§1 σ-premise

A ppm (premouse) is a structure satisfying all of the premouse axioms other than the initial segment condition. There are, in fact, different versions of this condition. The minimal initial segment condition (MIS) says that if $F = E^M_v \neq \emptyset$ and $\tau_r < \lambda'_r < \lambda$, and $\nu' = \lambda + \text{M}_{\lambda, \nu}$, then $\langle J^E_{\lambda'_r}, F \text{IX}' \rangle$ is not a ppm. This condition does not, however, have the preservation properties we want.

The full initial segment condition (IS) says that if $\nu, \lambda'$ are above $J^E_{\lambda, \nu} = J^E_{\lambda', \nu}$, and $E, \nu'$ are s.t. $\langle J^E_{\lambda'_r}, F \text{IX}' \rangle$ is a ppm satisfying MIS, then $\text{IX}' \in \text{M}_{\lambda, \nu}$.

(Note that we then have $\exists \tau : J^E_{\lambda'_r} \rightarrow J^E_{\lambda', \nu}$.)

[Note: Here $\kappa_r = \kappa^M_r = \text{crit}(E^M_r)$, $\lambda = \lambda^M_r = = F(\kappa_r) = \text{lh}(F)$, $\tau_r = E^M_r = \kappa_r + \text{M}_{\lambda, \nu}$. A t $M = \langle J^E_{\lambda'_r}, F \rangle$ we also s.t. $\kappa(M) = \kappa(F) = \kappa_r^M \lambda(M) = \lambda(F) = \lambda^M_r$, $\tau(M) = \tau(F) = \tau^M_r$.]

A premouse is a ppm satisfying IS.

Lemma 1.1 Every premouse satisfies MIS.

Proof.

Suppose not. Let $M = \langle J^E_{\lambda'_r}, F \rangle$ be a counterexample of minimal length. Hence there is $\lambda' \in (\tau_r, \lambda)$ s.t.
Let $\langle J^{E_i}, F_i \rangle$ be a p.p.m., where $\nu' = \chi' + M$.

Hence $\chi'$ is a limit cardinal in $M$. Hence $N \models \text{IS}$. Hence $\nu' \leq \chi' + M$ by the minimality of $M$. Hence $N \not\in M$.

Hence $\nu' < \chi' + M$. Hence $N \models \text{MIS}$. Q.E.D (Lemma 1.1)

Cor 1.2: The condition "$\langle J^{E_i}, F_i \rangle$ is a p.p.m." can be omitted from $\text{IS}$.

Proof:
Let $\langle J^{E_i}, F_i \rangle$ be a p.p.m. and let

$$\pi: J^{E_i} \rightarrow J^{E_i}, \quad \sigma: J_{\nu'} \rightarrow J_\nu$$

be defined by

$$\sigma(\pi'(f)(\alpha)) = \pi'(f)(\alpha) \quad \text{for } \alpha < \chi',$$

for $f: \nu' \rightarrow \nu$ in $\text{MII}_M$. Then $\nu' = \text{crit}(\sigma)$.

Hence $\chi'$ is a limit cardinal in $\text{MII}_M$.

Hence $N: = \langle J^{E_i}, F_i \rangle$ is a p.p.m. Hence $N \models \text{IS}$. Q.E.D (1.2).
However, there are narrower classes of premise satisfying their own initial segment condition. An example is the class of \( n \)-premises.

\[ \text{Def: Let } M = \langle J, G \rangle \text{ be an active } \]

\[ \lambda^M \ni \text{ the natural length of } F \]

\[ \lambda(M) = \lambda \cup \{ \emptyset \cup G \cup F \} \]

where \( G \cup F \) = The set of generation of \( F = \text{The set of } 3 \leq \lambda \) \( \text{ and } \)

\[ \emptyset \neq \Pi (f \setminus i^2) \text{ for any } f \in M \setminus (\lambda \setminus \emptyset) \text{ and any tuple } d_1, \ldots, d_m < 3 \]

\[ \text{Def: } \lambda(M)^+ = \lambda(M)^+ \]

\[ \text{Def: For } \nu \in N, E^N_\nu = \emptyset \text{ set } i \]

\[ \lambda^N(\nu) = \lambda(N \setminus \nu), \lambda^+(\nu) = \lambda^+(N \setminus \nu) \]
The minimal initial segment condition (S-MIS) says that if \( \mathcal{F} = \mathcal{E}^{-1}_N \neq \emptyset \) and \( \overline{v} \leq \overline{z} < \overline{2} \), then \( \overline{z} \in \mathcal{E}^{-1}_N \) if \( \overline{2} \leq \overline{z} \leq \overline{z} \) in \( N \) and \( z = \pi(N, z) \), then:

(a) If \( N_z \) satisfies S-MIS, then \( N_{z^1} \in N \)

(b) If \( N_z \) satisfies S-MIS and \( \overline{z} \leq \overline{z}^1 < \overline{2} \) in \( z \), \( N_{z^1} \) satisfies S-MIS

Then \( N_{z^1} \in N_z \).

Note Clearly \( \overline{z} = \pi(N, z) \). Let \( z = \text{lbs}(\overline{z}, \overline{z}^1) \).

Note (a) holds for any \( z \in [\overline{z}, \overline{2}] \) if

it holds for \( z = \pi(N, z) \).

Note 4 If \( \overline{z}, \overline{z}^1 \) are as in (b), then

\( N_{z^1} = (N_z \setminus \overline{z}^1) \).

By this last remark we easily get:
Lemma 2.1 Let \( N \) satisfy \( s \)-IS. Let \( z'_1 \leq z < z' \) in \( N \) s.t. \( N_{z'} \) satisfies \( s \)-MIS. Then \( N_{z} \) satisfies \( s \)-IS.

Moreover:

Lemma 2.2 Let \( N \) satisfy \( s \)-IS. Then \( N \) satisfies \( s \)-MIS.

Proof. Suppose not.

Let \( N = \langle J_{v}, F \rangle \) be a counterexample of minimal length. Then \( N \) satisfies \( s \)-MIS for all \( z < v \). Thus there is \( z \) s.t. \( z'_1 \leq z < z \) and \( J_{v} \cap E_{N_{z'}} = \emptyset \) where \( z' = z + n \).

We suppose w.l.o.g. That \( z(v) \) is minimal for counterexamples of length.

For the given \( N \) we also suppose \( z \) chosen minimally.

(1) \( N_{z} \) satisfies \( s \)-IS.

Proof. Assume w.l.o.g. \( z = \text{sup}(\{z \in \mathbb{N} : z < z' \}) \).

Then \( z = s(N_{z}) \). Let \( z \in \mathbb{N} \) s.t.

\( (N_{z})_z \) satisfies \( s \)-MIS. Then

\( (N_{z})_z = N \in N \). But \( N_{z} \leq z < z' = z' \in \mathbb{N} \).

in \( N \). Hence \( N_{z} \in \bigcup_{z' \in \mathbb{N}} = \bigcup_{z' \in \mathbb{N}} \subseteq N_{z} \).

QED(1)
There is $\sigma : N_3 \to \mathbb{E}_0$ defined by:

$\sigma(f \cdot (a^2)) = \pi(a \cdot (x^2))$ for $f : \mathbb{E}_v \to \mathbb{E}_v$,

$s \in N, a_1 \ldots a_m \in \chi_1$ where $\pi : J^E_0 \to J^E_0$ and $\pi : J^E_0 \to J^E_0$. Hence

$h_1(N) \leq v$. Moreover $s(N) = \bar{s} \leq \lambda(N)$. Hence $N_3$ satisfies $\tau - \text{MIS}$ by (1) and the minimality of $M$. Hence $N_3 \notin N_0$. But

$\bar{s} \leq s(N_3) \leq \bar{s} \leq \bar{s}' = \bar{s} +$ in $N$. Hence $N \in \bigcup_{\bar{s}' \in \mathbb{E}_v} J^E_0 = \bigcup_{\bar{s}' \in \mathbb{E}_v} N_3 \subset N_3$. Contrad!

QED (2.2)

Def. An $\alpha$-premone in a ppm satisfying $\text{MIS}$.

Lemma 2.2 corresponds to Lemma 1.1. However, the proof of Lemma 1.2 does not go through.

There can, in fact, be an $\alpha$-premone $M$ and $\bar{s}' < s(M)$ s.t. $M_3$ does not satisfy $\text{MIS}$. By Lemma 2.1, 2.2 however, we at least know that $M_3$ satisfies $\text{MIS}$ iff $M_3$ is an $\alpha$-premone.

Lemma 2.3 $\tau - \text{MIS} \to \text{MIS}$

pf. Trivial
Lemma 2.7 Let \( \xi \leq \gamma < \aleph_0 \) s.t. \( \gamma \) is a cardinal in \( N = \langle J^E, F \rangle \), where \( N \) is an active \( \mathfrak{a} \)-premonoe. Then \( N_\gamma \) is an \( \mathfrak{a} \)-premonoe.

\[ \text{Proof:} \]

Let \( 3 < \gamma \) s.t. \( N_3 = (N_\gamma)_3 \) satisfies \( \mathfrak{a} \)-MIS and \( 3 = \text{lut}(E \cup \text{gen}(n_3)) \). Then \( N_3 \in N. \) Hence \( N_3 \in \bigcup_{\gamma} N_\gamma \in N_\gamma. \)

Now let \( 5 < \gamma \) s.t. \( N_5 = (N_\gamma)_5 \) satisfies \( \mathfrak{a} \)-MIS. Then \( N_5 \in N_3 \), since \( N_5 = (N_5)_3 \) and \( N_3 \) is an \( \mathfrak{a} \)-premonoe.

Hence \( N_5 \in N_5 \in N_\gamma. \) QED (2.7)

Lemma 2.8 Every \( \mathfrak{a} \)-premonoe is a premonoe

\[ \text{Proof:} \]

Let \( N = \langle J^E, F \rangle \) be an active \( \mathfrak{a} \)-premonoe. Let \( \xi < \chi < \lambda \) s.t. \( N_\chi = \langle J^E, F \chi \rangle \) and \( J^E_{\chi} = J^E_{\chi}. \)

Claim \( F \chi' \in N. \)

Define \( \sigma : N_\chi' \to N \) by \( \sigma(f)(x) = F \chi(f)(x) \) for \( f \in N_\chi' \), \( x < \chi' \), s.t. \( F \chi' \in N_\chi' \)

Then \( \sigma \) is the identity, \( \sigma(\chi') = \chi. \) Hence \( \chi' \in \text{gen}(F) \) and \( \chi' \) is a limit cardinal in \( N. \) Hence \( N_\chi' \in N \) by Lemma 2.7.

QED (2.8)
Def Let $N$ be a p.p.m.

\[ C_N = \hat{C}_N = \text{the set of } z < z(N) \text{ s.t.} \]

\[ z = \text{lub} (\pi_v \cup \{ \text{sing}_v \}) \text{ and } N_z \text{ satisfies } z - \text{MIS}, \]

\[ C(\nu) = C_N^{\nu} = C_{N^{\|\nu}} \text{ for } E_v N \neq \emptyset, \]

Lemma 3.1 Let $N$ be an $\alpha$-p.p.m. 

Then $C_N$ is closed in $z = z(N)$.

Proof: 
Let $\gamma < z$ be a limit p.t. of $C = C_N$

At sufficient to show:

Claim $N_\gamma$ is an $\alpha$-p.p.m.

Clearly $C_N = \gamma \cap C_N$. For $z \in C_N$

let $\overline{\gamma}_v : N_\gamma \rightarrow \mathbb{Z}_+$ be defined

by $\overline{\gamma}_v (\tau (f) (d^v)) = \overline{\gamma}_v (f) (d^v)$ for

$d^v < \overline{\gamma}_v$, $f \in \text{N}_v (u^a_{\text{cr}})$, where

\[ \overline{\gamma}_v : J^E \rightarrow J^{EN_3}, \text{ where } \]

\[ \overline{N} = \langle J^E_\nu, F \rangle, \text{ and } N_3 = \langle J^E_{EN_3}, F \rangle. \]
Then \( \overline{2}^3 \| 3 = \text{id} \) and \( N_\overline{\gamma} = \bigcup_{\gamma \in C_{N_\overline{1}}} \gamma \cap C_{N_\overline{2}} \).

Let \( \overline{3} \in C_{N_\overline{2}} \), \( \overline{3}' \in C_{N_\overline{2}} \), \( \overline{3} < \overline{3}' \).

Then \( N_{\overline{3}} \subseteq N_{\overline{3}'} \), since \( N_{\overline{3}} = (N_{\overline{3}'} \cup \gamma) \cap C_{N_\overline{2}} \) satisfies MIS. Hence \( \overline{F} \overline{3} \subseteq N_{\overline{3}'} \).

Hence \( \overline{F} \overline{3} = \overline{\overline{3}'} \gamma \cap (\overline{F} \overline{3}) \in N_{\overline{3}'} \). Hence \( N_{\overline{3}} \subseteq N_{\overline{3}'} \). Now let \( \overline{5} < \overline{3} \), \( \overline{5} \in C_{N_\overline{2}} \).

Then \( N_{\overline{5}} = (N_{\overline{5}} \cap \overline{5} \in N_{\overline{3}} \subseteq N_{\overline{3}}) \).

QED (3.1)

**Lemma 3.2** Let \( N = \langle \bigcup E_i, F \rangle \) be an active \( \lambda \)-premonoid. Let \( \overline{\gamma} \in \text{gen } F \), \( \overline{\gamma} = \text{sub} (\bigcup E_i \cup (\gamma \cap \text{gen } F)) \) s.t. \( N_{\overline{\gamma}} \) is not a premonoid. Then \( \overline{\gamma} = \overline{\gamma} + 1 \), where \( \overline{\gamma} \in \text{gen } F \) and \( \overline{\gamma} = \text{sub} (\bigcup E_i \cup (\gamma \cap \text{gen } F)) \) is a limit cardinal in \( N \).

**Proof.** Suppose not.

Let \( \overline{5} \) be a counterexample. There is a unique cardinal \( \mu \) in \( N \) s.t.

\( \overline{5} \leq \mu < \overline{5} + N \). Let \( \sigma = \frac{\overline{3}}{3} \) where \( \overline{3} : N_{\overline{3}} \xrightarrow{\sim} N \) is the canonical map (i.e., \( \sigma(\overline{3}(f))(\overline{5}) = \text{pr}(f)(\overline{5}) \)).
for $f \in N\lambda(\kappa^\mu, \kappa^\mu), \exists \xi < \mu$. Choose $\sigma$ by

(1) $\sigma(\mu) = \mu$,

since otherwise $\mu \in \gamma \cap \eta = \eta$ is a limit cardinal in $N$ and $\mu = \delta$. Hence $N_\delta$ is an $\eta$-prespence by Lemma 3.1. Contradiction!

Hence:

(2) $\sigma(\delta) = \text{id}$, where $\delta = \mu + N\delta$.

(3) Let $\mu \in \delta < \delta \cap \eta$. $\delta_\eta(\mu) = \mu$.

Set: $\tilde{\delta} = \mu + N\delta$. Then $\tilde{\delta} < \delta$.

Proof: $\delta_\eta(\tilde{\delta}) = \text{id}$ as before. But then $\delta(\delta, \tilde{\delta}) \cap \gamma = \phi$, where $\delta < \delta$ and $\delta = \text{lab}(\delta \cap \eta \cap \eta)$. Hence $\tilde{\delta} < \delta$.

(4) Let $\gamma \leq \delta < \mu$. Then $(\bigcup_{\gamma<\delta}^E)^{\lambda_{\gamma}} \neq (\bigcup_{\gamma<\delta}^E)^{\lambda_{\mu}}$.

Proof:

$(\bigcup_{\gamma<\delta}^E)^{\lambda_{\gamma}} \neq (\bigcup_{\gamma<\delta}^E)^{\lambda_{\mu}}$ since $N$ satisfies M.S. But $\mu$ is a cardinal in $N$ and $\mu \in N\mu = \bigcup_{\gamma \in \mu}^E \mu$. $N$ is an $\eta$-prespence by Lemma 2.7. Hence $N\mu \in N$ and $\mu$ is a cardinal in $N\mu$.

Hence $(\bigcup_{\gamma<\delta}^E)^{\lambda_{\mu}} = (\bigcup_{\gamma<\delta}^E)^N$.

QED (4)
But $N_3$ does not satisfy MIS. Hence there is a $\xi < \tilde{\xi}$ such that $(\bigcup_{\xi+} E) N_3 = (\bigcup_{\tilde{\xi}} E) N_3$.

By (3), (4) there is only one possibility — i.e., $\tilde{\xi} = \bar{\mu}$.

(5) $\tilde{\mu} = \tilde{\xi}$ and $\bigcup_{\tilde{\mu}} E N_\mu = \bigcup_{\tilde{\xi}} E N_3$

where $\mu = \text{crit} (\bar{\mu})$.

But since $\tilde{\xi}$ is a counterexample, we have $\tilde{\xi} > \mu + 1$. Hence by (3):

(6) $\tilde{\mu} > \mu + 1 = \bar{\mu} + \text{N}_{\bar{\mu} + 1}$

But then $\text{N}_{\mu + 1}$ satisfies $\xi$-MIS by (6).

Hence $\text{N}_{\mu + 1} \in \text{N}$ in an $\xi$-premouse.

Since $\tilde{\mu} = (\text{N}_{\mu + 1})_{\bar{\mu}}$ is an $\xi$-premouse we conclude $\text{N}_{\mu} \in \text{N}_{\mu + 1}$. But $\bar{\mu} \bar{\mu} \in \mu$ in $\text{N}_{\mu + 1}$. Hence

$\tilde{\mu} = \mu + \text{N}_{\mu} < \bar{\mu} + 1$. Contr.$$

Q E D$ (Lemma 3.2)

(Thus $N_3$ can only fail to be an $\xi$-premouse in an unlikely seeming case. Martin Zeman has shown, however, that this case does occur at the level "strong past a measurable".)
Cor 3.3 Let \( \gamma \) be a limit point of \( \text{gen}_F \) where \( N = (J, E, F, \gamma) \) is an active preshove.
Then \( \gamma \) is a limit point of \( C_N \).

(Note \( \min(\text{gen}_F) = \nu_1 \), but if \( \exists \delta > \nu_1 \), \( \delta \in \text{gen}_F \), then \( \delta > \nu_1 \).)

Def An active preshove \( N = (J, E, F, \gamma) \) is of type 1 iff \( \tau(N) = \tau_1 \).

\( N \) is of type 2 iff \( \tau(N) = \delta + 1 \) for some \( \delta \).

\( N \) is of type 3 iff \( \tau(N) \) is a limit ordinal \( > \tau_1 \).

[Note if \( N \) is of type 2, there is a maximal \( \gamma \in C_N \) s.t. \( \gamma \leq \delta \), min \( \delta \) if \( \delta \) is a limit point of \( \text{gen}_F \), then \( \delta \in C_N \). Moreover, if \( \delta \in C_N \) and \( \gamma = \text{sup}(\delta \setminus \text{gen}_F) \), then \( \gamma \) is a limit cardinal in \( N_1 \), \( \gamma \in \text{gen}_F \), and \( \gamma = \gamma + N_3 \), where \( N_3 = N_{\gamma + 1} \) and \( J_{EN_3} = J_{EN} = J_{EN_3} \).]
Lemma 3.4 Let \( N \) be of type 3. Then 
\[
\varphi^N = \varphi^N.
\]

Let \( \varphi = \varphi^N \).

\[
\varphi^N \leq \varphi; \text{ since } h^N(\varphi) = N.
\]

We prove \( \varphi \geq \). Let \( \gamma \leq \xi \) and let \( \alpha \in \Sigma_1^N(\varphi) \).

Since \( N_\gamma = N \), we have \( N = \bigcup \gamma \varphi_\gamma \gamma \leq \alpha \),

where \( \varphi_\gamma : N_\gamma \rightarrow N \) is the canonical map (i.e. \( \text{rng } \varphi_\gamma = \text{the set of } \pi(f) \mid \alpha \leq \gamma \)),

\[\pi : J^E_{\varphi} \rightarrow J^E \text{ and } N = \langle J^E, F \rangle,\]

Pick \( \gamma > 3 \) such that \( \varphi \in \text{rng } \varphi_\gamma \), where \( \alpha \in \Sigma_1^N(\varphi) \).

Then \( \alpha \in \Sigma_1^N(N_\gamma) \)

in \( \varphi^{-1}(\varphi) \), where \( N_\gamma \in N \).

Hence \( \alpha \in \Sigma_1^N \).

QED (3.4)

Note. Still code \( N = \langle J^E_{\varphi}, F \rangle \) of type 3 by \( \tilde{N} := \langle J^E_{\varphi}, \tilde{F} \rangle \), where \( \tilde{F} = \{ (x, d) \mid d \leq \varphi(x) \} \),

where \( \varphi = h(N) \). \( \tilde{N} \) is then an amenable structure and is essentially the same as the reduct \( N^\varphi = N^1 \otimes \varphi \), since \( \varphi \).

\( \Sigma_1^N(\tilde{N}) = \Sigma_1^N(N^\varphi) \).
\[ f = f(g_1, g_2, \ldots, g_n) \]

where \( g_i \in (k, \mathbb{F}_p) \) and \( \omega(g_i) \in \mathbb{F}_p \).

Hence, \( \mathbb{F}_p \leq \mathbb{F}_q \).

Since \( \mathbb{F}_p \leq \mathbb{F}_q \), we have:

\[ \mathbb{F}_p \leq \mathbb{F}_q \]

Then, if \( \exists \alpha \in \mathbb{F}_p \) and \( \beta \in \mathbb{F}_q \),

\[ \mathbb{F}_p \leq \mathbb{F}_q \]

Let \( \mathbb{F}_p \leq \mathbb{F}_q \).

Hence, \( \mathbb{F}_p \leq \mathbb{F}_q \).

Since \( \mathbb{F}_p \leq \mathbb{F}_q \), we have:

\[ \mathbb{F}_p \leq \mathbb{F}_q \]

Thus, \( \mathbb{F}_p \leq \mathbb{F}_q \).

Hence, \( \mathbb{F}_p \leq \mathbb{F}_q \).

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Since \( \mathbb{F}_p \leq \mathbb{F}_q \), we have:

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\[ \mathbb{F}_p \leq \mathbb{F}_q \]

Thus, \( \mathbb{F}_p \leq \mathbb{F}_q \).

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Since \( \mathbb{F}_p \leq \mathbb{F}_q \), we have:

\[ \mathbb{F}_p \leq \mathbb{F}_q \]

Thus, \( \mathbb{F}_p \leq \mathbb{F}_q \).

Hence, \( \mathbb{F}_p \leq \mathbb{F}_q \).

Since \( \mathbb{F}_p \leq \mathbb{F}_q \), we have:

\[ \mathbb{F}_p \leq \mathbb{F}_q \]
Lemma 3.6 Let $\sigma : N \rightarrow N'$. Then
\[
\sup \sigma \circ \sigma (N) \leq \sigma (N') \leq \sigma (\sigma (N)).
\]
\[\text{pf.}\]
\[
\sup \sigma \circ \sigma (N) \leq \sigma (N') \text{ follows as before.}
\]
But the condition $\exists x \in \text{uniformly } TT_2 \in \exists \ ( \exists x \in \Lambda s \exists 3 \ s \in \text{gen } F).
\]
Hence $\sigma (\sigma (N))$ satisfies this condition in $N'$.
\[\text{QED (3.6)}\]

Lemma 4.1 Let $\pi : N \rightarrow N'$, where
\[N = \langle \cup E, F \rangle \ \text{is of type 1 and crit}(G) < \sigma (N). \]
Then $N'$ is of type 1.
\[\text{pf.}\]
By Lemma 3.5.

Lemma 4.2 Let $\pi : N \rightarrow N'$, where
\[N \ \text{is of type 1 and crit}(G) < \sigma (N).
\]
Then $N'$ is of type 1.
\[\text{pf.}\]
By 4.1 if crit$(G) = \sup N$. Otherwise by 3.6 and $\sigma (N) = \exists \nu \ \text{ (hence} \sigma (\sigma (N)) = \exists \nu \ \text{in } N').
\[\text{QED (4.2)}\]

*Def* Let $N = \langle \cup E, F \rangle$ be of type 2.
\[\tilde{q}_N = \max \tilde{C}_N^* \]
\[\text{With respect to } N.\]
Lemma 4.3 Let \( \pi : N \to \sigma N \) where \( N \) is of type 2 and \( \text{crit}(\pi) \leq \sigma N \). Then \( N \) is of type 2, \( \pi(\sigma N) = \sigma N', \) and \( \pi(\varphi) = \varphi N \).

Let \( N = \langle \cup E, F \rangle, N' = \langle \cup E', F' \rangle \). Then \( n = \omega + 1 \) for \( \omega = \max(\text{gen} N) \) and \( n' = \pi(\omega) + 1 = \sigma(\omega) \) by Lemma 3.5. We note:

Lemma 4.3.1 Let \( \gamma < \sigma N \). Then \( \gamma + \sigma N \gamma \leq \gamma + N \).

Let \( \delta < \gamma + \sigma N \gamma \). Let \( \alpha < \gamma \) code a well ordering of type \( \delta \). Then \( \alpha = \sigma(\gamma) N \gamma \in T \)

\( \text{QED (4.3.1)} \)

Now let \( \gamma = \max \sigma N \). Then \( \gamma' = \sigma'(\gamma) = \ell_{\omega \beta}(\text{gen} \sigma(N\gamma)). \) Moreover \( E_{\text{ht}} \sigma(N\gamma) = E' \) since the same \( T_{1} \) statement holds at \( \gamma \), \( \sigma(N\gamma) \) in \( N \). Moreover, \( \gamma < \sigma N \) and \( \sigma(N\gamma) \) is an \( \tau \)-pomove. Hence:

1) \( \gamma' \in \sigma N' \) and \( \sigma(N\gamma) = N' \gamma \)

Claim \( \gamma' = \max \sigma N' \).

Set \( \delta = \text{crit}(\gamma'), \delta' = \pi(\delta) \). Then \( \delta = \gamma \delta \gamma' \)

Case 1 \( N\gamma + 1 = \gamma' \)

Then \( \delta' = \gamma' ; \delta' = \gamma' + 1 \).

Case 2 \( N\gamma + 1 = N \gamma \) (i.e. \( \varphi(\gamma) = \gamma' \).
Then $\gamma = \pi_2 (f_1 (\delta))$ where $\delta < \gamma$, $f_1 : (\mu \cap \alpha) \cap \kappa \to \kappa$, and $\pi_2 : E \to \mathcal{P}(\kappa^+)$. Hence we have:

\[ \delta \in \pi \iff \delta < \gamma. \]

Hence the same $\Delta_1$-statement holds of $\pi (\kappa^+)$, $\pi (\delta)$, $\gamma$ in $\kappa$. Hence $\kappa \cup 1 = \kappa$ and $\alpha (\sigma, \kappa) = \kappa = \gamma + N'\gamma'$, where $\sigma, \kappa = \gamma + 1$.

**QED** (Case 2)

**Case 3.** The above fail.

$\kappa \cup 1$ does not satisfy $\text{MIS}$. Hence $\gamma \in \text{gen} N$, $\kappa$ is a limit cardinal in $N$ and $(\bigcup E)^N \gamma = (\bigcup E)^N \gamma$.

Thus $\delta \in \text{crit} (\gamma + 1, \kappa = \gamma + 1$ and $\kappa = \gamma + 1$.

At this point to show:

Claim 1: $(\bigcup E)^N = (\bigcup E)^N = \bigcup E$.

Claim 2: $\delta \in \gamma + N' \gamma + 1$. Then $\delta \gamma = \delta \gamma$.

Similarly:

**Lemma 4.4.** Let $\pi : N \to N', \gamma \in \kappa$.

Then $N'$ is of type 2 and $\text{crit} (\delta) = 1$. Then $N'$ is of type 2.

$\pi (\bigcup E) = \bigcup E$, $\pi (\kappa^+) = \kappa^+$. 

**QED** (Lemma 4.3)
Lemma 4.5. Let $N = \langle J^E, F \rangle$ be of type 3. Let $x = \pi(N) = \omega^\alpha_N$. Let $\delta: J^E \to J^E'$, where $G$ is an extender on $J^E$. Let $\delta': N \to N'$ be the canonical completion of $\delta$. Then $N'$ is an $\alpha$-premouse of type 3 and $\delta' = \pi(N)$.

Proof. Since $N = h_N(x')$, we have $N' = h_{N'}(x')$ hence:

1. $\delta' = \omega^\alpha_{N'}$.
2. $\delta' = \pi(N)$.

Proof. Let $x \in N$. Claim $x = \pi'(f)(x)$ for an $a < \delta'$ and an $f \in (k_{\nu'}, k_{\nu'})^{\lambda(N')}$ where $\pi': J^E' \to J^E'$ (and $N' = \langle J^E', F' \rangle$).

Let $x = h_{N'}(i, \delta')$, where $\delta < \delta'$, be positioned. Let $\delta < \delta(N')$, where $\delta < \delta'(3)$. The statement: $E|\bar{z} = E|\bar{z}$.
\[ \Pi_\delta' (N') \text{ in } N_3', \text{ hence the corresponding statement holds for } \delta'(N_3'), \delta'(3). \]

Hence \( N_3' \cap N_3 = \delta'(N_3) \). But

\[ \gamma_3 : N_3' \delta'(3) \to N' \text{ and } \gamma_3(x) = y. \]

Let \( x = \gamma_3^{-1}(x) = h' \delta'(3), \text{ Then } N_3' \delta'(3) \]

\[ \bar{x} = \Pi_\delta'(f)(x) \text{ for some } f \in \mathcal{N}(h', h') \cap N \]

and some \( x < 3 \), where

\[ \Pi_\delta'(3) : \bigcup_{\mathcal{E}'} \rightarrow \bigcup_{\mathcal{E}'} \]

Hence \( x = \gamma_3^{-1}(\Pi_\delta'(f)(x)) = \)

\[ = \Pi_\delta'(f)(x), \text{ QED (2)} \]

(3) \( N' \) is of Type 3.

It suffices to show that \( N_3' \cap N' \)

for arbitrarily large \( 3 < x' \), since

then \( N_3' \cap N' \) for all \( 3 < x' \). But

\[ N_3' \delta'(3) \cap N' \]

\[ \text{QED (Lemma 4.5)} \]

(Note: This means only \( \delta_1 \to \delta_2 \) can be...)
Lemma 4.6 Let $N = (\mathcal{E}, \mathbf{F})$ be of type $3$

Let $\delta : N \to N'$ be of $G$ on $N$, where $\text{crit}(G) < \omega_1(N)$,

Then $N'$ is of type $3$. Moreover,

$$\lambda(N') = \begin{cases} \sup \delta^\omega \lambda(N) \text{ if } \text{crit}(G) < \omega_1^N \\
\delta(\lambda(N)) \text{ if not} \end{cases}$$

Proof.

Set $\lambda = \lambda(N)$. Assume $\text{crit}(G) \geq \omega_1^N$. The result follows by Lemma 4.6.1. So assume $\text{crit}(G) < \omega_1^N$. Let $\lambda' = \delta(\lambda)$.

Then $\lambda' = \omega_1^N$. Hence $\lambda \leq \lambda(N')$.

The statement: $\Lambda \lambda < 2 \forall \lambda \in N$

is $\Pi^1_2(N)$ in $\mathfrak{m}$, $\mathfrak{m}_\lambda$, since it can be written as:

$$\Lambda \mathcal{F} \lambda \in \forall e \in \text{a function } e \in \text{dom}(e) \subseteq \mathcal{F}(\mathfrak{m}) \wedge$$

$$\forall x^0 \forall y^o \left( y^o = F(x^0) \rightarrow y^o \wedge \overline{\lambda} = e'((x^0)) \right)$$

$$\Pi^1_1$$
But $\delta \in \Pi_2^{(1)}$ preserving $i$ hence $F/\delta \notin N'$ for $3 < \alpha$. Hence $N_3 \notin N'$ for $3 < \alpha$ and it follows that $\alpha' = \chi(N')$. It also follows from this that $N'$ is of Type 3.

Q.E.D (Lemma 4.5.6)

Putting these together:

**Lemma 4.7** Let $N$ be an active $i$-premouse

Let $\pi : N \rightarrow N'$, where $\text{crit}(\delta) < \alpha$

Then $N'$ is an active $i$-premouse of the same type. Moreover:

(a) If $N$ is of Type 1 or 2, then $\pi(\alpha (N)) = \alpha(N')$

(b) If $N$ is of Type 2, then $\pi(\delta_N) = \delta_{N'}$

(c) If $N$ is of Type 3, then

\[ N'(1) = \begin{cases} \sup \pi^\alpha \alpha(N) & \text{if } \alpha \geq \text{crit}(\delta) \\ \pi(\alpha(N)) & \text{if } \alpha < \text{crit}(\delta) \end{cases} \]
By the remark following the proof of Lemma 3.2, we know that it is possible to have: \( \pi: \overline{N} \rightarrow N \), where \( N \) is an active \( \Sigma \)-premorse and \( \overline{N} \) is not. Nonetheless we do get:

**Lemma 5.1** Let \( N \) be an active \( \Sigma \)-premorse of type 1, where \( \sigma: \overline{N} \rightarrow N \). Then \( \overline{N} \) is an \( \Sigma \)-premorse of type 1.

**Proof:** \( \overline{N} \) is clearly a ppm. But \( \exists \xi \in \mathcal{N} \) for \( \exists \geq \overline{\mathcal{N}} \), since \( \exists \sigma(f) \mathcal{E} \mathcal{N} \). QED (5.1)

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**Lemma 5.2** Let \( N \) be of type 2, where \( \sigma: \overline{N} \rightarrow N \) and \( \xi \in \mathcal{R} \sigma(\overline{\mathcal{N}}) \). Then \( \overline{N} \) is of type 2 and \( \sigma(\xi^{-\overline{\mathcal{N}}} N) = \xi^{\overline{\mathcal{N}}} \).

**Proof:** Let \( N = \langle J^E N, F \rangle; \overline{N} = \langle J^E \overline{N}, F \rangle \). \( \overline{N} \) is then a ppm. Let \( \sigma(f) = \xi = \xi^\overline{\mathcal{N}} \). Let \( x = x_N = \xi + 1 \). We consider two cases!
Case 1. \( N_3 \) satisfies \( \alpha \)-MIS.

Then \( N_3 \in N \) and \( \bar{q} = \text{F} \bar{1} \bar{3} \). Hence \( \bar{s} = q(K_N) \in \text{rng}(\bar{s}) \). Let \( \delta(\bar{s}) = \bar{s} \).

Then \( \bar{s} \in \text{gen} F \), since \( \bar{s} \in \text{gen} F \).

Hence \( \bar{s} + 1 = \bar{s} = \delta^{-1}(\bar{s}) \leq \delta(\bar{N}) \). But \( \delta(\bar{N}) = \delta(\bar{s}) \)

If \( \bar{s} \geq \bar{s} \), then \( \bar{s} \geq \bar{N} \) and hence \( \bar{s} \in \text{gen} F \). Hence \( \bar{s} \in \text{gen} F \).

Hence \( \bar{s} \leq \bar{N} \). \( N_3 \) is coded by \( q \); hence \( N_3 \in \text{rng}(\bar{s}) \). Let \( \delta(\bar{N}') = N_3 \).

Then \( \bar{q} = F \bar{N}' / \bar{s} = F \bar{N}' / \bar{s} \) and

There is \( \bar{a}' : \bar{J}_{\bar{E}'} \rightarrow \bar{J}_{\bar{E}'} \).

where \( \bar{N}' = (\bar{J}_{\bar{E}'} : F') \). Hence \( \bar{N}' = \bar{N}' \) and \( \bar{q} = q_{\bar{N}} \). Hence \( N_3 \in N \) for \( s \leq \bar{s} \), where \( s(\bar{N}) = s + 1 \). Hence \( N \) is an \( \alpha \)-premonoe.

QED (Case 1)

Case 2. Case 1 fails.

Let \( \gamma = \max \mathcal{C}_{\bar{N}} = \max \text{gen} F \cap \bar{3} \).

Then \( \gamma \) is a limit cardinal in \( N \)

QED (Case 2)
and \((J^E \gamma)^N = (J^E \gamma^+)^{N_3}\), where 
\(\gamma = \gamma + N_3\). Moreover, \(N_3 = N_\gamma + 1\).

Then \(\gamma = F1\gamma = F N_\gamma \gamma\) and 
\(\gamma = 1 (N_\gamma)\). Since \(\gamma = \gamma + N_\gamma\) and 
\(N_\gamma\) is contracted from \(\gamma\), we have \(N_3 \in \text{rng} (\sigma)\). Let 
\(\sigma (\gamma, \overline{\gamma}, N', \overline{\gamma} - 1 = \gamma + 3, N_\gamma\) if \(N'\).

At follows exactly as in Case 1 that 
\(\overline{\gamma} = \overline{\gamma} + 1 = 2 (\overline{N} - 1)\), where \(\sigma (\overline{\gamma}) = 1\).

At also follows as before that 
\(\overline{N}' = \overline{N}_N\) and \(\overline{\gamma} = \gamma + \frac{N}{N}\). Hence 
\(\overline{N}_N \in \overline{N}\) for all \(\gamma \leq \gamma\), where 
\(3 \in \text{gen } F\) and \(\gamma = \max (\overline{\gamma} + \text{gen } F\).

An order to establish that \(\overline{N}\) is an 
\(\gamma\)-premome, it remains only to show 

Claim \(\overline{N}_N\) does not satisfy \(\gamma\)-mis

Proof.
Since \(\gamma < \overline{\gamma} < \gamma + N\), we know 
that \(\overline{\gamma} - 1 \overline{\gamma} = \text{id}\) with 
\(\overline{\gamma} : \overline{N}_N \to \overline{N}\) is the canonical map.
But $\bar{\gamma} \in \text{gen} F$, hence $\bar{\gamma} = \alpha \Gamma (\bar{\sigma})$, hence $\bar{\gamma} = \bar{\sigma} + \bar{\nu} \bar{\gamma}$.

Hence $\bar{\gamma} = \bar{\sigma} + \bar{\nu} \bar{\gamma}$, hence $\bar{\nu} \bar{\gamma} = \bar{\nu} \bar{\gamma}$, hence $\bar{\nu} \bar{\gamma}$ does not satisfy MIS. Q.E.D (Lemma 5.2)

For type 3 $\sigma$-premise the best we can do is:

**Lemma 5.3** Let $N$ be a type 3 $\sigma$-premise.

Let $\sigma : \bar{\nu} \rightarrow \bar{\gamma}$. Then $N \vdash *$ is a type 3 $\sigma$-premise.

**Proof.** Let $\sigma = \bar{\nu} (N) = \omega p^N$.

Then $\sigma^{-1} \in \omega p^N \leq \sigma (\bar{\nu} 1)$. But

for $\bar{\gamma} \in \bar{\gamma} = \sigma^{-1} \in \omega p^N$ we have:

$F^N | \bar{\gamma} \in N$, where $\bar{\gamma} = \sigma (\bar{\gamma})$. But

$F^N | \bar{\gamma} \in N$ in $\Sigma_1$ in $\bar{\gamma}$. $K_N$ can be expressed by:

$$\forall e \in \text{a function } \land x \in \text{dom}(e) \land e^e$$

$$\land \exists x \forall y_0. \exists x^0 (x^0 \rightarrow F(x^0) \rightarrow y_0 \land \bar{\gamma} = e^e (x^0))$$

Hence $F^N | \bar{\gamma} \in N$. At follows easily that $\bar{\gamma} = \omega (\bar{\nu} 1)$ and that $\bar{\nu} \bar{\gamma}$ is a type 3 $\sigma$-premise.

Q.E.D (5.3)
In some contexts it is useful to augment the structure \( N \), replacing \( N \) by \( \langle N, \exists \vec{z} \rangle \) if \( N \) is of type 2, and by \( \langle N, \emptyset \rangle \) if \( N \) is of type 1. For the so augmented \( \Sigma \)-premise, we then have:

- If \( \sigma : \overline{N} \rightarrow N \) and \( N \) is of type 1 or 2, then \( \overline{N} \) is of the same type.
- If \( \sigma : N \rightarrow N' \) and \( N, N' \) are \( \Sigma \)-premises, then they are of the same type.

We adopt this convention when defining the notions "standard parameter", "witness" (to \( \varphi_N \), where \( \varphi_N \) is the standard parameter) and "solidity".

The above lemmas then enable us to carry through the proof of solidity. (On the other hand, some versions of the condensation lemmas refer to the original non-augmented premise.)
In [CR] §1 ("Correction and Remark") we sketched a similar development for general premouse. In their case the initial segment condition is simply the principle IS stated earlier. For a premouse $N = \langle \mathcal{E}, P \rangle$ we then set:

$$C_N = \{ \lambda' \in \mathcal{V} : \exists \xi < \lambda' < \lambda \text{ and } \lambda' = \lambda \eta \} \quad (\text{i.e. } N_{\lambda'} \text{ has the form } \langle \mathcal{E}', F1\lambda' \rangle).$$

$N$ has type 1 if $C_N = \emptyset$. $N$ has type 2 if $\max C_N$ exists. ($C_N$ is easily seen to be closed in $\lambda_N$). If $\sup C_N = \lambda_N$, then $C_N$ has type 3.

For $N$ of type 2 we set $\gamma_N = F1\lambda'$, where $\lambda' = \max C_N$.

We then get:

1. If $\pi : N \rightarrow G$, exist $G < \lambda_N$, and $N$ is a premouse, then $N'$ is a premouse of the same type. Moreover $\pi(q_N) = q_{\lambda'}$ if $N$ has type 2.
(2) $\forall \sigma : \bar{N} \rightarrow \mathcal{Z}, N$ and $N \uparrow$ of type 1, then $\bar{N}$ is of type 1.

(3) $\forall \sigma : \bar{N} \rightarrow \mathcal{Z}, N$ is of type 2 and $\sigma(<) \in \text{rng}(\sigma)$, then $\bar{N}$ is of type 2 and $\sigma(\sigma^{-1}) = \sigma_N$.

(4) $\forall \sigma : \bar{N} \rightarrow \mathcal{Z}$, $N$ and $N \uparrow$ of type 3, then $\bar{N}$ is of type 3.

(Note: We also have $\lambda \sigma : \bar{N} = \text{wp}^1 \sigma \bar{N}$ if $\bar{N}$ is of type 3.)

For the purpose of defining and proving solidity, we then augment the premise in the same way as above.

As these facts suggest, $\alpha$-mice and general premice are special cases of a more general theory. There are in fact many interesting premouse classes lying between these two extremes. We develop