = \beta_\lambda = \delta^+$.

Define $Q, M, N$ exactly as before. We change the language $\mathcal{L}$ in exactly one respect: Replace $(\ast)$ by the conjunction of:

$(\ast)$ For each $\beta \leq \beta$ there are $\pi, r, u, i, t, v \in H_{\omega_1}$, $\pi : u \subseteq \langle Q, M, \beta \rangle$, and $\delta \in rny(\pi)$

$(\ast\ast)$ If $(\beta \beta) > \omega$

$\Delta_\pi, \Delta_i, \Pi_\pi, \Pi_\beta$ are defined exactly as before. § 3.1 Lemma 6 holds as before.
Lemma 1: $L$ is consistent.

Proof.

Just as before we may assume w.l.o.g. that
$
\lambda = \delta_0 \uparrow \omega$ is $\omega$-cofinal in $V$. Again let
$\mathcal{C} = \{C_i : i < \omega \} \text{ where } \langle C_i : i < \omega \rangle \in V$
be monotone and cofinal in $\lambda$. Set;
$\mathcal{B}' = \text{the inverse limit of } \langle 1B_\omega, i < \omega \rangle$.

$\mathcal{B}'$ is again subcomplete. Let $B$ be $\mathcal{B}'$-generic and set:
$\mathcal{M} = \langle H^+_\beta, B \rangle$. Claim $\mathcal{M} \models L$.

The only problematical axiom are
(* *) and (** *). To see (*), let $\exists \in \mathcal{M}$.
Let $\pi: \mathcal{M} \langle \mathcal{B} \rangle \subset \mathcal{M} \langle \mathcal{B} \rangle$ s.t. $\pi(\mathcal{B}) = B$
and $\exists, C \in \text{rng}(\pi)$ and $\mathcal{M} \in H_{\alpha_1}$.
Let $\pi(C) = C$, $\pi(Q) = Q$. Then
$\pi(\mathcal{Q}) = \vec{Q}, \pi(\mathcal{B}, \vec{C}) \subset \langle \mathcal{Q}, \mathcal{B}, \mathcal{C} \rangle$. Using
the fact that every $x \in \mathcal{M}$ has cardinality $\leq x$ in $\mathcal{M}$ we get:
$\pi$ in $\exists$-cofinal, where $\pi: \mathcal{M} \rightarrow \hat{\mathcal{M}}$
cofinally and $\pi(\exists) = \exists$. Hence
$\hat{\pi}: \langle \hat{\mathcal{Q}}, \hat{\mathcal{B}} \rangle \subset \langle \hat{\mathcal{Q}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \rangle$ and $\exists \in \text{rng}(\hat{\pi})$.
This proves (** *),
We now prove (**): \( \beta = \omega_1 \) in \( V[B] \) since \( \beta \subseteq \mathcal{L} \{ \mu_q(\pi_\alpha, \lambda, \mu, B) \mid \mu \in H_\alpha \land \mu \leq \beta, \lambda \leq \pi_\alpha, \lambda, \mu, B \} \).

Since GCH holds below \( \alpha \), we have \( \bar{\beta} \leq \beta \).

Hence \( \beta^{+} \) remains a cardinal in \( V[B] \). But then \( cf(\beta) = \omega_1 \) in \( V[B] \), since otherwise \( \beta^{+} \) is collapsed by §4 Lemma 4.1 of [ELF]. (Fact 11 of §3.2 in this paper). QED (Lemma 1)

Repeating our previous verbatim:

Lemma 2 Cor 1.1, Lemma 2, and Lemma 2.1
or §4.1 hold.

We then define \( \bar{\beta} \), \( \beta \) exactly as before (except that in the def. of \( \bar{\beta} \) we omit the clause: \( \sup_n \bar{\beta}_n = \beta \), even though \( \beta \) is regular).

Remark: We don't know whether forcing with \( IP \) is the same as forcing with the inverse limit of \( \langle \beta_i \mid i < \lambda \rangle \).

Repeating our previous proofs:

Lemma 3 Lemma 3.1 - 3.8 of §4.1 hold, as do the reversibility Lemmas 4.1 and 4.2.

Lemma 4.3 (IP adds no reals) goes through as well, but we shall need a stronger form of it. We first show:
Lemma 3.1 Let $G$ be $IP$-generic and $p \in G$. Then $\sup_p \pi^G_p \beta_p < \beta$. 

Proof: Assume to observe that $\Delta$ is dense above $p$, where $\Delta = \{ q \leq p \mid \sup_p \pi^G_p \beta_p < \beta_q \}$. This follows by § 4.1 Lemma 2. Q.E.D. (3.1)

Our stronger version of § 4.1 Lemma 4.3 now reads:

Lemma 3.2 Let $G$ be $IP$-generic. Let $f \in V[G]$ s.t. $f: \omega \to \beta$. There is $p \in G$ s.t. $f = \pi^G_p \bar{f}$ for all $\bar{f} \in H\omega_1$.

Proof: We can assume $f = f^G$, where $\sup_{f^G}: \omega \to \beta$.

As sufficient to show that the set $\Delta$ of $p \in IP$ s.t. $p \Vdash \pi^G_p \bar{f}$ for all $\bar{f} \in H\omega_1$, is dense in $IP$.

Let $\pi \in IP$. Set:

$N^* = \langle H_{\theta}, N, M, Q, IP, \pi, \ldots \rangle$, where $\theta > 2^\beta$ is big enough to verify the prudence of $\pi$ over $h$ for all $h \leq \pi \leq \pi$.

Let $p$ conform to $N^*$. Set:

$\bar{N} = N^*(p, N^*) = \langle \bar{H}, \bar{N}, \bar{M}, \bar{Q}, \bar{IP}, \pi, \ldots \rangle$.

(Hence $\bar{Q} = Q_p$, $\bar{M} = M_p$). Choose $\bar{G} \in \bar{N}$ s.t. $\bar{G}$ is $\bar{IP}$-generic over $\bar{N}^*$. Set $\bar{B} = \bar{B}^\bar{G}$.
By the second irreversibility lemma we can define \( q \in IP \) by:

\[
\bar{q}_0 = \langle \bar{a}, \bar{m}, \bar{B} \rangle, \quad \bar{q}_n = \bar{q}_n^* \quad \text{where} \quad \bar{B} = \bar{B}^G.
\]

Let \( \bar{f} = \bar{f}^G \). At sufficient to prove:

\underline{Claim}

(a) \( q \models \bar{f} = \pi^G \circ \bar{f} \), (b) \( q \) is compatible with \( q \).

We first prove (a). Suppose not. Let \( G \models q \) such that \( \bar{f} \neq \pi^G \circ \bar{f} \) where \( f = \bar{f}^G \).

Let \( i < \omega \), \( \bar{\gamma} = \bar{f}(i) \), \( \gamma = \pi^q(\bar{\gamma}) \) such that \( f(i) \neq \gamma \). Then \( \bar{q}_0 \models G \), where \( \bar{q}_0 = \bar{q}_0^* \), \( \bar{q}_1 = \bar{q}_1^* \), \( \langle \bar{q}_1, \bar{\gamma} \rangle \bar{q}_2 \).

Let \( q \models \bar{f}(i)^* \neq \gamma \). Let \( L \) be a solid model of \( L(q') \). Then \( L \models L(q') \) and \( \pi^L(\bar{\gamma}) = \gamma \), since \( \pi^L(q') = q' \), but \( \pi^L \) extends uniquely to \( \bar{\pi}^* : \bar{N}^* \rightarrow N^* \).

Let \( \bar{Q} \subseteq \bar{N}^* \) and \( \bar{Q} \subseteq \pi^* \bar{Q} \). Let \( \bar{Q} \in G \) such that \( \bar{Q}(i') \neq \gamma \). Then, letting \( \bar{q} = \pi^*(\bar{Q}) \), we have \( q \models \bar{f}(i') = \gamma \). Hence \( q, q' \) are incompatible. We derive a contradiction by showing that they are compatible cases in fact that \( \mathcal{M} = L(q) \) (hence \( L(q') \) is consistent).
\( \pi^C \triangleright_{\alpha} \langle m, \bar{a} \rangle \), since \( x_0 = \bar{x}_0 \) and \( \bar{a} \in \bar{G} \). But \( \langle m, \bar{a} \rangle = \omega \) and
\[ \pi^{\omega} : \omega \triangleright \langle m, \bar{a} \rangle \) where \( B = B^{\omega} \).

Set \( \pi = \pi^{\omega} \circ \pi^C \). Then
\[ \pi : \omega \triangleright \langle m, \bar{a} \rangle \), A+ remains only to show:
(1) \( \pi : \langle m, \bar{a} \rangle \triangleright \langle m, a \rangle \), whenever
\[ \langle a, \bar{a} \rangle \in F \).

Let \( a = \pi^{\omega}(a') \). Then \( \langle a, \bar{a} \rangle \in F \bar{a} \) and
\[ \pi^C : \langle m, \bar{a} \rangle \triangleright \langle m, a' \rangle \). But
\[ \pi^{\omega}(\langle m, a' \rangle) = \langle m, a \rangle \). Hence
\[ \pi = \pi^{\omega} \circ \pi^C : \langle m, \bar{a} \rangle \triangleright \langle m, a \rangle \), QED(a)

We now note that the last part of the proof shows that for any \( \bar{a} \in \bar{G} \),
\( a = \pi^{\omega}(\bar{a}) \) is compatible with \( \bar{g} \), hence with \( g \). But \( \bar{a} \in \bar{G} \) and
\( \bar{a} = \pi^{\omega}(\bar{a}) \) since \( \pi^{\omega} : \bar{G} \triangleright \bar{G} \),
QED (3.2)

By Lemma 3.2 and 3.1 we then easily get:

**Lemma 3.3** Let \( G \) be P-generic. Then
\[ H_{\omega_1} = H_{\omega_1}[G] \]
\[ \text{cf}(\beta) > \omega \text{ in } V[G] \]
Hence:
Lemma 3.4 §4.1 Lemma 4.4 holds, i.e., if $G \in \text{IP}$ is $\mathcal{P}$-generic and $B = B^G$, then
$< H^V[G], \mathcal{P} >$ models $L(p)$.

Proof:
The only problematical axioms were $H = H^\omega_1$ and $\text{cf}(\mathcal{P}) = \omega$, which are now seen to hold.

By a literal repetition of the proofs:
Lemma 3.5 §4.1 Lemma 4.5 — 4.9 hold.

We note that $\beta = \beta^\chi$ and $\text{cf}(\beta) = \omega$.

We can then repeat the proof of §4.1 Sublemma 4.10.1, to get
Lemma 3.6 BA(IP) has a dense subset of size $\beta^\chi$, (Hence $BA(\text{IP}) \leq \beta^+$
where $\beta = \beta^\chi$), Moreover $\beta^+$ remains a cardinal in $V[G]$ whenever $G \in$
(IP- generic.)

Repeating the relevant part of the proof (the case $\beta = \beta^\chi$) we get:
Lemma 3.7 §4.1 Lemma 4.12 holds. (i.e.,
if $G \in \text{IP}$- generic, then $\bar{\beta} = \omega_1 = \text{cf}(\lambda)$
in $V[G]$ and $\beta^+ = \omega_2$ in $V[G]$, )
We then choose $\sigma$, $IB_\lambda$ act

$\sigma : BA(1^P) \xrightarrow{\sim} IB_\lambda$

$k \uparrow$

$\cup I B_c$

and $IB_\lambda \subset H_{\beta^+}$.

By an almost literal repetition of the proof we then get:

Lemma 4, Lemma 5-10 of §4.1 hold,

(An particular $\langle IB_c, i \leq \lambda \rangle$ satisfies (a)-(h) of §2.3.)

This completes the second limit case.
§ 4.3 The third limit case

We now deal with the case that \( \lambda \) is not an \( \omega \)-point. Let \( \delta = \delta^* \). Then either

\( \lambda \) is an \( \omega_1 \)-point (i.e., \( cf(\lambda) < \delta \) and

\( cf(\lambda) = \omega_1 \) or \( cf(\lambda) \in \mathcal{A}_0 \), or

else \( \lambda = \delta^* \) is strongly inaccessible. In the first case we have \( \beta_\lambda = \delta^* \). In the second \( \beta_\lambda \) is undefined. In both cases we let \( IB_\lambda \) be a minimal completion of \( \bigcup_{i<\lambda} IB_i \), ensuring that \( IB_\lambda \subset V_{\delta^*} \).

If \( \lambda = \delta^* \) is not an \( \omega_1 \)-point, then

\( \bigcup_{i<\lambda} IB_i \) satisfies the \( \delta^* \)-CC. Hence

\( IB_\lambda = \bigcup_{i<\lambda} IB_i \subset V_{\delta^*} \). By the iteration theorem in \( \S 1 \) we have:

**Lemma 1.1** \( IB_\lambda \) is subcomplete

But then:

**Lemma 1.2** Let \( G \) be \( IP \)-generic.

(a) If \( \delta^* \) is an \( \omega_1 \)-point, then

\( \overline{\delta} = cf(\delta^*) = \omega_1 \) in \( V[G] \) and \( \delta^* = \omega_2 \) in \( V[G] \).

(b) If \( \delta^* \) is not an \( \omega_1 \)-point, then

\( \overline{\delta} = \omega_2 \) in \( V[G] \).
Proof.
Clearly each $\bar{y} < \bar{x}$ is collapsed to $\omega_{<i}$.
Since $\text{IB}_i \subseteq \text{IB}_\lambda$ for $i < \lambda$, hence $\bar{y} = \omega_{<i} = \omega_{<\lambda}$
if $x$ is an $\omega_{<\lambda}$-point. But, $x^+$ remains a cardinal, since $\text{IB}_\lambda$ has a dense
subset $\bigcup \text{IB}_i$ of size $\lambda$. Now let $\lambda = \aleph$ be strongly inaccessible. Then
$\text{IB}_\lambda$ satisfies $\delta$-cc, hence $\delta$ remains a cardinal.
QED (1.2)

Since $\bigcup \text{IB}_i$ is dense in $\text{IB}_\lambda$, any automorphism of $\bigcup \text{IB}_i$, extends uniquely
to an automorphism of $\text{IB}_\lambda$. Hence
Lemma 1.3 $\text{IB}_\lambda$ is symmetrical over $\langle \text{IB}_i | i < \lambda \rangle$.

Cor 1.4 $\text{IB}_\lambda$ is symmetrical over $\text{IB}_i$ for $i < \lambda$.

Let $\sigma : \text{IB}_i \rightarrow \text{IB}_i$ define $\sigma_i : \text{IB}_i \rightarrow \text{IB}_i$, for
$i \leq i' \leq \lambda$ by: $\sigma_0 = 0$; $\sigma_{i+1} : \text{IB}_{i+1} \rightarrow \text{IB}_i$,
s.t. $\sigma_i < \sigma_{i+1}$; $\sigma_i : \text{IB}_i \subseteq \text{IB}_i$. Let
$\sigma_i < \sigma_{j+1}$ for limit $\eta$. QED (1.4)

Thus remains to prove:
Lemma 2. Let $\mathfrak{A} \succeq \mathfrak{A}$. Then $\mathfrak{B} \succeq \mathfrak{B}$.
Otherwise $\mathfrak{B} \succeq \mathfrak{B}$.

Proof.

Let $\theta > 2^{\lambda}$ be big enough to verify the

prounderness of $\mathfrak{B}$ over $\mathfrak{B}$ for all $\mathfrak{A} \succeq \mathfrak{A}$.

Let $\mathfrak{B}$ be $\mathfrak{B} - \text{generic}$ and $\sigma$ - conforming,

where $\sigma \in V[\mathcal{B}]$. Let $\mathfrak{W} \subseteq \mathcal{W} = H_\theta$, $\mathfrak{W}$ is countable, $\sigma(\mathfrak{W}) = \mathfrak{W}$, and

$\sigma(\langle \mathfrak{B}, i : i \leq \lambda \rangle) = (\langle \mathfrak{B}, i : i \leq \lambda \rangle)$. Let

$\mathfrak{B}^\prime$ be $\mathfrak{B} - \text{generic}$ over $\mathfrak{W}$, $\mathfrak{B}^\prime \subseteq V$,

$\mathfrak{B}^\prime$ be $\mathfrak{B} - \text{generic}$ over $\mathfrak{W}$, $\mathfrak{B}^\prime \subseteq V$,

$\mathfrak{B}^\prime \subseteq \mathfrak{B}^\prime$. Let $\mathfrak{B}^\prime$ be $\mathfrak{B} - \text{generic}$ over $\mathfrak{W}$, $\mathfrak{B}^\prime \subseteq V$,

$\mathfrak{B}^\prime$ be $\mathfrak{B} - \text{generic}$ over $\mathfrak{W}$, $\mathfrak{B}^\prime \subseteq V$,

Claim. There exist $\mathfrak{B}^\prime$, such that

* $\mathfrak{B}^\prime \supseteq \mathfrak{B} \succeq \mathfrak{B} - \text{generic}$.
* $\sigma$ is an automorphism of $\mathfrak{B} - \text{generic}$.

* $\mathfrak{B} = \mathfrak{B}^\prime$.
Set: \( \bar{x} = \sup \sigma \) "\( \bar{x} \). Then \( \sigma (\bar{x}) = \omega \) in \( V[B] \).

Hence \( \bar{x} \) is not an \( \omega \)-point of the iteration. Hence \( \bar{x} \) is either an \( \omega \)-point or \( \bar{x} = \gamma_{\lambda} \) is strongly inaccessible in \( V \).

We handle these cases separately.

**Case 1: \( \bar{x} \) is an \( \omega \)-point**

We work in \( V[B] \). Pick \( \langle \bar{\gamma}_i \mid i < \omega \rangle \) monotone and cofinal in \( \bar{x} \) such that \( \bar{\gamma}_0 = \bar{x} \) and \( \bar{\gamma}_i = \sigma (\bar{\gamma}_i) \in A \) for all \( i \).

Set \( B'_{\bar{x}} = B' \cap B_{\bar{x}} \) for \( i \leq \bar{x} \) and \( B_i = B \cap B_{\bar{x}} \) for \( i \leq \bar{x} \). Since \( \bar{B}_{\bar{x}}^\lambda \) is monad over \( \Lambda B_{\bar{x}}^\lambda \), we can successively pick \( B'_{\bar{x}} \) such that

- \( B'_{\bar{x}} \supset B'_{\bar{x} + 1} \) with \( B_{\bar{x}}^\lambda \) as given.
- \( B'_{\bar{x}} \cap B_{\bar{x}}^\lambda \) is generic.
- \( V [B_{\bar{x}}^\lambda ] = V [B_{\bar{x}}^\lambda ] \)
- \( \sigma " B'_{\bar{x}} \subset B'_{\bar{x}} \)

Set: \( \tilde{B} = \bigcup_i B'_{\bar{x}} \) and \( \tilde{B} = \bigcup_i B_{\bar{x}}^\lambda \).

Since \( V [B_{\bar{x}}^\lambda ] = V [B_{\bar{x}}^\lambda ] \) for \( \lambda < \omega \), we have: \( H_{\lambda \bar{x}}^\lambda [\tilde{B}'] = H_{\lambda \bar{x}}^\lambda [\tilde{B}] \).
By §4.1 Lemma 7.3 we conclude that there are $B'_\lambda \supseteq \tilde{B}'$ and $\pi : IB'_\lambda \hookrightarrow IB'_\lambda \cong B'_\lambda$ s.t. $B'_\lambda$ is $IB'_\lambda$-generic, $\tilde{\pi} \in V$, and $\pi" B'_\lambda = B'_\lambda$. But then $\tilde{\pi}$ extends to $\pi : IB'_\lambda \hookrightarrow IB'_\lambda$, since $IB'_\lambda$ is symmetrical over $\bar{B}'_\lambda$. Set $B' = \pi " B'_\lambda$. Then $B'$ is $IB'_\lambda$-generic. Moreover, if $B^* = \bigcup_i \bar{B}'_\xi^i$, then $\sigma" B^* \subseteq \tilde{B}' \subseteq B'_\lambda \subseteq B'$.

Let $\tilde{B}' = \{ b \in \bar{B}'_\gamma \mid V_a \in B^* \text{ a } < b \}$. Since $\bigcup_i \bar{B}'_\xi^i$ is dense in $\bar{B}'_\lambda$. Hence $\sigma" \tilde{B}' \subseteq B'$. Q.E.D (Case 1)

Case 2 Case 1 fails.
Then $\tilde{\lambda} = \chi_\xi^\lambda$ is strongly inaccessible in $V$.
The second successor case applies to $IB''_{\lambda+1}$, since otherwise $\tilde{\lambda}$ would have cofinality $\omega_1$ in $V[B]$. Define
$$\langle \tilde{\xi}_i \mid i < \omega \rangle, \langle B'_i \mid i < \omega \rangle$$
eatly as before and set $B''_\lambda = \bigcup_i B'_i$. Then $B''_\lambda \cup IB''_{\lambda+1}$ is $IB''_{\lambda+1}$-generic by §3.2 Lemma 6.12.
Since $V[B'_i] = V[B'_i]$ for $i < \omega$ and $B''_\lambda = \bigcup_i B'_i$, we have
$$H''_\lambda \cap [B''_\lambda] = H''_\lambda \cap [B''_{\lambda+1}]$$
where
$$\tilde{\lambda} = \chi_\lambda^\lambda = \chi_\lambda^{\lambda+1}.$$
By §3.2 Lemma 7.2 we conclude:

There is a \( \overline{\pi} \in V \) such that \( \overline{\pi} \colon IB_{\lambda+1} \to IB_{\lambda+1} \)
and \( \overline{\pi}'' B_{\lambda} = B'_{\lambda} \). Hence \( B'_{\lambda+1} = \overline{\pi}' B_{\lambda+1} \cup IB_{\lambda+1} \),
which is generic and \( B'_{\lambda} \subseteq B_{\lambda+1} \). But \( \overline{\pi} '' \) extends to \( \pi : IB_{\lambda} \to IB_{\lambda} \).

Set \( B' = \overline{\pi}'' B' \).

Then \( B' \cup IB_{\lambda} \) is generic and \( \overline{\pi}'' B' \subseteq B_{\lambda} \subseteq B' \), and \( \overline{B} = \overline{\bigcup_{j=0}^{\infty} B_{j}} \).

QED (Lemma 2).

Thus we get:

Lemma 3 \( \{ IB_{i} \mid i \leq \lambda \} \) satisfies (a)-(b) of §2.3.

Proof:

(a) is immediate. (b) \( \cup \) by Lemma 1.2.

(c) \( \cup \) by Lemma 1.1. (d) \( \cup \) immediate.

(e) \( \cup \) vacuous; (g) \( \cup \) by Lemma 2.

(f) \( \cup \) by Lemma 1.3. (h) \( \cup \) immediate.

Now \( IB_{\lambda}/B \) is the direct limit of \( \{ IB_{i+1}/B \mid i < \lambda - h \} \) whenever \( B \in IB_{h} \) is generic. Thus the same proof can be carried out in \( V[B] \).

QED (Lemma 3)
Thus, if $G \in IB_k$ - generic, then $
 = \omega_2$ in $V[G]$ and every regular $\tau \in (\omega_1, \n)$ (in $V$) becomes $\omega$-cofinal in $V[G]$. Since $IB_k = \bigcup IB_i$ satisfies $\mu$-cc, every club subset $B \subset \n$ in $V[G]$ will contain a club $A$ lying in $V$. Hence all stationary subsets of $\n$ will remain stationary in $V[G]$.

This completes the proof.