§ 2 T-premise

In the following let $T$ be a function on active ppms $N \ni t$, $T_N = T(N) \subseteq [\lambda_N, \lambda_N]$ is closed in $\lambda_N + 1$ and is uniformly $\mathcal{J}_{\lambda_N}^{EN}$-definable in $\lambda_N$.

**Def** Let $N$ be an active ppm.

$t_N = t(N) = \left\{ \begin{array}{ll}
\min \{ \exists \exists \in T_N \mid \exists \leq \exists \} & \text{if such a } \exists \text{ exists}, \\
\lambda_N & \text{if not}
\end{array} \right.$

**Def** Let $N$ be an active ppm.

$C_N = C_N^{T} = \{ \exists \subseteq t_N \mid \exists \leq E_{\lambda_N} \}$

**Def** Let $N$ be an active ppm.

$C(N) = C_{t}(N) = \mathcal{C}_{M}^{T}(N) = \mathcal{C}_{M}^{T}(N) \subseteq \mathcal{C}_{M}^{T}(N)$

**Def** A ppm $M$ satisfies the minimal $T$-initial segment condition ($T$-MIS) iff whenever $E_{\lambda}^{M} \neq \emptyset$, $N = M_llv$, and $\exists \in C_{N}$, then

$(J_{\exists^{+}})^{E} \neq (J_{\exists^{+}})^{N_{\exists}}$

**Def** $C_N = \{ \exists \in C_N \mid N_{\exists} \text{ satisfies } T$-MIS $\}$

Similarly for $C(v) = C_{M}^{T}(v) = n_{d} C_{M}^{T}(v)$.
Def. A ppm $M$ satisfies the $T$-initial segment condition ($T-IS$) iff whenever $E_x^M \neq \emptyset$, $N = M|x$ and $x \in C_N$, then

(a) $N^3 \subseteq N$
(b) $\forall \bar{z} \in \exists x N_N \Rightarrow N^3 \subseteq N^3$

In this case we also call $M$ a $T$-promice

If we take $T_N = \{xN^3\}$, we get general promise. (For this reason we also call them $x$-promice.)

If we take $T_N = \emptyset$, we get the $x$-promice.

However, there are many intermediate possibilities.

Trivially:

**Lemma 1.1** Let $N$ satisfy $T-IS$. Let $T_N \leq \exists < t_N$ s.t. $\exists = t_N^3$ and $N^3$ satisfies $T-MIS$. Then $N^3$ satisfies $T-IS$.

**Lemma 1.2** Let $N$ satisfy $T-IS$. Then $N$ satisfies $T-MIS$.

Proof. Take $S \subseteq N$.

**Lemma 1.3** $(T-MIS) \rightarrow MIS$

Proof:
Suppose not. Let $x' \not\in x$ s.t. $x' = xN^3$, and $(\cup x')^N x' = (\cup x')^N$
Clearly \( x' = x \), \( x' = \omega x (x', 1, u) \), and \( x' \) is a limit cardinal in \( N \) and \( x' \notin \text{gen}(\bar{N}_x) \). Then \( x' < t_{\bar{N}_x} \).

But \( t_{\bar{N}_{x'}} < x' \), and \( N \) satisfies \( T\)-\( MIS \). Let \( \bar{s} = t_{\bar{N}_{x'}} \). Then

\[ \bar{N}_{x'} = \bar{s} \cap \bar{N}_{x'} \subseteq \bar{s} \]

and hence

\[ N_3 = \bar{N}_{x'} \cap \bar{s} = \bar{s}_{x'} \]. But then

\[ (\bigcup_{\bar{s} \in \mathcal{E}})^{N_3} = (\bigcup_{\bar{s} \in \mathcal{E}})^{N_{x'}} = (\bigcup_{\bar{s} \in \mathcal{E}})^{N} \].

\( \text{QED (1.3)} \)

**Lemma 1.4** Let \( t_{\bar{N}} \leq y < t_{\bar{N}} \) w.r.t. \( y \) is a cardinal in \( N = (\bigcup_{\bar{s} \in \mathcal{E}})\) and \( N \) is an active \( T\)-premonoid. Then \( N_3 \) is an \( \mathcal{E} \)-\text{premonoid},

Let \( \bar{s} < t_{\bar{N}_3} \) w.r.t. \( \bar{s} = t_{\bar{N}_3} + N_3 \) satisfies \( T\)-\( MIS \). Then \( \bar{s} < t_{\bar{N}_3} \leq y \), and so otherwise \( N_3 = \bar{N}_3 \), \( \bar{s} < t_{\bar{N}_3} < t_{\bar{N}_3} \).

Hence \( N_3 \in \mathcal{E}^N \subseteq \bar{N}_3 \). Similarly \( y \), \( \bar{s} < \bar{s}' < t_{\bar{N}_3} \), and \( N_3 \) satisfies \( T\)-\( MIS \). Let \( N_3' \notin \bar{N}_3 \), since \( N \) satisfies \( T\)-\( IS \). \( \text{QED} \).
Lemma 1.5  Every T-premonoe is a premooe.

Let $N = \langle S_i, E_i, F \rangle$ be an active T-premooe.
Let $\mathcal{E}_N \subseteq \mathcal{L}_N$ s.t. $\mathcal{L} = \lambda N\mathcal{L}$.

Claim $F(\lambda') \in N$.
Clearly $\mathcal{F}_x(\lambda') = \lambda$, $\lambda' = \text{crit}(\mathcal{F}_x)$.
Hence $\lambda'$ is a limit cardinal in $N$. But,

Hence $\lambda' = \text{crit}(\mathcal{F}_x)$, $\mathcal{F}_x \subseteq N_{\lambda'}$.

Hence $N_{\lambda'} = N_{\lambda}$, satisfies T-MIS
by Lemma 2.2 and Lemma 2.4. Hence $N_{\lambda'} \in N$.

Q.E.D (Lemma 1.5)

Lemma 2.1  Let $N$ be a tapi.

Then $\mathcal{E}_N'$ is closed in $tN$.

Proof
Let $y$ be a limit pt. of $\mathcal{E}_N$.

Case 1  There is no $y' \in y$ s.t. $y' \in \mathcal{E}_N$.

Case 1.1  $y = A_N y$

Then $y = tN y = A_N y$.

Case 1.2  $A_N y < y$.

Then $A_N y = A_N (y_n \wedge y_{n+1})$, s.t.

Let $y \in \mathcal{E}_N \wedge y$ s.t. $y < y$.
Then \( xN_3 = 1 \) and \( 3 = t_{N_3}^\ast > 2 \). Hence \( 3 = \text{the least } \bar{3} \in T_{N_3} \text{ s.t. } \bar{3} > 2 \).

Now let \( \bar{3} < \bar{3}' \in \bar{C}_{N_3} \). Then \( \bar{3}' = \text{the least } \bar{3} \in T_{N_3} \text{ s.t. } \bar{3} > 2 \).

Since \( \bar{3} \), \( \bar{3}' \), \( 1, 2 \) are all \( 3 \) and \( \bar{3} \),

\[
\bar{3} \bar{3}' : \bigcup_{\bar{3} \in N_3} \bigcup_{\bar{3}' \in N_3'} \subseteq \bigcup_{\bar{3} \in N_3} \bigcup_{\bar{3}' \in N_3'}
\]

we have:

\[
\bar{3} \bar{3}'(\bar{3}) = \bar{3}', \text{ where } \bar{3} \bar{3}' : \bar{3} = \text{id}.
\]

At follow that:

\[
\bar{3} \bar{3}'(\bar{3}) \geq \bar{3}' \bar{3} \bar{3}'(\bar{3}) \geq 3
\]

for \( \bar{3}' \in \bar{C}_{N_3} \) with \( \bar{3} > 3 \). Hence

\[
\bar{3} \bar{3}'(\bar{3}) \geq 3 \text{ and } \bar{3} \bar{3}'(\bar{3}) = \text{the least } \bar{3}' > 2 \text{ s.t. } \bar{3}' \in T_{N_3}.
\]

Case 2. Case 1 faith.

Case 2.1. There are arbitrarily large \( \bar{3} \in \bar{C}_{N_3} \) with \( \bar{3} = \text{exit } (\bar{3} \bar{3}'). \)

Let \( X = \text{the set of such } \bar{3} \).

Hence \( \bar{3} \bar{3}'(\bar{3}) > 3 \) for \( \bar{3} \in X \).
Hence \( N \) by claim (1).

\[ 3 = \frac{1}{\sqrt{N}} \Rightarrow \sqrt{N} = 3 \]

Case 2.2: Case 2.1 fails, we have \( 3 \notin T_N \) and hence \( 3 \notin N \).

Case 2.1: For sufficiently large \( N \),

\[ 3 \in T_N \Rightarrow \exists k \in \mathbb{N} \text{ s.t. } 3^k \equiv 1 \pmod{N} \]

\[ \Rightarrow 3 \in \mathbb{F}_{N^2} \]

\[ \Rightarrow 3 \notin T_N \]

Therefore, \( 3 \notin N \).

Case 1: \( N \) has a limit point in \( T_N \), hence \( 3 \notin N \).

\[ \Rightarrow 3 \notin \mathbb{F}_{N^2} \]

\[ \Rightarrow 3 \notin T_N \]

\[ \Rightarrow 3 \notin N \]

Hence \( N \).
Hence \( N^c_\gamma = \text{sub}(\gamma \cap \text{gen} N) = \gamma \).

Hence \( \gamma = t. N^c_\gamma \). \( \square \) \text{ E.D. (Lemma 2.1)}

**Lemma 2.2** Let \( N \) be an active \( T \)-premome. Then \( C_N \) is closed in \( t_N \).

**Proof.**
Let \( \gamma < t_N \) be a limit pt. of \( C_N \). Then \( \gamma \in C_N \) by Lemma 2.1. Repeating the proof of \( \S \) Lemma 3.1 we then get:

\( N^c_\gamma \) is in a \( T \)-premome. \( \square \) \text{ E.D. (2.2)}

**Lemma 2.3** Let \( N = \langle J, \mathcal{E}, F \rangle \) be an active \( T \)-premome. Let \( z \in C_N \setminus C_{N^c} \). Then \( z \) is the \( C_N \)-successor of \( \gamma \in C_N \) where \( \gamma \in \text{gen} N \) is a limit cardinal in \( N \).

**Proof.** Suppose not.
Let \( z \) be the least counterexample.

Let \( \mu = \) the unique cardinal in \( N \) s.t.
\( \overline{\gamma} \leq \mu \leq z < \mu + N \). Let \( s = z \).

(1) \( s \downarrow \mu = \mu \),

since otherwise \( \mu \in \text{gen} N \) is a limit cardinal in \( N \) and \( \mu = z \). Hence \( N^c_\gamma \) is in a \( T \)-premome by Lemma 1.4. \( \square \)
(2) \((J_{S^+}^E)^N S \neq (J_{S^+}^E)^N S\) for \(S \in \tilde{C}_N\) and \(\mu > \).

Proof:
\((J_{S^+}^E)^N S \neq (J_{S^+}^E)^N \) \(\mu \Rightarrow \) \((J_{S^+}^E)^N \) \(\mu \Rightarrow \) \((J_{S^+}^E)^N S\)

Since \(N\) satisfies T-MIS and \(\mu > \) is a cardinal in \(N\).

(3) Let \(\mu < \bar{S} < \bar{S}\) with \(\bar{S} \in \tilde{C}_N\). Then \(\mu + N_{\bar{S}} < \bar{S}\).

Proof:
Set \(\tilde{\mu} = \mu + N_{\bar{S}}\).
At \(\tilde{\mu} < \bar{S}\) there is nothing to prove, so we assume not. Then \(\overline{\tilde{\mu}}(\bar{S}) = \mu + \) and hence \(\overline{\mu} = \tilde{\mu}\). Hence \(\overline{\tilde{\mu}}(\bar{S}) = \bar{S}\). At sufficient for a how.

Claim: \([\bar{S}, \bar{S}) \cap \text{gen}_N \neq \phi\)
At \(\bar{S} = \) the least \(S \in \text{T}_{\bar{N}}\) s.t. \(S \supset \text{gen}_N = \).
But \(S = \overline{\tilde{\mu}}(\bar{S}) \in \text{T}_{\bar{S}}\), hence \(\tilde{\mu} = \overline{\tilde{\mu}}\), and \(S \supset \text{gen}_N = S \supset \text{gen}_N\).
Hence \(S = \bar{S} < \bar{S}\) if \(\bar{S} \cap \text{gen}_N \subseteq \bar{S}\). Q.E.D. (3)

(4) Let \(\mu < \bar{S} < \bar{S}\) with \(\bar{S} \in \tilde{C}_N\).
Then \((J_{S^+}^E)^N \bar{S} \neq (J_{S^+}^E)^N \bar{S}\).
Proof:
\(\mu + N_{\bar{S}} = \mu + (J_{S^+}^E)^N \bar{S} \neq \bar{S} \leq \mu + N_{\bar{S}}\).
Q.E.D. (4)
Hence by (2), (4):

(5) \( \mu \in \widetilde{C}_N \) and \((\cup^{E}_{\mu+}) N_\mu = (\cup^{E}_{\mu+}) N_3 \).

(6) \( \exists = \) the immediate successor of \( \mu \) in \( \widetilde{C}_N \)
Proof. Suppose not, let \( \gamma \) be least such that \( \mu < \gamma < \exists \) and \( \gamma \in \widetilde{C}_N \).
Then \((\cup^{E}_{\gamma+}) N_\gamma = (\cup^{E}_{\gamma+}) N_3 \neq (\cup^{E}_{\mu+}) E N_3 \),
and \( \mu + N_\gamma \leq \mu + N_3 < \exists \leq \mu + N_3 \).
\( \exists \) follow as in (2) that \( (\cup^{E}_{\exists+}) N_\exists \neq (\cup^{E}_{\exists+}) N_3 \) for \( \exists \in \widetilde{C}_N \).
Hence \( \exists \) satifies T-MIS. Hence \( N_\exists \in N \) and \( N_\mu \).
Moreover, \( N_\mu \in N_\exists \), since \( N_\mu \in \mu \) and \( N_\mu \in \cup^{E}_{\mu+} \).
Hence \( \mu + N_\gamma < \mu + N_3 \)
Since \( \mu + \exists \leq \mu \) in \( N_\mu \), Hence \( \mu + N_\gamma \leq \mu + N_3 \) in \( N_\mu \).
Hence \((\cup^{E}_{\mu+}) N_\mu \neq (\cup^{E}_{\mu+}) N_3 \), and \( \mu + N_3 \geq 3 > 2 \).
Contrad! QED (6)

(7) \( \mu = \operatorname{crit} (\widetilde{C}_N) \)
Proof. Suppose not, then \( (\cup^{E}_{\mu+}) N_\mu \neq (\cup^{E}_{\mu+}) N_3 \),
\( \mu + N_3 \geq 3 > \operatorname{crit} (\widetilde{C}_N) = \mu + N_\mu \). Contrad!, \( \mu \) (7).
But then $\mu = \text{crit}(\mathcal{A})$. At follows that $\mu \leq \text{gyp}_N$ and $\mu$ is a limit cardinal in $\mathbb{N}$. Hence $\mu \leq C_N$. QED (Lemma 2.3)

As a corollary of the proof:

**Corollary 2.4** Let $\xi$, $\eta$ be as in Lemma 2.3. Then $\eta + N_\xi = \eta + N_\eta$ and $\eta \leq \xi \leq \eta + N_\xi$.

(Hence if $\xi \in T_{N_\eta}$ and $T$ is n.t. $T_N$ is always a class of limit cardinals in $\mathbb{N}$, then the situation of Lemma 2.3 cannot occur.)

**Definition** Let $\mathbb{N} = \langle \mathcal{E}, \mathcal{F} \rangle$ be an active $T$-premonoid.

- $\mathbb{N}$ is of *Type 1* $\iff C_N = \emptyset$
- $\mathbb{N}$ is of *Type 2* $\iff C_N$ has a maximum.
- $\mathbb{N}$ is of *Type 3* $\iff \sup C_N = t_N$

(Note: By Lemma 2.3, every limit pt of $C_N$ is a limit pt of $C_N$.)

**Lemma 2.5** Let $\mathbb{N}$ be of Type 3. Then $\omega^N = t_N = \infty$. (proven like $\xi$ in Lemma 3.4)

**Lemma 2.6** Let $N, N'$ be active p.p.mrs. Let $\sigma : N \to N'$ n.t. $\sup \sigma^{-1}(A_N) \leq A_{N'} \leq \sigma(A_N)$. Then $\sup \sigma^{-1}(t_N) \leq t_{N'} \leq \sigma(t_N)$. Moreover $\sigma(t_N) = t_{N'} \iff A_N < t_N$.

We first prove two auxiliary lemmas.
Lemma 2.6.1 Let $N, N'$ be active ppms.
Let $\sigma : N \rightarrow N'$. Then $\sigma : J_{\Sigma}^E N \rightarrow J_{\Sigma}^E N'$.

Proof.
Let $N = \langle J_{\Sigma}^E, F \rangle$, $N' = \langle J_{\Sigma}^E', F' \rangle$.
Let $\bar{\pi} : N \rightarrow \tilde{N}$, $\bar{\pi}' : N' \rightarrow \tilde{N}'$.
There is $\bar{\sigma} : \tilde{N} \rightarrow \tilde{N}'$ defined by:
$$\bar{\sigma}(\bar{\theta}(\tilde{a})) = \bar{\pi}'(\sigma(\tilde{f})) \in (\sigma(\tilde{a})).$$
The usual methods then show: $\bar{\sigma}(\tilde{a}) = \tilde{a}'$, $\bar{\sigma}(J_{\Sigma}^E) = \tilde{\sigma}$.
Q.E.D (2.6.1)

(Note The proof of 2.6.1 does not require the well-foundedness of $\tilde{N}, \tilde{N}'$.)

Lemma 2.6.2 Let $N, N', \sigma'$ be as above.
Let $\bar{s} \leq \lambda_{N'}$ and $\bar{s} \subseteq \bar{z}$ and $\bar{t} \subseteq \bar{z}$, $T_{N'} \cap [\bar{z}, \bar{z}) = \emptyset$.
Let $\bar{s}' = \text{lub} \bar{\sigma}'(\bar{s})$, $\bar{z}' = \sigma(\bar{z})$.
Then $T_{N'} \cap [\bar{s}', \bar{z}') = \emptyset$.

Proof. Suppose not.
For each $\delta < \bar{s}$ we have:
$N' \not\subseteq \forall \delta \in T_{N'} \sigma(\delta) < \delta < \bar{z}'$.
Hence the same statement holds of $\bar{s}, \bar{z}$ in $N$. Hence there is $\delta \in T_N$ not:
$\delta < \bar{s} < \bar{z}$. Hence $\bar{s} < \bar{s}$. Hence $\bar{s} \in \lim \text{point of} T_{N}$. Hence $\bar{s} \in T_{N'} \cap [\bar{s}, \bar{z})$. Contrad (Q.E.D)
The proof of Lemma 2.6 is immediate. Let \( T_N \cap [a_N, \lambda + 1) = \emptyset \), then \( T_N' \cap [\sup \sigma \tau t_N', \lambda + 1) = \emptyset \) and hence \( t_N' = \lambda N' \). Otherwise \( \sigma(t_N) \in T_N ' \). If \( \lambda N < t_N \), then \( T_N \cap [a_N, t_N) = \emptyset \) and hence \( T_N' \cap [\sup \sigma \tau t_N', \sigma(t_N) \in T_N ' \). Thus \( T_N \cap [a_N, \sigma(t_N)] = \emptyset \)

\( \lambda N' \leq t_N \). Now let \( \lambda N = t_N \). Then \( \lambda N' \leq \sigma(t_N) \leq t_N \), hence \( t_N' = \sigma(t_N) \).

But \( \sup \sigma \tau t_N = \sup \sigma \tau a_N \leq \lambda N' \leq t_N \)

QED (Lemma 2.6)

As a pendant to Lemma 2.6.2 we get:

Lemma 2.6.3 Let \( N, N' \) be as in Lemma 2.6

Let \( \bar{a} \leq \lambda N' \), \( \bar{a} < \gamma' \in N' \) and \( T_N \cap [\bar{a}, \gamma') = \emptyset \).

Set \( \bar{a} = \text{lub} \sigma \tau a \bar{a} \), \( \bar{a} \leq \gamma' = \sigma(\gamma) \).

Then \( T_N \cap [\bar{a}, \gamma] = \emptyset \).

The proof is left to the reader.
Define \( C^*_N = \{ \bar{z} \mid \text{N}_3 \neq \text{N} \land \bar{z} \in \widetilde{T}_{\text{N}_3} \} \)
for active p.p.m. \( \text{N} \).

Note: It is clear that:
- \( C^*_N \subseteq \bar{S}_N \), since \( \text{N}_3 = \text{N} \) for \( \text{N}_3 \leq \bar{z} \).
- \( \bar{z} \in \widetilde{C}_N \) and \( \forall \gamma \geq \bar{z} \), \( \gamma \in \widetilde{T}_{\text{N}_1} \), then \( \bar{z} \in \widetilde{C}^*_N \).
- \( \bar{z} \in C^*_N \), \( \bar{z} = \bar{z}_{\text{N}_3} \), then there is \( \gamma \leq \bar{z} \) s.t. \( \bar{z} = \bar{z}_{\text{N}_3} \) and \( \gamma \in \widetilde{C}_N \).

**Lemma 2.6.4** Let \( \sigma : N \to N' \), where \( N, N' \) are active p.p.m. \( \forall C^*_N \cap [\gamma, \mu] = \emptyset \),
then \( C^*_N \cap [\sigma^{-1}(\gamma), \sigma^{-1}(\mu)] = \emptyset \).

**Proof.** Suppose not.
Let \( \bar{z}' \in C^*_N \cap [\gamma', \mu'] \) where \( \gamma' = \sigma(\gamma) \), \( \mu' = \sigma(\mu) \). Set \( \bar{z} = \text{lab} \sigma^{-1} \mu' \). Then \( \bar{z} \leq \sigma(\bar{z}') \). Define \( \sigma^* : \text{N}_3 \to \text{N}'_3 \) by:
\[ \sigma^*(\pi(f)(a)) = \pi'(\sigma(f)(\sigma(a))) \]
where \( f \in \text{N}(\text{N}'; \mu') \), \( u = \text{N}_N \), \( \alpha < \bar{z} \), and
\[ \pi : \text{J}_E \to \text{J}_E, \quad \pi' : \text{J}_E' \to \text{J}_E' \]
Then \( \sigma^*(\text{J}_3) = \text{id} \).

\[ \bar{z} = \sup \sigma^{-1} \bar{z} \text{. Then } \bar{z} \leq \bar{z} \leq \sigma(\bar{z}) , \]
Since $T_{N,3} \cap \{3, 3+1\} = \emptyset$, we have $T_{N,3} ' \cap \{3, 3+1\} = \emptyset$. Hence $\exists \bar{\sigma} \in T_{N,3} ' \cap \{3, 3+1\}$. Contrad QED (2.6.4)

Lemma 3.1 Let $N$ be of type 1. Let $\sigma : N \to N'$, where $\text{crit}(\sigma) < t_N$. Then $N'$ is of type 1 and $\sigma (t_N) = t_{N'}$.

Proof.

Case 1 $T_N = \emptyset$. Then $T_{N,1} = \emptyset$ by Lemma 2.6.1. Hence $t_N = \tau_N = \tau_{N',1} = t_{N'} = \sigma (t_N)$.

Case 2 $T_N \neq \emptyset$. Then $t_N = \min T_N$. Then $C_N = \emptyset$ and by Lemma 2.3 $C_N = \emptyset$. Hence $C_{N,1} = \emptyset$. Hence by Lemma 2.6.4, where $C_{N,1} \subseteq C_{N,1} \subseteq C_{N,1}$. Hence $N'$ is of type 2 and $t_N = \min T_{N,1} = \sigma (t_N)$. Hence $T_{N,1} \cap \sigma (t_N) = \emptyset$ by Lemma 2.6.2. QED (3.1)

Similarly:

Lemma 3.2 Let $N$ be of type 1. Let $\sigma : N \to N'$, where $\text{crit}(\sigma) < t_N$. Then $N'$ is of type 1 and $\sigma (t_{N'}) = t_N$.
Lemma 3.3 Let \( N \) be of type 2. Let 
\( \sigma : N \rightarrow N' \), where \( \text{crit}(\sigma) < t_N \). Then 
\( N \) is of type 2 and \( \sigma(t_N) = t_N' \).
Moreover, if \( \bar{s} = \max C_N \), \( \bar{s}' = \max C_{N'} \), then 
\( \sigma(\bar{s}) = \bar{s}' \) and \( \sigma(N_{\bar{s}}) = N'_{\bar{s}'} \).

**Proof:**

**Claim 1** \( \sigma(t_N) = t_N' \).

**Case 1** \( T_N \setminus s_N = \emptyset \) (hence \( t_N = s_N \)).

Then \( s_N' = \sup s' : s_N \) and \( T_N' \setminus s_N' = \emptyset \).

By Lemma 2.6.2, \( \sigma \) is a \( \text{Hec} \) isomorphism.

**Claim** \( s_N = \sigma(s') \) for some \( s' \),

since then \( s_N' = s'(s)+1 = \sigma(s_N) \).

Clearly, \( s_N > t_N \), since otherwise \( N \) has type 1. Suppose \( \tau \in (s_N) \). Then \( T_N \setminus \tau = \emptyset \)
for a \( \tau < s_N \), since otherwise \( s_N \) is a limit pt. of \( T_N \); hence \( s_N \in T_N \). Contradiction! QED (6)

**Case 2** Case 1 fails.

(1) \( T_N \setminus [s, t_N) = \emptyset \) for a \( s < s_N \).

Suppose not. Then \( s_N \) is a limit in \( T_N \) + hence \( s_N \in T_N \). Hence \( s_N = t_N \). But

Then \( \sup C_N = t_N \). To see this let

\( s < s_N \) and \( s \notin \tau \).

Set

\( s = \text{the least } s > s', \text{ s.t. } s \in y_{\tau} \in \tau \).

\( \tau = \text{ the least } \tau > \tau, \text{ s.t. } \tau \in T_N \).
Then \( s < t\_N \). But \( \sigma\_s (s) = s \), since otherwise \( T\_N \cap [s\_1, s] = \emptyset \) and hence \( T\_N \cap [s\_1, \sigma\_s (s)] = \emptyset \), by Lemma 2,6.2, where \( s < \sigma\_s (s) \). Hence \( s \notin T\_N \). Contr.

Since only successors of limit points in \( C\_N \) can fail to be in \( C\_N \), we have \( \sup \ C\_N = t\_N \). Hence \( N \) is of type 3. Contr! QED (Claim 1)

But then \( T\_N \cap [\sigma (s), \sigma (t\_N)] = \emptyset \), where \( \sigma (s) < t\_N \) and \( \sigma (t\_N) \in T\_N \). Hence \( \sigma (t\_N) = t\_N \). QED (Claim 1)

Claim 2 Let \( s = \max C\_N \). Then \( \sigma (s) \notin C\_N \) and \( \sigma (N\_s) = N\_\sigma (s) \).

Hence \( N\_s = N\_s \) we have: \( \sigma (N\_s) = \sigma (N\_s) \),

And we have:

\[ \forall x < s \forall x \in A (x) (a \in F (x) \leftrightarrow a \in F (x)) \]

where \( N = \langle J, E, F \rangle \), \( n = k \). Hence the same \( T\_N\) statement holds of \( \sigma (N\_s) \) in \( N' \). Hence \( \sigma (N\_s) = N\_\sigma (s) \). But \( \sigma (s) = t\_0 (N\_s) \) since \( s = t\_N \) Hence \( \sigma (s) \in C\_N \). QED (Claim 2)
Claim 3: Let $\bar{s} = \max C_N$. Then $\sigma(\bar{s}) = \max C_{N'}$ and $N'$ is a T-premeasure.

**Proof:** Set $\bar{s}' = \sigma(\bar{s})$.

**Case 1:** $\bar{s}' = \max C_{N'}$.

**Case 1.1:** $T_N \setminus (\bar{s}+1) = \emptyset$. Then $T_{N'} \setminus (\bar{s}'+1) = \emptyset$.

$\delta_N = \bar{s}+1$, where $\delta = \text{crit } (\tau_{\bar{s}}^N)$, since otherwise $\delta+1 \in C_N$. (Note it is easily seen that $\delta = \bar{s}$ or $\bar{s}+N_3$.) Clearly $\bar{s}' = \tau_{N'}^{\bar{s}'}$, $\delta' = \text{crit } (\sigma_{N'}^{\bar{s}'})$, where $\delta' = \sigma(\delta)$.

$\tau_{N'}^{\bar{s}'} = \sup_{\alpha < N'} \sigma^{\alpha} N_N = \delta' + 1$.

If $\delta = \bar{s}$, then $\delta' = \bar{s}'$ and hence $\bar{s}' = \max C_{N'}$. Hence for all $\alpha \in C_{N'}$,

$\left( N'_{\alpha} \right) = \left( N'_{\bar{s}'} \right) / \alpha \subseteq N'$ and $N'$ is a T-premeasure. Now let $\delta = \bar{s}+N_3$.

Then $\delta = \bar{s}'+N_3'$. But then $\delta' \leq \bar{s}'+N_3' \leq \bar{s}'+N_\delta = \bar{s}'$, since $\delta \leq \text{gen } N'$, Hence $\sigma_{\bar{s}'+1} N_\delta = \id$, and it follows that $(\bar{s}', \delta') \cap \text{gen } N' = \emptyset$. Hence $C_{N'} \cap (\bar{s}', \delta') \cap \text{gen } N' = \emptyset$, and if $\mu \in C_{N'} \cap (\bar{s}', \delta')$, then
\[ u = \text{sub}(\text{Janger}_{n}, 1) \text{, Contd} \]  
Hence \( \bar{s} = \max \mathcal{C}_{N} \) and we conclude as before that \( N' \) is a \( T \)-prehorn.  
QED (Case 1.2)

**Case 1.1** fails.
Then \( t_{N} \in T_{N} \) and \( C_{N}^{*} \cap (\bar{s}, t_{N}) = \emptyset \).
Hence \( t_{N} \notin T_{N} \) and \( C_{N}^{*} \cap (\bar{s}, t_{N}) = \emptyset \),
where \( \mathcal{C}_{N}' \subset C_{N}^{*} \). Hence \( \bar{s} = \max \mathcal{C}_{N}' \)
and it follows as before that \( N' \) is a \( T \)-prehorn.  
QED (Case 1.2)

**Case 2** Case 1 fails.
Then \( s \in \text{gen}_{N} \) is a limit cardinal in \( N \)
and \( s = \max \mathcal{C}_{N} \) the most largest element of \( \mathcal{C}_{N} \).
Moreover \( s \leq \delta \) where \( \delta = \bar{s} + N_{3} \),
But \( (J_{\bar{s}+})^{N_{5}} = (J_{\bar{s}+}^{E})^{N_{5}} \).
Since \( (J_{\bar{s}+}^{N_{5}})(\bar{s}) = \bar{s} \) we then have \( \delta = \text{crit}(\delta^{N_{5}}) \).
But \( \delta = \bar{s} + N_{3} \leq \bar{s} + N_{5} \leq \bar{s} + N_{5} = \delta \) for all \( \mu \in [\bar{s}, \delta] \). Hence \( (\bar{s}, \delta) \cap \text{gen}_{N} = \emptyset \).
We know that, letting \( s', \delta' = \sigma(\bar{s}, \delta) \), we have \( (J_{\bar{s}+}^{E})^{N_{5}'} = (J_{\delta'}^{E}) \)
and \( \delta' \in \text{gen}_{N'} \). Since \( \delta \in \text{gen}_{N} \),
Hence \( \delta' = \text{crit}(\sigma^{-N'}_\delta) \), since \( \delta' \cap (3', 3) = \emptyset' \). Hence \( \delta' = 3' + N_3' \leq 3 + N_3' \leq 3 + N_3' = \delta' \) for all \( \mu \in [3', \delta'] \). Hence \( \text{gen}^N_{3'}(3', \delta') = \emptyset' \), and \( \delta' = \text{crit}(\sigma^{-N'}_\mu) \) for \( 3' < \mu \leq \delta' \); hence \( (\bigcup_{3'}^\delta N_\mu' = \bigcup_{3'}^{\delta'} N_\mu' = (\bigcup_{3'}^\delta N_\mu)' \) for \( \mu' \in [3', \delta'] \).

Case 2.1 \( T_N \setminus (3'+1) = \emptyset \)

Then \( t_N = t_N = 3'+1 \), since otherwise \( 3'+1 \in \tilde{C} \).

Clearly \( T_N \setminus (3'+1) = \emptyset \). Hence \( t_N' = t_N' = \text{sup} \sigma^{-N'}_N = 3'+1 \). Moreover \( \tilde{S} = 3'+1 \) and \( \tilde{S}' = 3'+1 \). Hence \( \tilde{S}' \in \tilde{C}_N \) and \( \tilde{S}' \notin C_N \), by the above. Since \( (3', \delta') \) are \( N_3' = \emptyset' \), we have \( \tilde{S}' = \text{max} \tilde{C}_N' \).

Hence for each \( \mu \in \tilde{C}_N' \), either \( \mu = \tilde{S}' \) and \( N_{\tilde{S}'} \) does not satisfy T-MIS, or else \( N_{\tilde{S}'} = (N_{\tilde{S}'})' \), \( \in N' \)

Hence \( N' \) in a T-prime one. QED (2.1)
Case 2.2 Case 1 fails
Then \( S' = \sigma(S) = t_{N^{3+1}} \), the least \( S > 3' \)
\( \iff S' \in T_{N^{3+1}} \).

\( \text{At } S < S', \text{ then } S = \text{the least } S \in T_N \),
\( \text{at } S > 3 \text{ and } S' = \text{the least } S' \in T_{N'} \),
\( \text{at } S' > S \). Hence \( S' = t_{N^{3+1}} \),
\( \text{since } S' < \alpha N^{3+1}. \) At \( S = S' \),
Then \( \sigma_{S'}(S) = S + N' = \text{the least } S \in T_{N'} \),
\( \text{at } S > 3. \) Hence \( S' + N' = S + (S + N') = \)
\( \text{the least } S \in T_N \) at. \( S > S' \).
\( \text{Hence } \tilde{S}' = S' + N_{3+4} = (S^{3+1})^{-1}(S' + N'). \)
\( \text{the least } S' \in T_{N^{3+1}} \) at. \( S' > S' \).

But then \( S' \in \tilde{C}_{N'}, \) \( S' \notin C_{N'} \).

Case 2.2.1 \( T_N \setminus (S+1) = \emptyset \).
Then \( T_N \setminus (S+1) = \emptyset \). As before,
\( t_N = \alpha N = S+1, \) \( t_{N'} = \alpha N' = S'+1, \)
\( \text{hence } \sigma(t_N) = t_{N'} \). But \( \text{gen}_N \cap (S', S') = \emptyset. \)
\( \text{Hence } \tilde{C}_{N} \setminus (S+1) = \emptyset, \) since if \( m \in \tilde{C}_{N} \setminus (S+1), \) then \( m = \text{but}(m \cap \text{gen}_N) \).
But \( \tilde{C}_{N'} \cap (S', S') = \emptyset, \) since if \( m \in \tilde{C}_{N'} \cap (S', S'), \) then \( N' = N_S'. \)
and \( t_{N_{x}'} = t_{N_{z}'} = s' \). Hence \( s' = \max C_{N} \) and \( \mu \in C_{N} \).

Case 2.2.2 \( \mu \in C_{N} \) fact.

Then \( t_{N} \in T_{N} \) and \( T_{N} \cap (s', t_{N}) = \emptyset \), since \( t_{N} \). By the usual argument (Lemma 1.1), \( C_{N} \cap (s', t_{N}) = \emptyset \). Hence \( T_{N} \cap (s', t_{N}) = \emptyset \). But \( s' \leq s' \leq \mu \) and \( C_{N} \cap (s', t_{N}) = \emptyset \). Hence \( T_{N} \cap (s', t_{N}) = \emptyset \). \( C_{N} \cap (s', t_{N}) = \emptyset \) follows as in Case 2.2.1. Similarly, for \( C_{N} \cap (s', s'] = \emptyset \). But \( C_{N} \cap (s', t_{N}) = \emptyset \), since if \( \mu = \min (C_{N} \cap (s', t_{N})) \), then \( \mu \in C_{N} \) and \( C_{N} \cap (s', t_{N}) = \emptyset \). Hence \( s' = \max C_{N} \). And, as above, \( N' \) is a T-preimage.

QED (Lemma 3.3)

Similarly:

Lemma 3.4 Let \( \sigma : N \to N' \), where \( N' \) of Type 2 and \( \sigma (\mathcal{G}) < t_{N} \). Then \( N' \) of Type 2, \( \sigma (t_{N}) = t_{N}' \) and \( \sigma (s) = s' \), \( \sigma (N_{s}') = N_{s}' \), where \( s = \max C_{N} \), \( s' = \max C_{N} \).
Lemma 3.5. Let \( N = (J^E_N, E) \) be of Type 3. Let \( t = t_N \) (hence \( t = \pi_N = \pi_{J^E_N} \)). Let \( \delta : J^E_t \to \bigwedge J^E \), where \( \bigwedge \) is an extender on \( J^E_N \). Let \( \delta' : N \to \Sigma \) on \( N' \) be the canonical completion of \( \delta \). Then \( N' \) is a \( T \)-promome of Type 3. Concl. \( t' = t_N' \).

Moreover, \( \delta' : N \to \bigwedge N' \).

Proof.
As in §1 Lemma 4.5, we get:
(1) \( t' = \pi_N' = \pi_{J^E_N} \).
(2) \( \exists t' \to N' \in \bigwedge N' \).
(3) \( \exists t \to \delta(N'_3) = N_3 \).

But there are arb. large \( \exists t \) s.t.
\( \exists \in C_N \), i.e. \( N'_3 \) is a \( T \)-promome and \( \exists = t_N'_3 \). Hence \( N_3 \delta(3) \) is a \( T \)-promome and \( \delta(3) = t_N' \). Hence \( \delta(3) \in C_N \) and \( C_N \) is contained in \( t' \). But \( C_N \) is closed in \( t_N' \) and \( C_N \times t \) is \( \emptyset \) since \( t' = t_N \). Hence \( t' = t_N' \). Q.E.D. (3.5)

(Note: Using the fact that \( \delta' \mid t_N \) is cofinal in \( t_N' \), we can, in fact, show that \( \delta' : N \to \bigwedge N' \).)
Lemma 3.6 Let \( \mathcal{N} = \langle J^E, F \rangle \) be of type 3. Let \( \delta : N \to \mathcal{N} \) where \( \text{crit}(\alpha) < t_N \). Then \( N' \) is of type 3. Moreover,
\[
\tau_{N'} = \begin{cases} \sup \delta(t_N) & \text{if } \text{crit}(\alpha) = \omega \tau^2_N \\ \delta(t_N) & \text{if not} \end{cases}
\]

Proof. At \( \text{crit}(\alpha) = \omega \tau^2_N \), this follows by Lemma 3.

Now let \( \text{crit}(\alpha) < \omega \tau^2_N \). At follows as in Lemma 4.6 of \( \mathcal{N} \) that \( N_3' \in N' \) for
\( \delta < t' = \delta(t_N) \). Hence \( t' = \tau_{N'} = \omega \tau^3_{N'} \), as before. The statement:
\[
Q = N_3' \text{ is uniformly } \Pi^1_1(N') \text{ in } Q < 3, \text{ since } Q = N_3 \text{ iff } Q \text{ is a p.m. of } 3, Q < 3, Q \text{ s.t. } \sum_{N} \frac{F_{13}}{N} = F_{13} \]

Since \( N_3 \in \bigcup_{N'} N_3 \), for \( \delta < t' = \omega \tau^3_{N'} \),
\[
\tau \langle Q, 3 \rangle^{N'} = \left\{ Q = N_3' \right\} \text{ is elementary in } N'(4) = \langle J^E_{N'}, A^4_{N'} \rangle. \]

(Note: Since
\[
\tau_{N'} = \omega \tau^3_{N'} \text{ with parameter } \lambda \vdash C_{N'} = t' \text{ then expressible over } N'(4) \text{ by the } \Pi^1_2 \text{ statement:}
\]
\[
\forall \gamma \forall 3' > \gamma \left( N_3' \text{ is a } \Pi^1_2 \text{ p.m. of } 3 = t_{N_3'} \right),
\]

But the corresponding statement holds in \( N'(4) \) where \( \tau' : N'(4) \to N'(4) \).

(3, 4)
At is apparent from the above proof that
\[ \mathcal{E}(\bar{Q}, \bar{S}) \cap \mathcal{N} = \mathcal{N}_3 \] 
uniformly \( T_1(N) \) for active
ppms \( N \). Hence the set
\[ \mathcal{E}_N = \{ \bar{S} \mid \mathcal{N}_3 \subseteq \mathcal{N}_3 \} \]
is uniformly \( \mathcal{E}_1(N) \). Hence \( \sigma(\mathcal{E}_N) \subseteq \mathcal{E}_N \)
if \( \sigma : \mathcal{N} \rightarrow \mathcal{E}_1(N) \).

**Lemma 4.1** Let \( \sigma : \mathcal{N} \rightarrow \mathcal{E}_1(N) \) where \( N \) is of type
Then \( \mathcal{N} \in \mathcal{E}_1 \) and \( \sigma(tN) = tN \).

**Proof.**

**Case 1** \( T_N = \varnothing \)

Then \( T_{\mathcal{N}} = \sigma^{-1}(tN) = \varnothing \), Moreover \( \mathcal{E}_N = \varnothing \)
and hence \( \mathcal{E}_N = \sigma^{-1}(tN) = \varnothing \). Hence \( \sigma(tN) = tN \), \( \mathcal{E}_N = \sigma(tN) \), \( \mathcal{E}_N = \mathcal{E}_1(N) \)
where \( \sigma(tN) = tN \). Moreover, \( \mathcal{E}_N = \varnothing \), since if \( \exists \in \mathcal{E}_N \), then \( T_{\mathcal{N}} = \varnothing \) and hence \( \exists = \sup(\mathcal{E}_N (N) = 0 \) CONTR! Hence \( \mathcal{N} \) is a \( T \)-premonence vacuously.

**Case 2** Case 1 fails.

Then \( T_N = \min T_N \), since otherwise \( \exists \in \mathcal{E}_N \), \( \mathcal{E}_N = \min T_N \). Hence \( \sigma(tN) = t \), where \( t = \min T_N \). Moreover \( \mathcal{E}_N = \sigma(tN) \subseteq T \).
Hence \( T = tN \), but remains only to show:

**Claim** \( \mathcal{E}_N = \varnothing \).

Suppose not. Let \( \exists \in \mathcal{E}_N \). Then
\[ \bar{s} = t_{N_3}^N \text{ and hence } \sigma_{\bar{s}}^N(\bar{s}) = t_{\bar{N}} = \min T_{\bar{N}}. \]

Set \( \bar{3} = \sigma_{\bar{s}}^N(\bar{s}) \). Then \( \bar{3} \in \text{gen}_N \) and \( \bar{3} \in \text{gen}_{\bar{N}} \).

But \( \sigma_{\bar{s}}^N(\bar{3}) < t_{N_3} \), since otherwise \( \sigma_{\bar{s}}^N(\bar{3}) = t_{N_3} \) and hence \( \bar{3} = \min T_{N_3} = t_{N_3}^N \). But

\[ \tilde{C}_{N_3} = \bar{3} \cap \tilde{C}_N = \emptyset. \]

Hence \( N_3 \) satisfies \( T\text{-MIS} \) vacuously. Hence \( \bar{3} \in \tilde{C}_N \), \( \bar{3} < t_N \) Constr!

Since \( \sigma_{\bar{s}}^N(\bar{3}) < t_{N_3} \), there exist \( \gamma < t_{N_3}, f \in \mathcal{N} \) s.t.,

\[ \{ < \beta, \gamma > \in \mathcal{N} | f(\beta) = \gamma \} \subseteq F_{(\alpha, \gamma)}. \]

This is a \( \Sigma_1(N) \) statement about \( \bar{s} \).

Hence the same \( \Sigma_1(N) \) statement holds of \( \bar{3} \). Hence \( \sigma_{\bar{s}}^N(\bar{3}) < t_{N_3} \). Constr!

QED (Lemma 4.1)
Therefore $t_n = 2 t_{n-1}$.

\[ t_n = 2 t_{n-1} \]

**Proof:**

**Case 1:** $N = \max \{ \frac{3}{2}, 3 \}

\[ \frac{3}{2} = \frac{3}{2} \]

**Case 2:** $N = \max \{ \frac{3}{2}, 3 + t_1 \}

\[ \frac{3}{2} = \frac{3}{2} \]

Let $N$ be of type 2. Then $N = \max \{ \frac{3}{2}, 3 \}$, where $\frac{3}{2} = \frac{3}{2}$. Moreover, $\frac{3}{2} = \frac{3}{2}$. If $t_1 = \frac{3}{2}$, then $N = \frac{3}{2}$.

Therefore $\frac{3}{2} = \frac{3}{2}$. Otherwise $\frac{3}{2} = \frac{3}{2}$, where $\frac{3}{2} = \frac{3}{2}$. Moreover, $\frac{3}{2} = \frac{3}{2}$. Therefore $\frac{3}{2} = \frac{3}{2}$. Otherwise $\frac{3}{2} = \frac{3}{2}$, where $\frac{3}{2} = \frac{3}{2}$.
Hence $\sigma_{\bar{N}}^{-1}(\bar{S}) = \bar{S}$, Hence $\sigma_{\bar{N}}^{-1}(\bar{S}) + \bar{N}_3 = 1$.

Hence $\bar{S} = \text{crit}(\sigma_{\bar{N}}^{-1})$ is the least $\bar{S} \in \text{gen}_{\bar{N}}$ such that $\bar{S} = \bar{S}$. Hence $\bar{S} = \max \bar{C}_{\bar{N}}$, since if $\bar{S} < \bar{S} \in \bar{C}_{\bar{N}}$ we would have $\bar{S} = \text{crit}(\sigma_{\bar{N}}^{-1})$, $\bar{S} \in \text{gen}_{\bar{N}}$, $\text{min}_{\bar{N}} \bar{T}_N \backslash \bar{S} = \emptyset$.

Hence for all $\bar{S} \in \bar{C}_{\bar{N}}$ we have $1$.

$\overline{N}_S = (\overline{N}_3)_{\bar{S}} \in \overline{N}$, Hence $\overline{N}$ is a $T$-precone.

**Case 1.2** Case 1.1 fails.

At follows as before that if $\bar{S} \in \bar{C}_{\bar{N}}$,

$\sigma(\overline{N}_3) = N_3$, $\sigma(\bar{S}) = \bar{S}$, where $\bar{S} = \text{crit}(N_3)$

and $\bar{S} = \text{crit}(N_3)$. Clearly $\bar{N}_N \backslash (\bar{S}, T_N) = \emptyset$.

Since otherwise, letting $\bar{S}$ be the least $\bar{S} \in \bar{T}_N \backslash \bar{S} + 1$, we have $\sigma_{\bar{S}}^{-1}(\bar{S}) = 1$.

(Otherwise $\bar{S} < \sigma_{\bar{S}}^{-1}(\bar{S}) = \text{min}_{\bar{N}} \bar{T}_N \backslash \bar{S} + 1$).

Hence $\bar{S} = \bar{N}_S$, and $\bar{S} \in \bar{C}_{\bar{N}}$. Contd.

But then $\bar{T}_N \cap (\bar{S}, \bar{N}) = \emptyset$, where $\sigma(\bar{N}) = \bar{N}$ and $\bar{S} \in \bar{C}_{\bar{N}}$. Contd.

But $\bar{C}_{\bar{N}} \backslash \bar{S} + 1 = \emptyset$ by the argument of Lemma 4.1, $\text{min}_{\bar{N}} \bar{C}_{\bar{N}} \backslash \bar{S} + 1 = \emptyset$.

Hence $\bar{S} = \max \bar{C}_{\bar{N}} = \max \bar{C}_{\bar{N}}$ and $\bar{N}_S = (\overline{N}_3)_{\bar{S}} \in N$ for all $\bar{S} \in \bar{C}_{\bar{N}}$.

**MED (Case 1)**
Case 2: Case 1 fails.

Then $\delta \in \text{gen } \mathbb{N}$ is a limit cardinal in $\mathbb{N}$ and $\delta = \max \mathbb{C}_N$ is the next largest element of $\mathbb{C}_N$. Moreover $\delta \leq \delta$, where

$$\delta = \delta + \mathbb{N}^\delta.$$ 

But $(\bigcup E)^N\delta = (\bigcup E^{N_{\delta}})^{\mathbb{N}^{\delta}}$.

Hence $\delta = \text{crit}(N_{\delta}^\mathbb{N})$, since $N_{\delta}^\mathbb{N}(\delta) = \delta$.

Hence $\delta = \text{crit}(N_{\delta}^\mathbb{N})$. But

$$\delta = \delta + \mathbb{N}^\delta \leq \mathbb{N}^\delta \leq \delta + \mathbb{N} \delta = \delta$$

for all $\mu \in [\delta, \delta]$. Hence $(\delta, \delta)^{\text{gen } \mathbb{N}} = \phi$.

As before, $\delta \in \mathbb{C}_N$ and $\sigma(\mathbb{N}_{\delta}^\mathbb{N}) = \mathbb{N}_{\delta}^\mathbb{N}$,

where $\sigma(\delta) = \delta$. Moreover $N_{\delta}(\delta) = \delta$,

where $\delta = \delta + \mathbb{N}_{\delta}^\mathbb{N}$. \( \delta, \delta \in \text{gen } \mathbb{N} \)

and $(\delta, \delta)^{\text{gen } \mathbb{N}} = \phi$. Hence

$$\delta = \text{crit}(N_{\delta}^\mathbb{N})$$ for $\delta < \mu \leq \delta$. Clearly

$$\bigcup E_{\delta}^{\mathbb{N}_{\delta}^\mathbb{N}} = \bigcup E_{\delta}^{\mathbb{N}}$$ for $\delta < \mu \leq \delta$.

Since $E_{\delta}^{\mathbb{N}} \uparrow \delta^\mathbb{N}$.

Hence $(\bigcup E_{\delta}^{\mathbb{N}})^{\mathbb{N}_{\delta}^\mathbb{N}}$

$$= \bigcup E_{\delta}^{\mathbb{N}_{\delta}^\mathbb{N}} = \bigcup E_{\delta}^{\mathbb{N}_{\delta}^\mathbb{N}} = (\bigcup E_{\delta}^{N_{\delta}^\mathbb{N}})^{\mathbb{N}_{\delta}^\mathbb{N}}$$ for

$\mu \in (\delta, \delta]$. 

[- 29 -]

Case 2.1 \( T_N \setminus (3+1) = \emptyset \).

Then \( t_N = s_N = 3+1 \), since otherwise \( 3+1 \in \tilde{C}_N \). \( T_N \setminus (3+1) = \emptyset \). "Hence \( t_N = s_N = 3+1 \), since \( \overline{\text{gen}_N} = \sigma^{-1} \overline{\text{gen}_N} \). "Hence \( \sigma(t_N - 1) = t_N \).

Clearly \( s = 3+1 \). Let \( \overline{\sigma} = \overline{s} \). Then \( \overline{s} \in \tilde{C}_N \) by the above. But \( \overline{s} \notin C_N \), since \( \overline{T_N \overline{s}} \) does not satisfy \( \overline{T} \)-MIS by the above. Since \( (\overline{s}, \overline{\sigma}) \notin \overline{\text{gen}_N} \) = \( \emptyset \), we have \( \overline{s} = \max \tilde{C}_N \). Hence for each \( \overline{\mu} \in \tilde{C}_N \), either \( \overline{\mu} = \overline{s} \) or \( \overline{\mu} = (\overline{T_N \overline{s}})^{-1} \in \overline{N} \). Hence \( \overline{N} \) is a \( \overline{T} \)-premome. QED (Case 2.1)

Case 2.2 Case 1 fails.

Then \( s = \text{The least } s > 3 \text{ s.t. } s \in T_N \setminus (3+1) \).

But then \( s = \sigma(t_N - 1) \), where \( \overline{s} = t_N \).

1. If \( \overline{s} < \overline{s} \), then \( s = \text{the least } \overline{s} > \overline{s} \text{ s.t. } \overline{s} \in T_N \) ; hence \( \sigma(\overline{s}) = s \).

where \( \overline{s} = \text{the least } \overline{s} > \overline{s} \text{ s.t. } \overline{s} \in T_N \). Hence \( \overline{s} = \text{the least } \overline{s} > \overline{s} \)

s.t. \( \overline{s} \in T_N \setminus (3+1) \), hence \( \sigma(\overline{s}) = \overline{s} \).
At \( S = \delta \), then \( \sigma^{N}(\bar{S}) = \delta + N \in \text{arg}(\delta) \)

Hence \( \sigma^{N}_{3+1}(\bar{S}) = \frac{\delta}{3} + \bar{N} = (\frac{\delta - n}{3} + N) \).

The least \( \delta > \frac{\delta}{3} \) s.t. \( \bar{S} \in T_{\bar{N}} \).

Hence \( \bar{S} \in \bar{C}_{\bar{N}} \) and \( \bar{S} \notin C_{\bar{N}} \).

\text{Case 2.2.1} \quad T_{\bar{N}} \setminus (\bar{S} + 1) = \emptyset,

Then \( T_{\bar{N}} \setminus (\bar{S} + 1) = \emptyset \), as before,

\( t_{\bar{N}} = \sigma^{N}_{\bar{N}} = \bar{S} + 1 \), \( t_{\bar{N}} = \sigma_{\bar{N}} = \bar{S} + 1 \);

hence \( \sigma(t_{\bar{N}}) = t_{\bar{N}} \).

But \( \text{gen}_{\bar{N}} \cap (\bar{S}, \bar{S}) = \emptyset \)

Hence \( \bar{C}_{\bar{N}} \setminus (\bar{S} + 1) = \emptyset \), since for

\( m \in \bar{C}_{\bar{N}} \setminus (\bar{S} + 1) \) we would have \( m \leq t_{\bar{N}} \)

and \( m = \text{lin} \cap (\text{gen}_{\bar{N}} \cap m) \).

Obviously \( \bar{C}_{\bar{N}} \cap (\bar{S}, \bar{S}) = \emptyset \), since if \( m \in \bar{C}_{\bar{N}} \cap (\bar{S}, \bar{S}) \),

then \( \bar{N} = \bar{N}_{\bar{S}} \) and \( t_{\bar{N}} = t_{\bar{N}_{\bar{S}}} = \bar{S} \).

Hence \( \bar{S} = \max \bar{C}_{\bar{N}} \) and for

\( m \in \bar{C}_{\bar{N}} \) we have either \( m = \bar{S} \)

and \( \bar{N}_{\mu} \) does not satisfy \( \text{T-MIS} \)

or else \( \bar{N}_{\mu} = (\bar{N}_{\bar{S}} \mu) \in \bar{N}_{\bar{S}} \).

Hence \( \bar{N} \) is a \( \text{T-precis} \), QED (Case 2.2.1)
Case 2.2.2  Case 2.2.1 fails.

Then \( t_N \in T_N \) but \( T_N \cap (s, t_N) = \emptyset \), since if \( \mu \in T_N \cap (s, t_N) \) is minimal, then \( \mu \notin \tilde{C}_N \) by the usual argument.

Hence \( T_N \cap (\bar{s}, t_N) = \emptyset \). But \( \bar{s} \leq \bar{t} \leq \text{gen}_N < t_N < \sigma^{-1} t_N \).

Hence \( t_N = (\text{the least } t \in T_N \text{ s.t. } t > \bar{t}' = \sigma^{-1} (t_N)) \), \( \tilde{C}_N \cap (\bar{s}, \bar{t}) = \emptyset \) follows as in Case 2.2.1. But then \( \tilde{C}_N \cap (\bar{s}, t_N) = \emptyset \) by the argument of Lemma 4.1. Hence \( \bar{s} = \max C_N \), and for all \( \mu \in \tilde{C}_N \), either \( \mu = \bar{s} \), where \( \bar{N}_{\bar{s}} \) does not satisfy \( T_- \) - MIS, or else \( \mu = (\bar{N}_{\bar{s}})_{\mu} \leq \bar{N} \).

Hence \( \bar{N} \) is a \( T_- \) - prime core.

QED (Lemma 4.2)
The best we can do for type 3 mice is:

**Lemma 4.3** Let \( N \) be of type 3. Let \( \sigma: \overline{N} \xrightarrow{\Sigma_{1}^{(1)}} N \). Then \( \overline{N} \) is of Type 3 and \( \overline{t_{N}} = \sigma^{-1}\overline{t_{N}} \).

**Proof.**

Let \( \overline{N} = \langle J_{\overline{N}}, F \rangle \), \( \overline{N} = \langle J_{\overline{N}}, F \rangle \).
Let \( \overline{s} = \omega_{\overline{N}} \overline{s} = \overline{s}_{N} \), \( \overline{t} = \overline{s}_{N} \).

**Claim 1** \( \overline{s} = \overline{s}_{N} \).

(\( \leq \)) \( \exists \overline{e} \in \text{gen}_{\overline{N}} \to \sigma(\overline{e}) \in \text{gen}_{N} \)
\( \to \sigma(\overline{e}) < \overline{s} \leq \sigma(\overline{e}) \)
\( \to \overline{s} \leq \overline{t} \).

(\( \geq \)) \( \forall \overline{e} \text{ s.t. } \overline{s} < \overline{e} \).
An \( \overline{N} \) exists:
\( \forall \overline{e} > \sigma(\overline{e}) \exists \overline{e} \in \text{gen}_{\overline{N}} \).

Hence the corresponding \( \Sigma_{1}^{(1)} \) condition holds of \( \overline{e} \) in \( \overline{N} \). Hence there is \( \overline{s} > \overline{e} \text{ s.t. } \overline{s} \in \text{gen}_{\overline{N}} \).

QED (Claim 1)

**Claim 2** \( \overline{s} = t_{N} \).

**Care 1** \( T_{N} \setminus \overline{s} = \emptyset \).
Then \( T_{N} \setminus \sigma^{-1}(\overline{s}) = \emptyset \); hence \( T_{N} \setminus \overline{s} = \emptyset \).

Hence \( t_{N} = \overline{s}_{N} = \overline{s} \).
Case 2. Case 1 fails.

We must prove \( \exists \overline{z} \in T_{\overline{N}} \).

At \( \text{ but } T_{\overline{N}} \cap \overline{z} = \emptyset \), this is immediate.

Otherwise there is \( \gamma < \overline{z} \) s.t. \( T_{\overline{N}} \cap (\sigma(\gamma), \sigma(\overline{z})) = \emptyset \).

Hence \( T_{\overline{N}} \cap (\sigma(\gamma), \sigma(\overline{z})) = \emptyset \), where \( \sigma(\gamma) < z \leq \sigma(\overline{z}) \) and \( z \in T_{\overline{N}} \).

Hence \( \sigma(\overline{z}) = z \in T_{\overline{N}} \). Hence \( \overline{z} \in T_{\overline{N}} \).

QED (Claim 2)

Claim 3. \( \overline{z} = \sup C_{\overline{N}} \) and \( \overline{N} \) is a T-premonoe.

Let \( \gamma < \overline{z} \). Then in \( N \) we have:

\[ V_{\overline{z}} > \sigma(\gamma) \forall Q'(Q = N_3 \in \text{a T-premonoe } \overline{z} = t_{\overline{Q}}) \]

Hence the same \( \Sigma_1^{(n)} \) statement holds of \( \gamma \) in \( \overline{N} \). Hence there is \( \overline{z} > \gamma \) s.t. \( \overline{N}_3 \in \overline{N} \) (hence \( \overline{z} < \overline{z} = \sigma(\overline{N}), \overline{z} = t_{\overline{N}_3} \) and \( \overline{N}_3 \) in a T-premonoe. Hence \( \overline{z} \in C_{\overline{N}} \).

Hence \( \text{ but } C_{\overline{N}} = \overline{z} \). Moreover \( \overline{N}_3 = (\overline{N}_3) \gamma \in N \). Hence \( \overline{N} \) is a T-premonoe. QED (4.3)
We now consider the iterates appropriate to $T$-premice. At these iterations, if $\gamma = \gamma_i$, indexes an extender used at stage $i$, then $t_i = t(\gamma)$ will play the role of $\lambda_i$, and $t_i^+ = t^+(\gamma)$ will play the role of $\nu_i$.

As with $\kappa$-premice we set:

\[ t^+(N) = t(N) + N \]
\[ t^+(\gamma) = t^+(N \cup \gamma) \text{ for } E_{\gamma} \neq \emptyset. \]

**Lemma 4.** Let $N$ be a $T$-premice set. Then $\nu \neq \nu' \implies t^+(\nu) \neq t^+(\nu')$.

Moreover, if $\nu < \nu'$, then $t^+(\nu)$ is not a cardinal in $\text{M}(\nu')$, since $t(\nu) \leq t(\nu) < t^+(\nu')$, whereas $t^+(\nu')$ is.

Q.E.D (4.7)
Let $J = \langle \langle M_i \rangle, \langle v_i \rangle_{i \in D}, \langle \gamma_i \rangle, \langle \pi_{ij} \rangle, D \rangle$ be an iteration of $M$ of length $\theta$, where $M$ is a $T$-premouse. We call $J$ a normal $T$-iteration iff, letting $t_i = t(v_i)^{M_i}$, \( t_i^+ = t^+(v_i)^{M_i} \),

(i) $J$ is standard

(ii) $v_i > t_i^+$ for $i \in D$, $h \in D n i$

(iii) $T(i+1) =$ the largest $\bar{z} \in D$ s.t.

\[ \sup t_h \leq \pi_i \text{ for } i \in D \text{ (hence } v_i < t_i) \]

\[ h < \bar{z} \]

(iv) At $i \in D$ there is no $v > v_i$ s.t.

$E^M v_i \neq \emptyset$ and $t^+(v) < t_i^+$ in $M_i$

Note (iv) is called the applicability condition. $\bar{v}$ with $E^M v \neq \emptyset$ is called applicable or free in $M_i$ iff there is no $v$ s.t.

$E^M v_i \neq \emptyset$ and $t^+(v) < t^+(\bar{v}) \leq \bar{v} < v$.

In the following we develop the proof of normal $T$-iterations. The initial lemmas do not even assume that $M$ is a $T$-premouse, but merely a $\kappa$ $M$.}
Lemma 5. Let $i \in D(n)$. Then $J_{E_{+}^{M_{i}}} = J_{E_{+}^{M_{i}}}$ and $t_{i}^{+}$ is a cardinal in $M_{i}$.

Proof (w.l.o.g. assume that $i$ is direct). Suppose not. Let $i$ be a minimal counterexample. Then $i > 0$.

Case 1: $i = \lambda$. Fix $\lambda$. It suffices to show: Claim: There is $h < T_{+}(\lambda)$, $i < i + 1 \leq T_{+}(\lambda)$.

Suppose not. Pick $h < T_{+}(\lambda)$, $i < h > i$ and $h$ is simple above $h$ for $h < i < \lambda$.

Let $h = T_{i}(i+1)$, $i+1 < T_{i}^{+}$.

Then $h_{i} = \text{crit}(\mathcal{P}_{h_{i}}^{M_{h_{i}}}) < T_{i}^{+}$, but $T_{i}^{+} \leq h_{i}$. Since $\tau_{i}$ is a cardinal in $M_{h}$, $J_{\tau_{i}}^{E_{M_{h}}} = J_{\tau_{i}}^{E_{M_{i}}}$, and $t_{i}^{+}$ is a cardinal in $M_{h}$, we have $\tau_{i}^{+} \leq T_{i}^{+}$, but $J_{T_{i}^{+}}^{E_{M_{h}}} = J_{T_{i}^{+}}^{E_{M_{i}}}$; hence no element of $t_{i}^{+}$ is a cardinal in $M_{h}$. Hence $\tau_{i} = T_{i}^{+}$, and $h_{i} = T_{i}$.

Now let $i+1 = T_{i}(i+1)$, $i+1 < T_{i}^{+}$.

The main argument above: $h_{i} = T_{i}$.

Hence $T_{i}(i+1) = h < i+1$. Contradiction.
Case 2. \( i = h+1 \). Then \( \kappa_h \) is a cardinal in \( M_i = \bigcup_{h} E^M_j = \bigcup_{h} E^M_i \), while \( t^+_h \leq \kappa_h \).

Hence it holds for \( i = h \). Now let \( i < h \). Then \( t^+_i \) is a cardinal in \( M_h \), and \( \bigcup_{h} E^M_j = \bigcup_{h} E^M_i \) = \( \bigcup_{h} t^+_h = \bigcup_{h} t^+_i \), since \( t^+_i < \kappa_h \).

Q.E.D (Lemma 5)

As a corollary of the proof of Case 1 we have:

**Cor 5.1** Let \( i < h \leq i < \frac{T}{1+i} \) \( h, i, \ell \in D \), where \( h \) is simple above \( h \). Then \( \text{crit}(\prod_{h} G_{h}) \geq t^+_i \).

If \( \kappa_i \geq t^+_i \), we are done. Otherwise \( \kappa_i = t^+_i \) and hence \( \kappa_{\ell} \geq t^+_i \) by the above argument.

Q.E.D (Cor 5.1)

**Cor 5.2** Let \( \lambda \leq \Theta \), \( \text{Lim}(\lambda) \), \( \text{sup} \, D \cap \lambda = \lambda \).

Set \( \tilde{\kappa} = \text{sup} \{ \kappa_i \mid i \in D, i \leq \lambda \} \). Then \( \tilde{\kappa} = \text{sup} \kappa_i = \text{sup} t^+_i = \text{sup} t^+_i \), \( i \in D \cap \lambda \).

**Cor 5.3** Let \( i \in D \), \( \frac{T}{1+i} = T(i+1) \). Then \( t^+_i \leq t^+_i \).
Lemma 6. Let \( j, i \in D, j \leq i \). Then \( t_i^+ \leq t_j^+ \).

Proof. Suppose not.

Assume w.l.o.g. that \( j \) is direct. Let \( i' \) be the least counterexample. Then \( i' > c \) and \( i' < i \). By Cor 5.1 we have:

1. \( \mu_{i'} = h + (M_{i'}^h) \), since otherwise \( t_i^+ \geq \sigma(\mu_{i'}) \geq t_i^+ \).

2. \( \mu_h > \mu_{i'} \), since otherwise \( t_i^+ > t_i^+ \geq \mu_{i'} = \mu_h \).

C. Locally: \( t_h \leq t_3^+ \leq t_3^+ (\mu) \) in \( M_h^* \), since \( \mu_h < t_3^+ \) and \( t_3^+ < t_3^+ (\mu) \) if \( \mu \neq \mu_3 \) by the applicability condition. Hence:

3. \( t_h = t_3^+ (\mu) \) in \( M_h^* \).

Otherwise \( t_h \leq t(\mu) \), since \( t_h \) is a cardinal in \( M_h^* \) and the interval \( (t(\mu), t_3^+ (\mu)) \) has no cardinals. But
Then \( t^+_h \leq \nu_h = \text{sup} \, \pi^+_3 \nu \), \( \tau^+_h \leq \text{sup} \, \pi^+_3 \nu \), \( \tau^+_h \leq \tau^+_i \leq \tau^+_i \).

Hence:

(4) \( \nu^+_3 = \nu \) (hence \( t^+_3 = \tau^+_i \) in \( M_h^i \)),

since otherwise \( \tau^+_h \leq t^+_3 < \tau^+_i \) by the applicability condition.

(5) \( \lambda_h < \tau_i \) (since \( \lambda_h = \pi^+_3 \nu, \forall h \) < \( \text{sup} \, \pi^+_3 \nu \), \( t^+_3 \leq \tau_i \)).

(6) \( t^+_h = \lambda_h \), since otherwise \( t^+_h < \lambda_h < \tau_i < t^+_i \). But Then:

(7) \( t^+_h < \tau_i \) by (5)

Hence \( t^+_h \leq t^+_i \), since \( t^+_h, t^+_i \) are the successor cardinals of \( t_i, t_i \) in \( M_i \). QED (Lemma 6)

(Note: The proofs of (1) - (7) are only: \( i = h+1 \) and \( t^+_i > t^+_h \).

As a corollary of the proof of Lemma we get: \( i \).
Cor 6.1 Let $t_i^+ = t_h^+$ for an $h < i$. Then $h = \max (D \cap i)$ and $\lambda_h = t_h < t_i^+$. (Hence $\lambda_i > t_i^+$ since $\nu_i = t_h^+ = t_i^+$ and hence $\lambda_h < t_i^+ < \nu_i$, hence $t_i^+$ is not a cardinal in $M[\nu_i]$.)

Proof.
Assume w.l.o.g. that $i$ is largest. Our claim then reduces to

**Claim** $h + 1 = i$ and $\lambda_h = t_h < t_i^+$.

Clearly $i > 0$. By Cor 5.4, let $i' = h + 1$. Then $t_{i'}^+ = t_h^+$ by Lemma 6.

Using only the fact that $t_i^+ \neq t_h^+$ we can repeat the proofs of (1) - (7), hence $t_h = \lambda_h < t_i^+$. As above, we conclude that $\lambda_i > t_i^+$. Now suppose there were a $j < h$ s.t. $t_j^+ = t_h^+$. As before, we can take $h = j + 1$. We can then repeat the proofs of (1) - (7) with $i, h$ in place of $i', h$. But then as above we can conclude $\lambda_h > t_h$. Contr! \[Q.E.D \ (Cor \ 6.1)\]

Cor 6.2 $t_h < t_i^+$ for $h < i$, $h, i \in D$
proof of Cor 6.2
By Cor 6.1 for $t_h^+ = t_i^+$, Otherwise $t_h^+ < t_i^+$ where $t_h^+$ is a cardinal and $t_i^+$ the cardinal successor of $t_i$ in $\bar{\kappa}^E_{t_i}$ (or $t_i^+ = \kappa_i$). Hence $t_h^+ \leq t_i^+ \leq t_i$.
QED (Cor 6.2)

**Note** Lemmas 5 - 6.2 hold for arbitrary p p.m c.

**Note** that in a $T$-iteration we don't necessarily have $\lambda_i \leq \lambda_j$ for $i < j$.
We can squeeze out a bit more information about these iterations by defining:

**Def** For $h < i < \ell h(y)$, $h \in D$ set:

$$\bar{\lambda}_{hi} = \inf \{ \lambda_l | h \leq l < i \land l \in D \}.$$

**Lemma 7**

1. Let $h \leq l < i$, $h \in D$. Then $\bar{\lambda}_{h_{hi}} = \bar{\lambda}_{h_{hi}}$
   and $\bar{\lambda}_{hi}$ is a limit cardinal in $M_i$.

2. Let $h = T(i+1) < i$, $i \in D$. Then $\ell_i < \bar{\lambda}_{hi}$

3. $\bar{\lambda}_{h_{hi}} = \bar{\lambda}_{h_{hi}}$, if $h = T(i+1)$, $i \in D$

4. $h \leq i < \iota \rightarrow \bar{\lambda}_{hi} \leq \bar{\lambda}_{i\iota}$

5. $h < i \leq \iota \rightarrow \bar{\lambda}_{hi} \leq \bar{\lambda}_{i\iota}$
proof of Lemma 7, (w.l.o.g. T is direct)
(a) follows by a straightforward modification of the proof of Lemma 5.
(b) \( \eta_i \leq t \leq t^+ \leq \lambda \) and \( \bar{\eta}_i \leq t^+ \leq t^+ \leq \lambda \) for \( h \leq k < 1 \).
Hence \( \bar{\eta}_i = \eta_i^+ + \mathcal{E} M^h \leq \lambda + \mathcal{E} M^h \leq \lambda \).
(c) \( \bar{\eta}_i \in \bigcup \mathcal{E} M^h = \bigcup \mathcal{E} M^h \), where \( \bar{\eta}_i = \lambda \),
and \( \bar{\eta}_i \) is a cardinal in \( \bigcup \mathcal{E} M^h \).
Hence \( \gamma_i \subseteq \bar{\eta}_i \).
(d) and (e) are trivial. QED (Lemma 7).

Note This too holds for \( T \) - iteration of an arbitrary p.p.m. The same is true of:

Lemma 8 Let \( Y = \langle M_i, V_i, \gamma_i, \eta_i, \bar{\eta}_i \rangle \)
be a normal \( T \) - iteration of \( M_i \). If \( i \in D \), then \( E_{\gamma_i} \) is close to \( M_i^x \).
(Here \( M_i^x = \text{ht} M_{\text{T}(i+1)} \).)

proof. (Assume w.l.o.g. that \( Y \) is direct.)
The case \( \gamma_i \in M_i \) is trivial, since then \( E_{\gamma_i} \in M_i \).
Hence \( E_{\gamma_i} \notin \mathcal{P}(\bar{\eta}_i) \cap M_i \),
and \( E_{\gamma_i} \notin \mathcal{P}(\bar{\eta}_i) \cap M_i^x \) for \( i < \lambda_i \).
Hence it suffices to prove:

**Lemma 8.1** Let \( \kappa_i = \text{ht}(M_i) \). Let \( A \subseteq C_i \) be \( \Sigma_n(M_i) \). Then \( A \subseteq \Sigma_n(M_i^\kappa) \).

**Proof.** Suppose not.
Let \( J \) be a counterexample of minimal length \( \Theta \). We derive a contradiction.
Clearly \( \Theta = i+2 \), where Lemma 8.1 fails at \( i \) and holds at all \( j < i \). Let \( \delta = T(i+1) \). Then \( \kappa_i = \text{ht}(M_i) \) and \( \delta < i \), since otherwise Lemma 8.1 would hold at \( i \). It is easily seen that \( i = h+1 \) for some \( h \). (Otherwise pick a \( j < i \) s.t. \( j > \delta \), \( j+1 \supseteq C_i \), \( j+1 \) simple above \( j+1 \) in \( J \), and \( A \subseteq \Sigma_n(M_i) \) in \( p \in \text{rng}(\text{Tr}_{j+1,i}) \).
Then \( \kappa_i = 2^\delta > \kappa_i \). Hence \( 2^\delta > \Sigma_i \), since \( \kappa_i \) is cardinal in \( \kappa_i' \). Hence \( A \subseteq \Sigma_n(M_i) \) in \( p = \text{Tr}_{j+1,i}^{-1}(p) \).
Define an iteration \( \mathcal{J}' \) of length \( j+3 \) by setting \( \mathcal{J}'(i+2) = M_{i+2}^i \) and \( \kappa_i' = \text{ht}(M_i+1) \). This is well-founded, since there is a
canonical map $\sigma : M_i^{j+1} \to M_i^*$.

But clearly $T'(j+1) = j$ and $A \in \Sigma_i(M_i^*)$, where $l(h(Y)) < \theta$. Hence $A \in \Sigma_i(M_i^*)$ by minimality, since $M_i^* = M_i^{j+1}$. Conclusion.

Let $S = T(h+1)$ where $h+1 = i$. Lemma 8 holds at all $j \leq h + 1$ and hence so does Lemma 8. Hence $\overline{\pi_i} \in \Sigma^*$-preserving whenever $h \leq j \leq i$ and $j$ is simple above $k$. Then:

1. $\kappa_i < \kappa_h$ (hence $\overline{\pi_i}^+ \circ \overline{\tau_i}^+ = \text{id}$, where $\overline{\tau_i}^+ = \overline{\tau_i}^+$)

proof

Let $\kappa' = \overline{\pi_i}^+ \circ (\kappa_i) = \kappa_i + (E_{M_i}^{M_i^*})$. At

Note: We show $\kappa' < \kappa_h$. At note,

$\kappa = \overline{\pi_i} \circ (\kappa_i) \geq \overline{\pi_i} \circ (\kappa_h) = \lambda \kappa \geq \theta_i$.

Hence $\delta = \iota$. Conclusion!

2. $\delta \leq \overline{\pi_i}$, since $\kappa_i < \kappa_h < \theta_i$.

3. $\omega P_i^1 \leq \overline{\pi_i}:

\text{Suppose not. Let } A \in \Sigma_i. \text{ be } \Sigma_i(M_i^*)$. Then $A \in \#(\Sigma_i) \cap M_i \subset \bigcup E_{M_i}^{M_i^*} = \bigcup E_{M_i}^{M_i^*} = \bigcup \Sigma_i^*$. Hence $A \in M_i^*$. Conclusion!
(4) \( \omega_{\mu_n}^* \leq \tau_i \)

\[ M_{n}^* \]

\[ \text{mod}_{\tau_i} \pi_{\tau_i} \in \Sigma^* - \text{preserving and } \pi_{\tau_i} \mu_i(\tau_i + 1) = \]

(5) \( \forall (\kappa_h) \land \Sigma \land (M_c) \subset \Sigma \land (M_n^*) \)

\[ \text{prf.} \]

\[ \tau_i : M_n^* \rightarrow E_n \]

\[ \text{is a } \Sigma_0 \text{ ultrapower by } \mu_i \]

and (1). The conclusion follows by [NFS] ("A New Fine Structure Theory for Higher Core Models") §1 Lemma 8 and the closure of \( E_n \) to \( M_n^* \).

(6) \( \tau > \delta \)

\[ \text{prf. Suppose not. Then } \tau = \delta, \mu_i + \gamma \leq \gamma_i \]

\[ \text{since } \tau_i < \kappa_h < \tau_i, \mu_i \] Then

\[ \forall (\tau_i) \land \Sigma \land (M_c) \subset \Sigma \land (M_h^*) \subset \Sigma \land (M_n^*) \]

\[ \text{since } M_h^* = M_c^* (\gamma_i), \text{ Contr. !} \]

(7) \( M_n^* = M_\delta \text{ (i.e. } \gamma = ht (M_\delta) \)

\[ \text{prf. Suppose not. Then } \]

\[ \tau_i < \delta_{\delta, i}, \] where \( \delta_{\delta, i} \) is a limit cardinal

in \( M_\delta \). Let \( A \in T_i \) be \( \Sigma \land (M_c) \). Then

\[ A \in \forall (\tau_i) \land \Sigma \land (M_n^*) \subset \forall (\tau_i) \land \Sigma \land M_g = \forall (\tau_i) \land \Sigma \land M_c \subset \]

\[ = \forall (\tau_i) \land \Sigma \land (M_n^* \subset M_{\mu_i}^* \subset M_\delta, \text{ Contr. !} \]

\[ \text{QED (7)} \]

We now define a new iteration \( \bar{\gamma} \) of length \( \tau + 2 \). Set: \( \bar{\gamma}_{\tau + 2} = \gamma_{\tau + 1} \)
and $\overline{\nu}_3 = \text{ht}(M_3)$. Then $\overline{\nu}_3 = \kappa_i$ and $\overline{M}_3^* = M_i^*$. $M_3^{s+1}$ is well-founded, since there is a canonical $\varphi: M_3^{s+1} \to \Xi M_{i+1}$ defined by:

$$\varphi(\overline{\nu}_3, f_1(\alpha)) = \overline{\nu}_d, i + 1 (f) (\overline{\nu}_d, (\alpha)),$$

and $\langle \text{id} \overline{M}_i^*, \overline{\nu}_3 \rangle : \langle \overline{M}_3^*, E_3 \rangle \to \langle M_i^*, E_i \rangle$ (where $\omega_{\overline{M}_3^*} = \kappa_i < \omega_{\overline{M}_i^*}$). Now let $A \subseteq \mathcal{E}_i$ be $\Xi_1(M_3^*)$. Then $A \in \Xi_1(M_3^*)$ by (5-1). But $\forall \gamma \in A$ of length $s + 2 < i + 2$. Hence Lemma 8.1 holds at $\overline{\nu}_3$ and $A \in \Xi_1(M_3^*)$, where $\overline{M}_3^* = M_i^*$. Contra! Q.E.D (Lemma 8)

**Def:** $M$ is normally $T$-iterable (up to $\overline{\nu}_3$) if there is a successful strategy for normal $T$-iterations of $M$ (of length $< \overline{\nu}_3$)

The concept of a good $T$-iteration is defined in the usual way. (A.o. there can be decomposed into a linear sequence of normal $T$-iterations.)
Def. \( M \) is \( T \)-iterable up to \( \alpha \) iff there is a successful strategy for good \( T \)-iterations (of length \( \leq \alpha \)).

Def. Let \( \gamma = \langle \langle M_i \rangle, \langle \nu_i \rangle, \ldots, T \rangle \) be a normal iteration. \( i \) is singular in \( \gamma \) iff \( E_i \neq \emptyset \) and \( t_i^+ = t_i^+ \). This means that the situation in Cor. 6.1 occurs at \( i \) if \( \nu_i = t_i^+ \), \( \lambda_i = t_i \), \( E_i = E_i^+ \), \( \nu_i < t_i^+ \). At \( i \) is singular, \( i < j \) and \( t_i^+ = t_j^+ \), then \( j \) is not singular and \( h \not\in D \) for \( h \in \gamma_{ij} \).

Def. Let \( M^0, M^1 \) be premises which are normally \( T \)-iterable up to \( \alpha \leq \infty \) with successful strategies \( S_0, S_1 \), resp. The \( T \)-correspondence of \( M^0, M^1 \) up to \( \alpha \) given by \( \langle S_0, S_1 \rangle \) with correspondence indices \( \langle t_i^+, i + 1 < \alpha \rangle \) in the pair \( \langle y^0, y^1 \rangle \) defined by:

\[
(a) \quad J^h = \langle \langle M^h_i \rangle, \langle \nu^h_i \rangle | i \in D^h \rangle, \langle \pi^h_i \rangle, T^h \rangle
\]
in a normal \( T \)-iteration of \( M \) of length \( \Theta \leq \alpha \) (\( h = 0, 1 \)).
Def. Let $M = \langle J, E, F \rangle$ be a $T$-premise, $\gamma \leq \check{\eta}$.

\[ \hat{E}_\gamma = \bigcup \{ E_\nu : E_\nu \neq \emptyset \text{ and } \gamma = t^+_{\nu} \} \]

or $\emptyset$ if no such $\nu$ exists.

(b) Let $M_i^n$ be defined, $i < \check{x}$. Set:

\[ t^+_i = \text{the least } \gamma \leq \max \{ \text{ht}(M_i^0), \text{ht}(M_i^1) \} \]

such that $E_{M_i^n} \neq E_{M_i^n}$.

(c) We set: $i \in D^h \iff E_{M_i^n} \neq \emptyset$

and $x_i = \text{that } \nu \text{ s.t. } E_{M_i^n} = E^{t^+_i}_{M_i^n}$, except in the case that this

convention would make $i$ singular in $Y^h$

but not in $Y^{1-h}$. If that happens,

we set $i \in D^h$, $E_{M_i^n} = E^{t^+_i}_{M_i^n}$, but $i \notin D^{1-h}$.

(d) $Y^h$ is closed.

Note: The normality of $Y^0, Y^1$ follows easily.

Lemma 9. Let $\bar{M}^0, \bar{M}^1 < \check{x}$, where $u$ is regular and $M^0, M^1$ are normally $T$-iterable up to $\check{x} + 1$. Then the $T$-coiteration terminates below $\check{x}$.

Proof. Suppose not.

Let $\gamma = \langle \gamma^0, \gamma^1 \rangle$ be the coiteration.
Let $t = n$. Let $X \preceq H^x$ s.t., $\bar{X} \preceq n$.

Then $M_i^h = M_i^h$, $\pi_{i, k}^h = \pi_{i, k}^h$, for $i, i < k = k \cap X = \text{crit}(X)$. Then:

$$\bar{M}_{i, k}^h, \langle \pi_{i, k}^h \rangle_{i < k} = \lim_{i \to \infty} \langle M_i^h, \pi_{i, k}^h \rangle_{i < k}$$

Hence:

1. $\bar{M}_{i, k}^h = M_{i, k}^h$, $\pi_{i, k}^h = \pi_{i, k}^h$,

2. $\sigma \cap M_{i, k}^h = \pi_{i, k}^h$,

Let $x \in M_{i, k}^h$, $x = \pi_{i, k}^h (\bar{x})$. Then:

$$\sigma (\bar{x}) = \sigma (\pi_{i, k}^h (\bar{x})) = \pi_{i, k}^h (\bar{x}) = \pi_{i, k}^h \pi_{i, k}^h (\bar{x}) =$$

$$= \pi_{i, k}^h (\bar{x}), \quad \text{Q.E.D.} (2)$$

Any truncation in the branch $b^h = \{ i \mid i \leq n, M_i^h \}$ must occur below $\bar{\kappa}$. It follows that $
\Phi (\bar{\kappa}) \cap \bar{M}_{i, k}^h = \Phi (\bar{\kappa}) \cap M_{\bar{\kappa}}^h$. Otherwise, we have:

$$\Phi (\bar{\kappa}) \cap M_{\bar{\kappa}}^h = \Phi (\bar{\kappa}) \cap M_{\bar{\kappa}}^h$$

Now let $i, i < k$, s.t., $\bar{\kappa} = T(i, n) + 1$.

Since $\bar{\kappa}_{i, k} = \bar{\kappa}_{i, k+1}, \bar{\kappa}_{i, k}, \bar{\kappa}_{i, k+1}$ and $\text{crit}(\sigma) = \bar{\kappa}$,

we have:

3. $\text{crit}(\sigma) = \bar{\kappa}$,

$E (x) \cap T_i^h = \sigma (X) x \cap T_i^h$, for:

$$E = E_{T_i^h}, x \in \Phi (\bar{\kappa}) \cap M_i^h.$$
From this we derive a contradiction.

Case 1: $i_0 = i_1$. Let $i = i_0 = i_1$.

At $t_i^o = t_i^1$, then $\hat{E}_i^o = \hat{E}_i^1$. Contradiction!

Let e.g., $t_i^o < t_i^1$. Set \( \tilde{M}^h = M^h \| v_i^h \).

Then $\tilde{M}^o = (\tilde{M}^h)^{t_i^o}$, $t_i^o = t_{\tilde{M}^0}$. Hence $\tilde{M}^0 \subset \tilde{M}^1$. Hence $t_i^+ = t_{\tilde{M}^0} < t_{\tilde{M}^1} = t_i^+$. Contradiction!

Case 2: $i_0 \neq i_1$. Let e.g., $i_0 < i_1$.

Then $t_{i_0}^+ = t_{i_1}^+$.

Case 2.1: $t_{i_0}^o < t_{i_1}^o$.

We obtain a contradiction exactly as in Case 1, using the fact that, setting $\tilde{M}^h = M^h \| v_i^h$, we would have $\tilde{M}^o \subset \tilde{M}^1$, although $E_{i_0}^+ = (t_{i_0}^o)^+ \tilde{M}^0$.

Case 2.2: Case 2.1 false.

Then $t_{i_1}^+ = t_{i_0}^+$ since otherwise $t_{i_0}^o < t_{i_1}^+ \leq t_{i_1}^1$. Hence $t_{i_0 + 1}^+ = t_{i_0}^1$.

Since $i_0 \geq i_0 + 1$, Hence $E_{i_0}^{M_{i_0 + 1}^o} \neq E_{i_0}^{M_{i_0 + 1}^1}$.

So $i_0$ must be singular in either $M^o$ or $M^1$. But if $i_0 \in D^1$, 


it must be singular in both. It follows easily that \( i_0 + 1 \) is not singular in
\( Y^0 \) or \( Y^1 \), hence that \( \hat{E}^{M^0_{i_0 + 1}} = \hat{E}^{M^1_{i_0 + 1}} = \emptyset \).

Hence \( t_{i_0 + 1}^+ > t_{i_0}^+ \). Hence \( i_1 = i_0 + 1 \).

At \( t_{i_1}^0 < t_{i_0}^0 \) we would obtain a contradiction exactly as in Case 1. Hence
\( t_{i_1}^1 = t_{i_0}^0 = t \) and \( \hat{M}^0 = \hat{M}^1 = \hat{M} \). Thus
\( \hat{E}^{M^1_{i_0}} \neq \emptyset \), since otherwise \( \hat{E}^{M^0_{i_0 + 1}} = \emptyset \).

Hence \( i_0 \in D^1 \), since otherwise \( \hat{M} =
M^1_{i_0} \Pi V^1_{i_0} \) and \( \hat{E}^{M^0_{i_0}} = \hat{E}^{M^1_{i_0}} \). Contr!

But then \( t = t_{i_0 + 1}^+ > t_{i_0}^+ = t \). Contr!

Q.E.D. (Lemma 9)
Similar lemmas can be worked out for "double rooted iterations", though we have not checked this in detail. It should then be possible to prove solidity and condensation lemmas for iterable T-premice. The proofs are likely, however, to be somewhat more complicated.

For now we take this for granted and consider criteria of iterability.

**Def** By a weakly T-iterable T-premice (or weak T-mouse) we mean a T-premice M with the property:

\[ \exists f : \overline{M} \rightarrow \overline{M} \]

where \( \overline{M} \) is a countable T-premice. Then \( \overline{M} \) is T-iterable up to \( \omega_1 + 1 \).

By a Löwenheim–Skolem argument it will follow that weak T-mice satisfy as much solidity and condensation as fully iterable T-mice. We assume that this will be enough to carry out the iterability arguments developed later in this paper.

We shall obtain weak T-mice by Steel's technique of constructing what we call "arrays".
Def. A T-array is a sequence \(<N_i:i<\Theta>\) of T-premice s.t.
(a) \(N_i\) is a weak T-mouse for \(i+1<\Theta\),
(b) \(N_0 = \langle \emptyset, \emptyset \rangle\)
(c) Let \(i+1<\Theta\), core \((N_i) = \langle J^E_{\beta}, E_\omega \rangle\).
Either \(N_{i+1} = \langle J^E_{\beta+1}, \emptyset \rangle\) or \(E_\beta = \emptyset\)
and \(N_{i+1} = \langle J^E_{\alpha}, F \rangle\), where \(F \neq \emptyset\),
\(\beta = t^+, N_{i+1}\) and \(J^E = J^E_{\beta}\).

Def. For \(3 \leq i \leq \Theta\) let:
\(\kappa_3 = \kappa_3, i = \kappa_{3+1}^+ \cup \{\omega^{\kappa_3}_{\omega^{\kappa_3}_h} \mid 3 \leq h < i \leq 3\};\)
\(\mu_3 = \mu_3, i = \mu_{3+1}^+ + \kappa_3\) (adapting the convention that \(\mu_3 = \text{ht}(N_3)\) if \(3 \leq i \leq \Theta\), i.e., \(\kappa_3 = \text{ht}(N_3)\) or \(\kappa_3 = \text{the largest cardinal in } M_3\).

(d) Let \(\lambda < \Theta\), Linh (\(\lambda\)). Then
\(J^E_{N_3} = J^E_{N_i}\) for \(3 \leq i \leq \lambda\) and
\(\mu_3 = \mu_3, i = \mu_{3+1}^+ \cup \{J^E_{N_i} \mid 3 \leq i \leq \lambda\};\)
\(N_{\lambda} = \langle \bigcup_{3 < \lambda} J^E_{N_i}, \emptyset \rangle\).

We also let: \(M_i = \text{core}(N_i)\) if \(i < \Theta\)
and \(N_i\) is a weak T-mouse.
Note $3 < i \leq \Theta \rightarrow \kappa_{3i}^{4} \leq \kappa_{3i}^{5}$.

Note Suppose $\Theta = i+1$ and that $N_{i}$ is a weak $T$-mover. We can extend our array to one of length $\Theta + 1$ simply by setting $N_{\Theta} = \langle J_{d+1}^{E}, \emptyset \rangle$ where $M_{i} = \langle J_{d}^{E}, E_{d} \rangle$. And $E_{d} = \emptyset$. There may be another alternative: Suppose that $F$ is an extender on $M_{i}$ with critical point $\kappa$. Let $\tau = \kappa^{+}$ and let $\pi: J_{\tau}^{E} \rightarrow J_{\tau}^{E'}$. Set $F' = \pi N_{\Theta}(\kappa)$. And $N = \langle J_{\tau}^{E'}, F' \rangle$ is a $T$-premorse with $\tau_{N} = \tau$; we may set $N_{\Theta} = N$.

We summarize the main facts about arrays:

**Fact 1** Set $\mu_{h} = \mu_{hc}$. For $h \leq i < i \leq \Theta$:

(a) $\mu_{h} \leq \mu_{i}$

(b) $J_{\mathcal{E} N_{h}} = J_{\mathcal{E} N_{h}} = J_{\mathcal{E} N_{i}}$.

(c) $(\kappa_{h} = \kappa_{i} = \omega_{p}^{(\omega)}$ and $i < k < i) \rightarrow \mu_{h} < \mu_{k}$

(d) $N_{h} \neq N_{i}$ if $h < i$. 

Note: If \( \lim(\theta) \) it follows that we can extend the array by setting:

\[
N_\theta = \left< \bigcup_{3 < \theta} J \in \mathcal{N}_3, \emptyset \right>,
\]

If \( \theta = \infty \), we extend a wearred by setting:

\[
N = \bigcup_{3 < \infty} J \in \mathcal{N}_3.
\]

Def: Let \( 3 \leq \theta \leq \infty \), \( \lim(\theta) \), \( \lambda = \text{ht}(N_\theta) \).

Let \( 1 < \alpha < \mu \) s.t. \( \lambda \) is a limit ordinal and \( \beta \) cardinal, absolute in \( N_\theta \), \( \langle i, \alpha \rangle \). If \( i < \lambda \) is a cardinal in \( J \in \mathcal{N}_\theta \), then it is a cardinal in \( N_\theta \). Set:

\[
\delta = \delta(\lambda) = \delta(\lambda, \delta) = \sup \{ \delta < \alpha \mid \mu, \delta < \lambda \}.
\]

Fact 2: \( \delta \) is a limit ordinal.

Fact 3: \( \mu_i \delta = \mu_i \delta \) for \( i < \delta \).

Hence:

Fact 4: \( N_\delta = \left< \bigcup_{i < \delta} J \in \mathcal{N}_{i, \delta}, \emptyset \right> = \left< \bigcup_{i < \delta} J \in \mathcal{N}_{i, \delta}, \emptyset \right> \).

Fact 5: \( M_\delta = N_\delta \) and \( \mu_\delta = \lambda = \text{ht}(M_\delta) \).

Fact 6: \( \lambda \) is a limit cardinal in \( N_\delta \),

then \( \mu_\delta = \kappa_\delta = \lambda \).
We gave rather garbled proofs of these facts in §10 of [NFS]. A better proof is contained in §1 of [MOI].

In Steel’s arrays, every extender $E^N_i$ can be traced back to its point of origin as the top extender of some $N_{i+1}$ by means of the “resurrection sequence”. In $T$ arrays, $N_{i+1} = (J^E, \beta, \phi)$ may be much longer than $M_i = (J^E, \phi)$, so extenders $E_i^{N_{i+1}}$ with $\lambda \leq \tau < \omega^2 \beta$ cannot be traced. But these extenders are not applicable, since $\lambda = t^+$ is then a cardinal in $J^E$ and hence $t^+ < t^+_\beta < \tau < \beta$.

In fact every applicable extender does have a resurrection sequence. Using this we can imitate the known constructions to get $T$-iterable premice (e.g., for 1-small $T$-premice).
**Def.** Let \( N \) be a \( T \)-premouse, \( \nu \leq \text{ht}(N) \), \( \nu \) is free in \( N \) iff there is no \( \nu' \leq \text{ht}(N) \) s.t. \( E^N_{\nu'} \neq \emptyset \) and \( E^N_{\nu'} \subset \nu \subset \nu' \) in \( N \).

[Thus, if \( E^N_{\nu} \neq \emptyset \), \( \nu \) is free iff \( \nu \) is applicable.]

**Def.** Let \( N \) be a \( T \)-premouse. Let \( \nu \leq \text{ht}(N) \) be free in \( N \). Set:

\[
\beta = \beta(N, \nu) = \text{the maximal } \beta \in [\nu, \text{ht}(N)] \text{ s.t. } M_{N, N, \beta} \overset{\|3}{\models} \text{ for all } z \in [\nu, \beta].
\]

[\( \beta \) is then the place where \( w^\omega_\nu \) becomes minimal for \( \beta \in [\nu, \text{ht}(N)] \).]

**Fact 7.** There is exactly one \( \gamma \leq \iota \) s.t. \( N_i \|_{\nu} \beta = M_{\gamma} \) (\( \beta = \beta(N_i, \nu) \)).

**Def.** Let \( M \) be a \( T \)-premouse. Let \( \nu \leq \text{ht}(M) \) be free in \( M \).

\[
\beta^+ = \beta^+(M, \nu) = \text{the maximal } \beta \in [\nu, \text{ht}(M)] \text{ s.t. } M_{M, M, \beta} \overset{\|3}{\models} \text{ for all } z \in [\nu, \beta].
\]

**Fact 8.** If \( N_i \) is a weak mouse, then there is exactly one \( \gamma \leq \iota \) s.t. \( M_i \|_{\nu} \beta^+ = M_{\gamma} \) (\( \beta^+ = \beta^+(M_i, \nu) \)).

**Fact 7** and **Fact 8** are proven by simultaneous induction on \( \iota \). The proof is virtually the same as in §10 of [NFS].
Def. Let $\tilde{s} < \theta$, $E_{r}^{N_{3}} \neq \emptyset$, where $\nu$ is free in $N_{3}$. The recursion sequence:

$$S(r, \tilde{s}) = \langle \langle \gamma_{1}, \beta_{1}, \sigma_{1} \rangle, \ldots, \langle \gamma_{\tilde{p}}, \beta_{\tilde{p}}, \sigma_{\tilde{p}} \rangle \rangle$$

(Where $\tilde{p} = \tilde{p}[\tilde{s}, \nu] < \omega$) is defined by:

**Case 1** $\nu = h^{+}(N_{3})$, $S(r, \tilde{s}) = \emptyset$ (hence $\tilde{p} = 0$).

**Case 2** $\nu < h^{+}(N_{3})$, Set:

$\beta_{1} = \beta(N_{3}, \nu)$, $\gamma = \text{the } \gamma \text{ s.t. } \gamma \in \gamma_{\tilde{s}}$. $N_{3} \upharpoonright \beta_{1} = M_{y}$

$\gamma_{a} = \text{the core map } \sigma : M_{y} \rightarrow N_{y}$. $\tilde{s}(r, \tilde{s}) = \nu^{+} \langle \gamma_{a}, \beta_{1}, \sigma_{a} \rangle \rightarrow S(\sigma_{a}(\nu), \gamma_{a}).$

(here $\sigma_{a}(\nu) = h^{+}(N_{y})$ if $\nu = h^{+}(M_{y})$.)

We also write:

$\gamma_{h}[r, \tilde{s}] = \gamma_{h}$, $\beta_{h}[r, \tilde{s}] = \beta_{h}$, $\sigma_{h}[r, \tilde{s}] = \sigma_{h}$.

For $1 \leq h \leq \tilde{p} = \tilde{p}[r, \tilde{s}]$. An additional we set:

$\gamma_{0} = \gamma_{0}[r, \tilde{s}] = \tilde{s}$, $\beta_{0} = h^{+}(N_{3})$, $\sigma_{0} = \text{id } M_{N_{3}}$, $\gamma_{0}^{m} = \sigma_{m}^{m}(r, \tilde{s}) = \nu^{+} \sigma_{m}^{m} \cdots \sigma_{0}$ for $m \leq \tilde{p}$

$\sigma_{\tilde{p}^{*}} = \sigma(\tilde{p})$. Then

$S(r, \tilde{s}) = \langle \langle \gamma_{a}, \beta_{1}, \sigma_{a} \rangle, \ldots, \langle \gamma_{m}, \beta_{m}, \sigma_{m} \rangle \rangle$

$\rightarrow S(\sigma_{m}(\nu), \gamma_{m})$

for $0 \leq m \leq \tilde{p}$. Hence:
Fact 10: \[ \langle \gamma_{m+n}, \beta_{m+n}, \bar{\gamma}_{m+n} \rangle = \langle \gamma_n, \beta_n, \bar{\gamma}_n \rangle \left[ \sigma^{-m}(\nu), \gamma_m \right] \]

Def: Let \( N \) be a \( T \)-premodel. Let \( E_{\rho} \neq \emptyset \) and \( \nu \in N \) where \( \nu \) is free in \( N \). \( \beta_\iota \left[ \nu, N \right] \) (\( i \leq \rho = \rho \left[ \nu, N \right] \)) is defined by:

\[ \beta_0 = h^+ \left( N \right) \text{, } \beta_{i+1} = \beta \left( \nu, N \| \beta_i \right) . \]

Then:

Fact 11: \[ \bar{\beta} = \rho \left[ \nu, N \right] \text{, } \beta_n = \sigma^{-n}(\bar{\beta}_n) \text{ (} 1 \leq n \leq \rho \text{)} \]

Fact 12: \[ \beta_\rho = \nu \text{, } \beta_\rho = \sigma^{-\rho}(\nu) \]

Def: \[ \sigma^* = \sigma^* \left[ \nu, \mu \right] = \sigma \left( \mu \right) \]

\[ \gamma^* = \gamma^* \left[ \nu, \mu \right] = \gamma_\rho \]

Fact 13: \[ \sigma^* : N_\mu \| \nu \rightarrow N_\mu \| \nu \]

Fact 14: Let \( \lambda < \nu \) be a cardinal in \( N_\lambda \), \( m \leq \rho \). Then \( \sigma^{-m} \langle \lambda \rangle = \id \).

Fact 15: Let \( \lambda < \nu \) be a successor cardinal in \( N_\lambda \). Then \( \sigma^{-m} \langle \lambda + 1 \rangle = \id \).

(Facts 9-13 are straightforward. Facts 14 and 15 are proven in §10 of [NFS].)
In forming arrays we shall employ formation rules which are intended to guarantee that each \( N_i \) is iterable (or at least a weak mouse). We will also want to know that each \( N_i \) is unique: As \( N_i = \langle J_{P_i}^E, \emptyset \rangle \) we might find two different possible extensions \( N_{i+1}^h = \langle J_{P_i}^E, F^h \rangle \) \((h = 0, 1)\). To show that this cannot happen we use the theory of bicephali. The proof that \( N_{i+1}^h \) is iterable should also give us the iterability of the pair \( \langle N_i^0, N_i^1 \rangle \). (Roughly speaking, this means that we can do normal iterations of the pair, choosing an extender from one structure or the other and applying it to both.) Such a pair is called a bicephalus. By coiterating it against itself we then discover that, in fact, \( N_i^0 = N_i^1 \). The appropriate iterations for this purpose are \( E_0 \)-iterations — i.e., we take only \( E_0 \) ultraproducts until a truncation occurs. In the following pages we first develop the \( E_0 \)-iterations of \( T \)-premise. Then we introduce bicephali and develop their iteration theory.
Lemma 10. Let $N$ be a $T$-promouse, $\sigma : N \rightarrow N'$, where $\text{crit}(\sigma) < t_{N'}$. Then $N'$ is a $T$-promouse of the same type.

Proof.
For $N$ of type 1 this follows by Lemma 3.1.
For $N$ of type 2 this follows by Lemma 3.3.
For $N$ of type 3 we can quote the proof of Lemma 3.5 (or use Lemma 3.5* and the remark following it.) QED

Def. An $\Sigma_0$ iteration $^\gamma = \langle \langle m_\ast \rangle, \ldots, T \rangle$ of $N$ is defined as before except that we take $E_{i+1}^{\gamma_3, i+1} : N_{i+1} \rightarrow N_i$ whenever $i \in D$ and $i+1$ is simple in $^\gamma_j$, and otherwise $E_{i+1}^{\gamma_3, i+1} : N_{i+1} \rightarrow E_i$. (This motion is developed in [NFS].) Normal and good $\Sigma_0$ $T$-iterations are defined correspondingly.

Lemma 5-7 then go through for $\Sigma_0$ $T$-iterations of $T$-promouse. Lemma 8 holds only in the form: If $i \in D$ and $i+1$ is not simple in $^\gamma_j$, then $E_i$ is close to $M_i$. 
At $N^0, M^1$ are $T$-premise which are normally $E_0$ $T$-iterable, we define the $E_0$-coteration exactly as before. The corresponding version of Lemma 9 goes through as before. From this we get:

**Lemma 9'** Let $N^0, M^1$ be $T$-premise which are preserved and normally $E_0$ $T$-iterable. Let $\tilde{N}^0, \tilde{M}^1$ be the coterater. Either $N^0$ is a simple iterate of $M^0$ and a segment of $\tilde{M}^1$, or conversely.

Note Lemmas 9, 9' hold under the assumption that $M^0, M^1$ are $\beta + 1$-iterable, where $\overline{M^0, M^1} \leq \beta$.

Note Lemmas 9, 9' also hold for mixed coterations where one side is a $E_0$ iteration and the other a $\star$-iteration.
Bicephali

In this section we develop a technique which we shall use to show that the extenders choices in the arrays developed in §3 and §4 are unique. The promises used there will have the property:

\[ E^m \neq \emptyset \text{; then } t^m \text{ is a cardinal in } M. \]

\[ \text{If we didn't have this property we would presumably have to use the method of EMS7 §411 as well as the "bicephalic method".} \]

**Def.** Let \( N \) be an active \( T \)-premouse.

\( N \text{ is of type } A \) iff \( t_N \) is a limit cardinal in \( N \) and \( N_3 \subseteq N \) for arbitrarily large \( 3 < t_N \).

Otherwise \( N \) is of type \( B \).

**Lemma 11.1.** Let \( N \) be of type \( A \). Then

\[ s_N = t_N. \]

\textit{pf.} Trivial.
Lemma 11.2 Let \( N \) be of type B.

Let \( \sigma : N \to N' \) where \( \text{crit}(\sigma) < t_N \).

Then \( t^+_N = \sigma(t^+_N) \) and \( N' \) is of type B.

Proof:

Let \( x \) be the cardinal predecessor of \( t^+_N \).

If \( t_N > x \), then:

\[ \sigma(x) < \sup \sigma'' t_N \leq t_N' \leq \sigma(t_N) < \sigma(t^+_N) , \]

hence \( t^+_N = \sigma(t^+_N) \).

Now let \( x = t_N \). If \( x \) is a successor cardinal then \( \sigma(x) = \sup \sigma'' x \leq t_N' \leq \sigma(x) \).

Hence \( \sigma(x) = t_N' \), \( \sigma(t^+_N) = t^+_N' \).

Now let \( x \) be a limit cardinal.

There is \( y < \omega_N \) s.t. \( (y, x) \cap T_N \neq \emptyset \), since otherwise \( x = \sup T_N \) and \( x \); hence

\[ \sup C_N = x \] and \( N \) is of type A. But then

\( \alpha \in T_N \), since otherwise \( x = x \), \( \sup C_N = \sigma(x) \).

Hence:

\[ \sigma(x) < \sup \sigma'' x \leq t_N' \leq \sigma(x) \in T_N' , \]

where \( (\sigma(x), \sigma(x)) \cap T_N' = \emptyset \). Hence

\[ \sigma(x) = t_N' \), \( \sigma(t^+_N) = d^+ N' = t^+_N'. \]

QED (11.2)
Def By a T-prebicerebrum (T-pb) we mean a triple \( P = \langle P^0, P^1, \tau \rangle \) s.t.

(i) \( P^0, P^1 \) are T-preme

(ii) \( \tau \) is a successor cardinal in \( P^0, P^1 \)

and \( \tau^P_0 = \tau^P_1 \)

(iii) \( \ell < d \) be the cardinal predecessor of \( \tau \) in \( P^h_0 \) (\( h = 0, 1 \)). Then \( \ell \) is a limit cardinal in \( P^h_0 \) (\( h = 0, 1 \)) and one of the following holds:

(A) \( \alpha = \tau^P_0 = \tau^P_1 \) and \( P^h_0 \in P^h_1 \) for arbitrary large \( \ell < d \).

(B) \( P^0, P^1 \) are of type B and \( \tau = \tau^P_0 + (h = 0, 1) \), \( \ell \) is a T-pb of type A if (A) holds and otherwise of type B.

We set: \( t^+_p = t \), \( d = d \).

We use obvious abbreviations like:

\( \beta^+ P = \beta^+ P^h_0 \) (for \( \beta < t^+_p \)), \( \tau^P_0 = \tau^P_0 (\text{for } \beta < t^+_p) \),

\( \#(\alpha) P^h_0 = \#(\alpha) P^h_0 \) (for \( \alpha < t^+_p \)), etc.

Def A \( \text{strict } \) T-pb \( \in \) a T-pb s.t.

\( t^+_p = t^+_p \) for \( h = 0, 1 \). (Thus if \( \ell \) is

\( \text{strict and of type A, then } P^0_0, P^1_0 \) are of type A.)
We are primarily interested in strict pb's. However, we shall have to consider the structures which arise from them by iteration.

**Lemma 11.3** Let \( P \) be a pb. Let \( \sigma : P \rightarrow Q \) where \( \text{crit}(\sigma) < t \). Then:

(a) \( \sigma \uparrow t^+ \uparrow 1 = \sigma \uparrow t^+ \uparrow 1 \pmod{P} \)

(b) If \( P \) is of type A, then \( Q = \langle Q^0, Q^1, t \rangle \) in a T-pb of type A, where:

\[ \alpha = \lambda \uparrow \sigma^0 \uparrow \tau \pmod{P}, \quad \tau = \alpha \uparrow Q^h \quad (h = 0, 1) \]

(c) If \( P \) is of type B, then \( Q = \langle Q^0, Q^1, t \rangle \) in a T-pb of the same type, where \( t = \sigma^h (t^+ \pmod{P}) \quad (h = 0, 1) \).

The proof is left to the reader.

\( \sigma^0, \sigma^1, P, Q \) are as above.

Note: We also use obvious abbreviations like \( \sigma \uparrow J^E \) = \( \sigma^h \uparrow J^E \pmod{P} \) (\( \tau \leq t^+ \)) and \( \sigma \uparrow J^E (\tau) = \sigma^h \uparrow J^E (\tau) \pmod{P} \).
Def: an extender \( e \in P \) iff \( e \) is either a top extender of \( P^o \) or \( P^a \) or \( e = E^p \gamma^+ \) for some \( \gamma < t^+P \).

\( e \) in applicable in \( P \) iff \( e \) is a top extender or \( e = E^p \gamma^+ \) and \( \gamma < t^+P \) is applicable in \( P \).

(Note that there is no \( \gamma < \text{ht}(P) \) with \( t^+_P = t^+P \), since then \( t^+_P \) would not be a cardinal in \( P^h \).

For \( 3 \leq t^+_P \), \( h < 2 \) set:

\[
E^h_P = \left\{ \begin{array}{ll}
E^P_\gamma & \text{if } \gamma < t^+_P, \, E^P_\gamma \neq \emptyset, \, 3 = t^+_P, \\
E^P_{\text{ht}} & \text{if } 3 = t^+_P, \\
\emptyset & \text{otherwise}
\end{array} \right.
\]

(Note \( E^h_3 = E^P_{\text{ht}} \) for \( E^P_\gamma \neq \emptyset, \, 3 = t^+_P, \) unless \( P \) is of type A and \( 3 = t^+_P < t^+_P \).

If \( Q \) is a \( T \)-premouse set: \( Q^o = Q^4 = Q \).

If \( Q \) is active, let \( t^+Q \) have its usual meaning.

Otherwise set: \( t^+Q = \text{ht}(Q) \).

For \( 3 \leq t^+_Q \) define \( E^h_3 \) (\( h = 0, 1 \)) as above.

(Thus \( E^h_3 = E^P_\gamma \) if \( E^P_\gamma \neq \emptyset \) and \( 3 = t^+_P \).

If \( P \) is a \( T \)-premouse then set:

\[
U^P_3 = \{ E^P_0, E^P_1 \} \text{ for } 3 \leq t^+_P.
\]
Def. Let $P$ be a $T$-path or premise. Let $e \in U^p_3$, $e \neq \emptyset$.

$v_e = v(e, p) = v$ where $e = p^h$

$\lambda_e = \lambda(e, p) = \lambda h(e) = \lambda p^h v$

$\tau_e = \tau(e, p) = \tau p^h v = \tau p^h$.

$t_e = t(e, p) = \sum dp$ if $P \cup \emptyset \subseteq T_b$ of type A and $\emptyset = t^p$;

$\tau p^h v$ otherwise.

$t_e^+ = (t_e^+)^p$.

(Note that $v_e$, $t_e$, etc., depend only on $e$, $p$ and not on the $h < 2$ in the definition.)

Def. Let $P$, $Q$ be $T$-paths or $T$-premises.

$P$ is a segment of $Q$ (Prox $Q$) iff either $P_0 = P^h = Q_0 = Q^h$ or else $P_0 = P^h = Q_1 i \exists$ for a $i < T^+$.

At is easily checked that:

Lemma 11.3. At $\tau (\text{Prox} Q)$ and $\tau (\text{Asyn} P)$, then there in $\exists \leq \min (t^+_P, t^+_Q)$ s.t.

there are $e \in U^p_3$, $e' \in U^q_3$ with $e \neq e'$.
We are at present uncertain how to deal with iterations of $T$-problems if the situation described in Cor 6.4 is permitted, so we shall make an assumption which forbids it:

From now on assume that all of the premises we deal with satisfy one of the following conditions:

1. If $E_\nu^M \neq \emptyset$, then $\mathcal{M}_{11} \mathcal{Z}$ in a cardinal in $\mathcal{M}_{11} \mathcal{Z}$.

2. If $E_\nu^M \neq \emptyset$, then $\mathcal{M}_{11} \mathcal{Z} \subseteq \mathcal{M}_{11} \mathcal{Z}$.

This limits the applicability of our theory, since e.g. very large $\mathcal{M}_{11}$-mice may not satisfy (1) or (2). On the other hand, if we take $T_N = \{ \mathcal{M}_{11} \mathcal{Z} \mid \mathcal{M}_{11} \mathcal{Z} \} \subseteq \mathcal{M}_{11} \mathcal{Z}$ which are cardinals in $\mathcal{M}_{11} \mathcal{Z}$, we get a very general class of premises satisfying (2). In any case our assumption does not endanger the applications we shall make in §3, §4, since all of the premises used there satisfy (1).
We now define the notion of a normal iteration of a $T$-$pb$ $D$. This is a structure $\mathbf{Y} = \langle \langle p_\alpha \rangle, \langle \epsilon_i \mid i \in \mathcal{O} \rangle, \langle \mathcal{W}_i \rangle, T \rangle$ which is like a normal iteration of a $T$-premouse with the following differences:

- $p_\alpha$ may be either a premouse or a $pb$.
- $e^\alpha_i$ is some extender applicable in $p_\alpha$.
- Set $t_i = t_{p_\alpha}$ if $e^\alpha_i$ is a top extender of $p_\alpha$. Otherwise set $t_i = t_{p_\alpha} \upharpoonright \mathcal{W}_i$, where $E_{\mathcal{W}_{i+1}}$, $e^\alpha_i < t_{p_\alpha}$. The $t_i$ are determined $T(i+1)$.

- $p_\alpha$ is a $pb$ if $i$ is simple in $\mathbf{Y}$ and in otherwise a premouse; i.e., when we truncate, we truncate to a premouse.

- If $i+1$ is simple in $\mathbf{Y}$ and $s = T(i+1)$, then $\mathcal{W}_{i+1} : p^*_s \rightarrow e^* p_{i+1}$. (Hence this is really a $S_0$-iteration.)

- If $i+1$ is not simple in $\mathbf{Y}$, then

\[ \mathcal{W}_{i+1} : p^*_s \rightarrow e^* p_{i+1} \]
The formal definition follows:

**Def** Let \( P \) be a \( T \)-pb. By a normal iteration of \( P \) of length \( \Theta \) we mean

\[
\mathcal{I} = \langle \langle p_i \mid i < \Theta \rangle, \langle e_i \mid i \in D \rangle, \langle \pi_i \mid i \in \Theta \rangle, T \rangle
\]

\( T \cdot \Theta \) is an iteration tree.

(a) \( T \cdot \Theta \) is an iteration tree

(b) \( P \) is either a \( T \)-pb or a \( T \)-premonoe

and \( P_0 = P \).

(c) At \( i \in D \), then \( i + 1 < \Theta \) and \( e_i \) is applicable

in \( P_i \).

**Def**

\[
t_i = t_{e_i} = \begin{cases} t_{P_i} & \text{if } e_i \text{ is a top extender;} \\ t_{P_i} \upharpoonright \nu_i & \text{if not, where } e_i = E^h_{r_i}. \end{cases}
\]

\[
\nu_i = \nu_{e_i} = \nu \text{ where } e_i = P_i \upharpoonright \nu_i \quad (h = 0, 1)
\]

\[
\lambda_i = \lambda_{e_i} = > \nu_i \\
\kappa_i = \kappa_{e_i} = \kappa_i + P_i
\]

(Clearly \( t_i, t_i^+, \ldots, \kappa_i \) are uniquely determined by the pair: \( P_i, e_i \).)

(d) At \( P_i \) is a \( T \)-pb, then for all \( h \leq_T i \), \( P_h \) is a \( \Theta \)-pb and \( \pi_{hi} : P_h \rightarrow \Sigma_i P_i \).

(i.e. \( \pi_{hi}^l : P_h^l \rightarrow \Sigma_i P_i^l \) and \( \pi_{hi}^t : P_h^t = \pi_{hi}^t \downarrow t^+ \))
\( \text{Note } \sigma : P \to A \text{ mean } \sigma = \langle \sigma^h, \sigma^* \rangle \text{ with } \\
\sigma^h : P^h \to A^h \text{ and } \sigma^* t_p = \sigma^h t_p, \\
\sigma^h t_p \in t_{A^h} \text{ for } p \in P \text{ and } A, \text{ similarly for } \frac{1}{2}, \frac{3}{2}, \text{ etc.} \)

Def: \( i \) is simple in \( J \) iff \( \theta_i \) is total for all \( h \leq i \). \( i \) is simple above \( k \) in \( J \) iff \( \theta_i \) is total for \( k \leq h \leq i \). (Hence \( i \) is simple if \( P_h \) is a p b.)

(\( \theta \)) The \( \theta_i \) commute, i.e.,
\( i \leq^T k \leq^T i \rightarrow \theta_i \circ \theta_k = \theta_k \circ \theta_i \)
(An obvious abuse of notation.)

(f) \( \text{At } \text{Lim} \lambda, \text{ then } \lambda \) is simple above \( i \) for some \( i < \lambda \). More over:
\( M_{\lambda} \langle \theta_i \mid i \leq^T \lambda \rangle = \text{The direct limit of} \\
\langle M_i \mid i \leq^T T \rangle, \langle \theta_i \mid i \leq^T i < \lambda \rangle \).

(g) \( \text{At } i \notin D, \text{ } i+1 \in D, \text{ then } i = T(i+1) \), \( P_i = P_{i+1} \).
\( \theta_i \circ i+1 = id \)

(h) \( \text{At } i \in D, \text{ then } T(i+1) = \text{the least } z \in D \text{ s.t.} \\
\mu_i < z \) (hence \( z \leq i \)).
Def. Let $i \in D$, $\bar{s} = T(i+1)$

$$P^* \in \begin{cases} P_{\bar{s}} & \text{if } \tau_i \text{ is a cardinal in } P_{\bar{s}} \\ P_{\bar{s}} \upharpoonright \gamma & \text{if not, where } \gamma = \gamma_i \text{ is maximal s.t. } \tau_i \text{ in a cardinal} \\ \text{in } P_{\bar{s}} \upharpoonright \gamma \end{cases}$$

(Note) We must show that this definition makes sense if $P_i$ is a $T$-pb. Clearly $\tau_i \leq t^+_P$, since $\tau_i \leq t^+_P$ for $h = 0, 1$. If $\tau_i$ is not a cardinal in $P_i^h$, then $\tau_i < t^+_P$ is not a cardinal in $H^\uparrow P_i^h$, hence $t^+_P$ is a cardinal in $P_i^h$. Hence $\gamma_i < t^+_P$.

(i) If $\bar{s} = T(i+1)$ and $P_i^* \neq P_{\bar{s}}$ or $\bar{s}$ is not simple in $J$, then $\bar{s}_{j+1} : P_i^* \rightarrow P_i^*$. If $\bar{s} = T(i+1)$ and $P_i^* = P_{\bar{s}}$ and $\bar{s}$ is simple in $J$, then $\bar{s}_{j+1} : P_{\bar{s}} \rightarrow P_{\bar{s}}$. If $i \in D$, then $\gamma_i > t^+_P$ for all $i \in Dn i$.

(Note) If $P_i$ is a $p$-b and $\bar{s}_{j+1} : P_i^h \rightarrow P_i^h$, then $\pi_{i+1} : P_{\bar{s}}^0 \rightarrow P_i^0$, $\pi_{i+1}^h : P_{\bar{s}}^h \rightarrow P_i^h$, where $\text{crit}(\pi_{i+1}^h) = t^+_P$ for $h = 0, 1$. Hence we may set $\gamma_i^0, \gamma_i^h = \text{crit}(\pi_{i+1}^h)$.)
An examination of the proofs of Lemma 5 - Lemma 7 show that with slight modification they go through for normal iterations of $T$-pre-epiloph: In particular we get: $i' < i \rightarrow t_i^+ < t_i^t$. Hence $t_i^o, t_i^t < t_i^t \leq t_i^o, t_i^t$, since there is no cardinal between $t_i^t$ and $t_i^t$ in $P_i$.

The proof of Lemma 8 then yields:

**Lemma 8'** Let $Y = \langle \langle P_i^o \rangle, \langle P_i^t \rangle, \langle \pi_i^t \rangle \rangle, T \rangle$ be a normal iteration of the $T$-pb $P$. At $i+1$ is not simple in $Y$ and $i \in D$, then $P_i^o$ is close to $P_i^t$.

**Cor 8.1** At $i$ is not simple in $Y$ and $i \notin$ simple above $i$, $i \geq i\hat{1}$, then $\pi_i : P \rightarrow P_i^t$.

We also have:

**Lemma 11.4** Let $Y$ be a normal it. of a $T$-pb. $P$ at $P^o \neq P^t$. Let $i$ be simple in $Y$. Then $P_i^o \neq P_i^t$.

**M. Suppose not.** Then $\pi_i^o \uparrow P_i^t : P_i^+ \rightarrow P_i^t$. Hence, letting $o^*, e^1$ be the top extenders
of $P$ and $e^*_i$, $e^+_i$, the top extenders of $P_i$, we have: $e^*_i \upharpoonright t^+_P = e^+_i \upharpoonright t^+_P$. Since for $2 < t^+_P$ we have:

\[ d \in e^+(x) \iff \pi_{e^+}(d) \in e^+(\pi_{e^+}(x)) = e^+(\pi_{e^+}(x)) \iff d \in e^+(x). \]

Since $2 < t^+_P$ ($h = 0, 1$), it follows that $e^0 = e^1$; hence $P^0 = P^1$, and

\[ P^h = \langle U^E, e^h \rangle, \quad \varphi_i: U^E \rightarrow e^h \cup e^h. \]

QED (Lemma 19.4)

*→

Def: A normal iteration strategy $S$ for the T-pb $P$ is a partial function on normal iteration $\gamma$ of $P$ of limit length $\alpha$. Let $b = S(\gamma)$ in a cofinal branch in $T_\gamma$ with a well-founded (hence transitive) limit model $P_b$, whenever $S(\gamma)$ is defined.

By an $S$-iteration we mean an $\gamma$-act, whenever $\gamma < \text{lh}(\gamma)$, $\lim(\gamma)$, then

\[ \exists i \in \gamma : \gamma^3 = S(\gamma^i \gamma). \]

* At is clear that each $P_i$ has the same type (A or B) as $P$ if $P_i$ is a T-pb,
Def. We say that $S$ is a **successful iteration strategy** (up to $\Theta$) iff whenever $Y$ is an $S$-iteration (of length $< \Theta$), then $Y$ can be continued in the following sense:

- If $lh(Y) = h+1$, $h+2 < \Theta$, and $e \in \text{applicable in } P_h$ with $e(0) > e_l^+$ for all $l < h$ s.t. $l \in P$, then $Y$ can be extended to $Y'$ of length $h+2$ by setting: $e_h = e$.
- If $lh(Y) = \lambda < \Theta$, $\text{Lim}(\lambda)$, then $S(Y)$ is defined.

Def. $P$ is **normally iterable** (up to $\Theta$) iff $P$ possesses a successful iteration strategy (up to $\Theta$).

- $P$ is **weakly iterable** iff whenever $\sigma : P < P$ and $P$ is countable and transitive, then $\overline{P}$ is normally iterable up to $\omega_1 + 1$.

Def. By a **T-binarymachine**, we mean a weakly iterable $T$-pb $P$ s.t. $P^h$ is preotic for $h = 0, 1$. 
The main result on bicephalism is that they are trivial.

**Lemma 12** Let \( P \) be a T-bicephalous.

Then \( P^0 = P^1 \).

**proof.** Suppose not.

By a L"owenheim–Skolem argument we may assume \( P \) to be countable. Hence \( P \in \omega_1 + 1 \)-iterable. Fix an iteration strategy \( S \) which is successful up to \( \omega_1 + 1 \). We form an \( S \)-coiteration of \( P \) — i.e., a pair \( \langle Y, Y^* \rangle \) of normal \( S \)-iteration of \( \langle \psi, \phi \rangle \) up to \( \omega_1 + 1 \).

\[ Y^h = \langle P^h_i, e^h_i | i \in D^h \rangle, \langle \pi^h_i, \tau^h \rangle \]

with \( P^h_i = \langle P^h_0, P^h_1 \rangle \) if \( P^h \) is a p.h.

(a) \( P^h = P \) (\( h = 0, 1 \))

(b) Let \( i \in \omega_1 \) s.t. \( P^h_i \) is defined (\( h = 0, 1 \)). Set \( t_i \) the least \( \gamma \leq \min(t_{P^h_i}, t_{P^h_i}) \) s.t. there are \( e^0, e^1 \in U^p_i \) with \( e^0 \neq e^1 \). Choose such \( e^0, e^1 \) and let \( e^0_i, e^1_i \in D^h, e^h_i \neq \emptyset \). At \( i \in D^h \), let \( e^h_i = e^h_i \).
Claim The coiteration terminates below \( \omega_1 \).

pf. Suppose not.

Then \( h^*(\Sigma^h) = \omega_1 + 1 \). Let \( X \subseteq H_{\omega_2} \) s.t.,
\( \gamma_0, \gamma_1 \in X \) and \( X \) is countable. Let
\( \sigma : \Pi \ni \gamma \rightarrow X \), \( \Pi \) transitive. Then
\( \sigma \upharpoonright H_{\omega_1} = \text{id} \). Let \( d = \omega_1 \). Then
\( d = \omega_1 + 1 \), \( \sigma(d) = \omega_1 \). Let
\( \sigma(\langle p, \gamma^h \rangle) = \langle p, \gamma^h \rangle \), where
\( \gamma^h = \langle \langle \alpha^h \rangle, \langle \varepsilon^h \rangle, i \in D^h \rangle, \langle \pi^h_{\gamma^h} \rangle, \overline{T^h} \rangle \),
Then \( \alpha^h = \alpha^h \) for \( i < d \), \( \varepsilon^h = \varepsilon^h \) for \( i < d \).
\( D^h = d \cap D^h \), \( \overline{T^h} = T^h \cap d^2 \), and
\( \pi^h_{\gamma^h} = \pi^h_{\gamma^h} \) for \( i \leq d \).

But if \( b^h = \Sigma_{i \leq \omega_1} T^h \), \( \overline{b^h} = \sigma^{-1}(b^h) \),
Then \( \overline{b^h} = b^h \cap d \). Clearly \( d \subseteq b^h \), since
\( d \) is a limit pt. of \( b^h \); hence \( \overline{b^h} =
= \{ i \mid i \leq \omega_1 \} \subseteq T^h \cap d^2 \). Then
characterizes \( T^h \). Since \( i \),
\( \overline{p^h}_d, \langle \pi^h_{\gamma^h} \rangle_{i \leq \omega_1 \cap d} \rangle = \text{the direct}
limit of \( \langle p^h \rangle_{i \leq T^h \cap d} \rangle, \langle \pi^h \rangle_{i \leq T^h \cap d} \rangle \),
we conclude \( \overline{p^h}_d = p^h, \pi^h_{\gamma^h} = \pi^h_{\gamma^h} \).
We know that any truncation on the branch $b^n$ occurred below $d$. Hence

\[ \pi_{\omega_1}^h : P_d^h \rightarrow P_d^h \]. Let $x \in P_d^h$,

\[ x = \pi_{\omega_1}^h (\overline{x}) \), \]  
$c < d$. Then $\pi_{\omega_1}^h (x) = \pi_{\omega_1}^h (\overline{x}) = \sigma (\pi_{\omega_1}^h (\overline{x}) ) = \sigma (\pi_d^h (\overline{x}) ) = \sigma (x)$. Hence,

(1) $\sigma \cap P_d^h = \pi_d^h, \omega_1 (\text{hence } \sigma (\pi_d^h, \omega_1) = d)$. Now let $\gamma + 1 < T^h \omega_1 \ni t$, $d = T^h (\gamma + 1)$.

Then $\pi_d^h, \gamma + 1 : P_d^h \rightarrow P_d^h \gamma + 1$ (or $\rightarrow e^{\gamma + 1}$ if $P_d^h$ is not a ph), But

Then $d = k e_{\gamma}$ $\leq t_d$ and $T = T e_{\gamma}$ $\leq t^+_d$,

where $t^+_d$ is a cardinal in $P_d^h$ for $h, d < \omega$.

and $\bigcup_{t^+_d} E_{d}^h = \bigcup_{t^+_d} E_{d}^h$ (or $\rightarrow \gamma$). But then

(2) $T = d + P_d^h = \pi_d^h \gamma$ (h = 0, 1) and

(3) $\bigcup_{t^+_d} E_{d}^h = \bigcup_{t^+_d} E_{d}^h$ (hence $\pi_d^h, \omega_1 \pi_d^h = \pi_d^h$).

Let $x \in \pi_d^h, \omega_1 \pi_d^h$. Let $z < \min (t^+_z, \gamma, \gamma)$

(whence $t^+_z = h \pi_d^h$). Then

(4) $z \in \pi_d^h (x) \leftrightarrow z \in \pi_d^h (x) \gamma + 1 \leftrightarrow z \in \pi_d^h, \omega_1 (x)$

$\leftrightarrow z \in \sigma (x)$.
$$\text{since } \text{crit}(\pi^h_{\gamma + 1}, \omega_1) \geq t^h_{\gamma}.$$  Thus

$$E^{0}_{\gamma, 3} = E^1_{\gamma, 3}$$  where \(3 = \min(t^0_{\gamma}, t^1_{\gamma}).$$

From this we derive a contradiction.

Case 1 \(\gamma_0 = \gamma_1 = \gamma.$$

Case 1.1 \(t^0_\gamma = t^1_\gamma.$$. Then \(E^0_\gamma = E^1_\gamma \neq \emptyset,$$

since \(t^h_\gamma \geq 2^\gamma \cdot \beta^h.$$. Contr.$$

Case 1.2 \(t^0_\gamma \neq t^1_\gamma$$ (let e.g. \(t^0_\gamma < t^1_\gamma$$)

Case 1.2.1 \(\text{P}^1_\gamma$$ is a type A pb.

Then \(\tilde{P}^0_\gamma = (\tilde{P}^0_\gamma)_{t^0_\gamma} = (\tilde{P}^1_\gamma)_{t^0_\gamma} \in \text{P}^1_\gamma.$$

Hence \(\tilde{P}^0_\gamma \in \text{P}^1_\gamma \cap t^+.$$. Hence \(t^+ =

\quad = (t^0_\gamma)^+ \tilde{P}^0_\gamma < t^+.$$. Contr.$$

Case 1.2.2 Case 1.2.1 fails and \(t^0_\gamma = t^0_{\tilde{P}^0_\gamma}.$$

Then \(\tilde{P}^0_\gamma = (\tilde{P}^1_\gamma)_{t^0_\gamma}$$ where \(t^0_\gamma < t^1_\gamma = t^0_{\tilde{P}^1_\gamma};$$

hence \(\tilde{P}^0_\gamma \in \text{P}^1_\gamma$$ and we get a contradiction as before.

Case 1.2.3 The above fail.

Then \(\tilde{P}^0_\gamma$$ is a type A pb. Hence if

\(\text{P} \cap \tilde{P}^0_\gamma$$ were a pb it would
also have to be of type A + hence Case 1.2.1 would apply. Hence $P_i^\gamma$ is a premonse. Since $t_{i}^\circ = t_{P_i^\circ} t_{\bar{P}_i^\circ} < t_{\bar{P}_i^\circ}$, we have $t \in T_{\bar{P}_i^\circ}$ where $t = t_{\bar{P}_i^\circ}$. But $T_{\bar{P}_i^\circ}$ is bounded in $t_{i}^\circ$, since otherwise $t_{i}^\circ \in T_{\bar{P}_i^\circ}$ and $t_{i}^\circ = t_{\bar{P}_i^\circ}$. Hence there is $3 < t_{i}^\circ$ s.t. $(3, t_{i}^\circ) \cap T_{\bar{P}_i^\circ} = \emptyset$. Since $\bar{P}_i^\circ = (\bar{P}_i^{\circ})^\circ$, there is a canonical $\sigma : \bar{P}_i^\circ \to \bar{P}_i^{\circ}$ s.t. $\sigma 1_{\bar{P}_i} = \text{id}$. Hence $\sigma(t_{i}^\circ) \in T_{\bar{P}_i^{\circ}}$ and $(3, \sigma(t_{i}^\circ)) \cap T_{\bar{P}_i^{\circ}} = \emptyset$.

Since $t_{i}^\circ \leq t_{i}^\circ = t_{P_i^\circ} < t_{i}^\circ$, But $t_{i}^\circ$ is a cardinal in $\bar{P}_i^\circ$ since $P_i^\circ$ is of type A. Hence $t_{i}^\circ \leq t \leq \sigma(t_{i}^\circ)$ and $\sigma(t_{i}^\circ) = t_{i}^\circ < t_{i}^\circ$. Contradiction! QED (Case 1)

Case 2 $\gamma \neq \gamma'$ $\gamma' < \gamma_1$

Then $t_{\gamma_0}^+ < t_{\gamma_1}^+$ and $t_{\gamma_0}^h < t_{\gamma_1}^h$ (h, k = 0, 1)
Case 2.1: $P_1^\alpha$ is a type $A$ path.

Exactly like Case 1.2.1.

Case 2.2: Case 1.2.1 fails and $t_{\gamma_0}^\circ = t_{\gamma_1}^\circ$.

Exactly like Case 1.2.2.

Case 2.3: The above fail.

We again have: $P_0^\alpha$ is a type $A$ path and $P_1^\alpha$ is a premouse. Set $\gamma_1$.

$b = \{ i | i < \omega_1 \omega_1 \}$. Then $\alpha, \gamma_0 \in b$ and any truncation in $b$ must occur below $\alpha$.

Let $i + 1 \leq_{\gamma_0} \gamma_1$ be minimal s.t. $\gamma_0 \leq i$. Then $\gamma = T_{\gamma_0}(i + 1) \leq \gamma_0$.

Case 2.3.1: $\kappa_1^\alpha < t_{\gamma_1}^\circ$.

(1) $\kappa_1^\alpha > \kappa_{\gamma_0}^\circ = \kappa_{\gamma_1}^\circ$.

Suppose not. $P_1^\alpha$ is an active premouse. Set $\kappa = \kappa P_1^\alpha$. Then

$\kappa_{\gamma_1}^{\alpha + 1} = \kappa_{\gamma_1}^\alpha \leq \kappa_{\gamma_1}^\alpha = \kappa_{\gamma_0}^\circ$. 

Hence \( \kappa^* > \bar{\kappa} \), since otherwise

\[
\kappa^*_{i+1} = \lambda^* \geq t^* \geq \kappa^0 > \kappa^* = \kappa^* \leq \kappa^*_{i+1}.
\]

Contradiction! Hence \( \bar{\kappa} = \kappa^*_{3,1+1} = \kappa^*_{1+1} \).

But \( \text{crit}(\bar{\kappa}_{3,1+1}, \gamma) \geq t^* \geq \gamma^* > \kappa^0 \) = \gamma^*.

Q.E.D. (1)

But then, since \( \text{crit}(\bar{\kappa}^*, \gamma) = \kappa^* \), we have \( e^*_{3} \\gamma^*_{1} = e^*_{3} \\gamma^*_{1} = e^*_{3} \\gamma^*_{1} \).

Let \( N = (\bar{\kappa}^*)^* \gamma_{1}^* \). Then \( N \in \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^* \).

\[
= \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^* = \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^* = \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^*.
\]

and \( N = (\bar{\kappa}^*_{1})^* \gamma_{1}^* = (\bar{\kappa}^*_{1})^* \gamma_{1}^* \), Let

\[
\bar{\kappa}^*_{1}, \ldots (N) = N^*. Then N^* = \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^* \text{ where } \gamma^* = \bar{\kappa}^*_{1}, (\kappa^*_{1}) \geq \lambda^* \geq t^* \geq \gamma_{1}^*.
\]

Hence

\[
\bar{\kappa}^* = (\bar{\kappa}^*_{1})^* \gamma_{1}^* \in \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^* = \exists (\bar{\kappa}^*_{1})^* \gamma_{1}^*.
\]

Contradiction! Q.E.D. (Case 2.3.1)
Hence $t^0 < t^1$. Hence $3 = \gamma = 1$, and

otherwise $\gamma_1 < t^1 < t^2 < t^0$. But $\gamma_1 < t^0$, hence $t^0 < t^1$ and $\gamma^0 | t^0 = \gamma^1 | t^0 = \gamma_1 | t^0$, since $\text{crit} (\gamma, \gamma) = \gamma_1 \geq t^0$. But this leads to a contradiction just as in Case 1.2. A.E.D. (Claim)

Now let $P^0$, $P^1$ be the ultimate co-iterates. Using presolitarity it follows as usual that at least one $P^h$ is a simple iterate of $P$. Moreover, $P^h$ is a subsegment of $P^{1-h}$ if $P^{1-h}$ is not simple. Let $Q = P^h$, where $Q$ is a simple iterate of $D$ and a subsegment of $P^{1-h}$. Then $Q^0 = Q^1$. Using the iteration map $\pi : P \rightarrow Q$ we conclude that $e^0 | t^0 = e^1 | t^0$, where $e^0, e^1$ are the top extenders of $P$. Hence $P^0 = P^1$. Contrs!

A.E.D. (Lemma 12)
Remark Lemma 12 can be proven without assuming (*1) or (*2).

Def Let $P$ be a $T$-pb. Let $3 \leq t_P$ and let $e \in U_3^P$. $e$ is superstrong in $P$

\[ \forall e = 3 \text{ (hence } \lambda_e = t_e^P \text{, as it easily seen).} \]

Note A $\models \forall Y = \langle \langle P_i \rangle, \langle e_i \mid i \in D \rangle, \langle \pi_i \rangle \rangle$, $T$

is a normal iteration of a $T$-pb, then $t_i^+ = t_i^+$, $i < j$ can only occur if $e_i$ is superstrong in $P_i$, since then the situation of Cor 6.1 must be present.

Def Let $Y$ be an above, $i$ is superstrong in $Y$ iff $e_i$ is superstrong in $P_i$.

At it easily seen that $i$.

Fact Let $i$ be superstrong in $Y$,

\[ \forall e \in U_3^{P_i, t_i} \text{, } e \text{ is superstrong in } P_i^{t_i} \text{.} \]
We now modify (b) in the definition of coiteration to read:

(b') Let $i < \omega$, s.t. $P_i^h$ is defined ($h = 0, 1$).

Set: $t_i^+ = \min(t_{p_0}^+, t_{p_1}^+)$

s.t. there are $e^0 \in U^0_{p_0}, e^1 \in U^1_{p_1}$ with $e^0 \neq e^1$. If possible, choose $e^0, e^1$ s.t. both are superstrong or both are not superstrong. Set:

$i \in D_i^h$ iff $e_i^h \neq \emptyset$, unless $e_i^{1-h}$ is superstrong in $P_i^{1-h}$ and $e_i^h$ is not superstrong in $P_i^h$, in which case we set: $i \notin D_i^h$.

Set: $e_i^h = e_i^h$ if $i \in D_i^h$.

The proof of Case 1 remains unchanged.

The proof of Case 2 remains unchanged for $t_{p_0}^+ < t_{p_1}^+$.

Now let $t_{p_0}^+ = t_{p_1}^+$; if $t_{p_0}^+ \neq t_{p_1}^+$ we get a contradiction exactly as in case 1, i.e. $t_{p_0}^+ = t_{p_1}^+$. 
Hence $\gamma = \gamma_0$ and $\beta = \beta_0$.

Since $t^+_0 \geq t^+_1$, we have $t^+_0 + 1 = t^+_0$.

Hence $\gamma$ is superstrong in $Y^0$ or $Y^1$. But if it were superstrong in $Y^1$ it would have to be so in $Y^0$ as well, since $\gamma \in D^0$. Hence $\gamma \in \text{superstrong in } Y^0$.

Claim: $\gamma + 1$ is not superstrong in $Y^0 \cap Y^1$.

Proof: Suppose not. $\gamma + 1$ is not superstrong in $Y^0$ by the above Fact. Hence it is superstrong in $Y^1$. Hence $\gamma \in D^1$ by the above Fact.

Thus $P_0^1 = P_1^1$. Let $e = e^1_\gamma \gamma + 1$.

Since $e \in U_{P_0^1}$ is superstrong and was not chosen at $e^1_\gamma$, we must have $e = e^0_\gamma$. Since the iteration did not terminate at $\gamma$, there must be $e' \in U_{P_0^1}$ not $e' \in \text{not } P_0^1$. 


supers strong in $P_{\gamma}$. But since $e'$ was
not chosen as $e_{\gamma+1}$, we must
have: $e' = e_{\gamma+1}$. We then have the
following situation: Set $Q = P_{\gamma}$,
$F_0 = e$, $F_1 = e'$. Then $Q$ is a $T_\mu$-
$Q = \langle Q_0, Q_1, t \rangle$ with $e.g. F_1$ the
top extender of $Q$ and
\[ \pi: Q_0 \to Q_1. \]
But then $\pi^t Q_1 \subseteq \lambda Q_0$ and hence
\[ \pi^t Q_1 \supseteq \lambda Q_0 \supseteq t = t_+ Q_1. \]
This is impossible in prebiconhali,
contradiction! \hfill QED (Claim 1)

But then $P_{\gamma+2}^h = \emptyset$ for $h = 0, 1$.
Hence $t_+ > t_+ = t_+^{\gamma}$. Hence
\[ \gamma_1 = \gamma + 1 \quad \text{and} \quad e_{\gamma+1} = e_{\gamma}^0 . \]
$P_{\gamma+1}^0 = P_{\gamma}$. This is impossible
since $e_{\gamma+1}$ is not supers strong in $P_{\gamma+1}$
contradiction! \hfill QED