Applications of Differential Calculus to Quasilinear Elliptic Boundary Value Problems with Non-Smooth Data

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Abstract

This paper concerns boundary value problems for quasilinear second order elliptic systems which are, for example, of the type

$$\begin{aligned} \partial_j\left(a^{ij}_{\alpha\beta}(u,\lambda)\partial_i u^\alpha + b^j_\beta(u,\lambda)\right) + c^i_{\alpha\beta}(u,\lambda)\partial_i u^\alpha &= d_\beta(u,\lambda) \quad \text{in } \Omega, \\ \left(a^{ij}_{\alpha\beta}(u,\lambda)\partial_i u^\alpha + b^j_\beta(u,\lambda)\right)\nu_j &= e_\beta(u,\lambda) \quad \text{on } \Gamma_\beta, \\ u^\beta &= \varphi^\beta \quad \text{on } \partial\Omega \setminus \Gamma_\beta. \end{aligned}$$

Here Ω is a Lipschitz domain in $\mathbb{R}^N$, ν_j are the components of the unit outward normal vector field on $\partial\Omega$, the sets Γ_β are open in $\partial\Omega$ and their relative boundaries are Lipschitz hypersurfaces in $\partial\Omega$. The coefficient functions are supposed to be bounded and measurable with respect to the space variable and smooth with respect to the unknown vector function u and to the control parameter λ. It is shown that, under natural conditions, such boundary value problems generate smooth Fredholm maps between appropriate Sobolev-Campanato spaces, that the weak solutions are Hölder continuous up to the boundary and that the Implicit Function Theorem and the Newton Iteration Procedure are applicable.

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1 Introduction

This paper concerns boundary value problems to quasilinear second order elliptic equations and systems in divergence form. We consider boundary value problems which have non–smooth data in the following sense: First, they are posed on a domain $\Omega \subset \mathbb{R}^N$, the boundary of which is non–smooth, but only Lipschitz, in general. Second, we deal with mixed (changing type) boundary conditions, where the "Dirichlet" and the "Neumann" boundary parts can touch. Third, the right hand sides are not functions, but only distributions in $W^{-1,2}(\Omega)$, in general. And fourth, the coefficient functions are not continuous with respect to the space variable, but only measurable and bounded, in general. On the other hand, the coefficient functions are smooth with respect to the unknown function u and to the control parameter λ. Remark that we do not impose any growth conditions on the coefficient functions with respect to u and λ.

The aim of our paper is to apply the Implicit Function Theorem and the Newton Iteration Procedure to the boundary value problems. We state conditions such that, roughly speaking, the following is true:

Let $u = u_0$ be a bounded weak solution to the boundary value problem with control parameter $\lambda = \lambda_0$, and suppose that zero is the only weak solution to the formal linearization of the boundary value problem at $u = u_0$ and $\lambda = \lambda_0$. Then there exists a $\gamma \in (0, 1)$ (depending on u_0 and λ_0, in general) such that the following holds:

(i) For all λ close to λ_0 (in a certain normed vector space of control parameters) there exists a weak solution $u \in C^{0,\gamma}(\overline{\Omega}) \cap W^{1,2}(\Omega)$, it depends C^1–smoothly (in the sense of $C^{0,\gamma}(\overline{\Omega}) \cap W^{1,2}(\Omega)$) on λ and it is the only solution close to u_0 in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$. In particular $u_0 \in C^{0,\gamma}(\overline{\Omega})$.

(ii) Let u_1 be sufficiently close to u_0 in $L^\infty(\Omega) \cap W^{1,2}(\Omega)$, and let $u_2, u_3, \ldots$ be the Newton iterations determined by means of the formally linearized boundary value problem in $u = u_1, u_2, \ldots$ and $\lambda = \lambda_0$. Then $u_l \in C^{0,\gamma}(\overline{\Omega})$ and $u_l \to u_0$ in $C^{0,\gamma}(\overline{\Omega}) \cap W^{1,2}(\Omega)$ for $l \to \infty$.

In order to formulate our results more precisely, let us consider the following model problem:

$$\left.\begin{array}{rcll} -\nabla \cdot (\rho a(u) \nabla u) + b(u) &=& -\nabla \cdot f & \text{in } \Omega, \\ \rho a(u) \nabla u \cdot \nu + c(u) &=& f \cdot \nu + g & \text{on } \Gamma, \\ u &=& 0 & \text{on } \partial\Omega \setminus \Gamma. \end{array}\right\} \quad (1.1)$$

In (1.1), Γ is a relatively open subset of $\partial\Omega$, $\nu : \partial\Omega \to \mathbb{R}^N$ is the unit outward normal vector field on $\partial\Omega$, and $\rho \in L^\infty(\Omega)$. Suppose that the nonlinearities $a, b, c : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, and the right hand sides satisfy $f \in [L^p(\Omega)]^N$ and $g \in L^{p-1}(\Gamma)$ for some $p > N$. The control parameter is, for example,

$$\lambda = (\rho, f, g) \in L^\infty(\Omega) \times [L^p(\Omega)]^N \times L^{p-1}(\Gamma).$$

The formal linearization of (1.1) at $u = u_0$, $\rho = \rho_0$, $f = f_0$ and $g = g_0$ is

$$\left.\begin{array}{rcll} -\nabla \cdot (\rho_0 \left(a(u_0) \nabla u + a'(u_0) u \nabla u_0\right)) + b'(u_0) u &=& 0 & \text{in } \Omega, \\ \rho_0 \left(a(u_0) \nabla u + a'(u_0) u \nabla u_0\right) \cdot \nu + c'(u_0) u &=& 0 & \text{on } \Gamma, \\ u &=& 0 & \text{on } \partial\Omega \setminus \Gamma. \end{array}\right\} \quad (1.2)$$

Now, suppose that the essential infimum of $\rho_0 a(u_0)$ is positive, that zero is the only weak solution to (1.2) and that a mild assumption concerning the relative boundary of Γ in $\partial\Omega$ is satisfied. Then the assertion (i) above is true. If, moreover, a', b' and c' are locally Lipschitz continuous, then assertion (ii) is also true. Here, for a given Newton iteration u_l, the next Newton iteration u_{l+1} is determined by means of the inhomogeneous linear boundary value problem

$$\begin{aligned} -\nabla\cdot\left(\rho_0\left(a(u_l)\nabla u_{l+1}+a'(u_l)u_{l+1}\nabla u_l\right)\right)+b'(u_l)u_{l+1} &= \\ =-\nabla\cdot\left(\rho_0 a'(u_l)u_j\nabla u_l\right)+b'(u_l)u_l-b(u_l)-\nabla\cdot f_0 \quad &\text{in } \Omega, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \rho_0\left(a(u_l)\nabla u_{l+1}+a'(u_l)u_{l+1}\nabla u_l\right)\cdot\nu+c'(u_l)u_{l+1} &= \\ =\rho_0 a'(u_l)u_j\nabla u_l\cdot\nu+c'(u_l)u_l-c(u_l)+f_0\cdot\nu-g_0 \quad &\text{on } \Gamma, \end{aligned} \tag{1.4}$$

$$u_{l+1}=0 \quad \text{on } \partial\Omega\setminus\Gamma. \tag{1.5}$$

Let us use the model problem (1.1) also for explaining the main ideas of the proofs in our paper.

As usual, we denote by $W_0^{1,2}(\Omega\cup\Gamma)$ the space of all $u\in W^{1,2}(\Omega)$ such that $u=0$ on $\partial\Omega\setminus\Gamma$ in the sense of trace, and $W^{-1,2}(\Omega\cup\Gamma)$ is its dual space. A weak solution to (1.1) is a function $u\in L^\infty(\Omega)\cap W_0^{1,2}(\Omega\cup\Gamma)$ such that

$$\int_\Omega \left(\left(\rho a(u)\nabla u-f\right)\cdot\nabla v+b(u)v\right)dx+\int_\Gamma \left(c(u)-g\right)v\,d\Gamma=0 \tag{1.6}$$

for all $v\in W_0^{1,2}(\Omega\cup\Gamma)$. It is easy to see that this variational equation is equivalent to the operator equation

$$\mathcal{F}(u,\rho,f,g)=0, \tag{1.7}$$

where $\mathcal{F}$ is a continuously differentiable map from $\left(L^\infty(\Omega)\cap W_0^{1,2}(\Omega\cup\Gamma)\right)\times L^\infty(\Omega)\times[L^p(\Omega)]^N\times L^{p-1}(\Gamma)$ into $W^{-1,2}(\Omega\cup\Gamma)$ and is defined such that $\langle\mathcal{F}(u,\rho,f,g),v\rangle$ is the left hand side in (1.6). Here $\langle\cdot,\cdot\rangle$ is the dual pairing on $W_0^{1,2}(\Omega\cup\Gamma)$. Moreover, the weak formulation of (1.2) is equivalent to

$$\frac{\partial\mathcal{F}}{\partial u}(u_0,\rho_0,f_0,g_0)u=0.$$

But, unfortunately, the partial derivative $\frac{\partial\mathcal{F}}{\partial u}(u_0,\rho_0,f_0,g_0)$ is not a Fredholm operator from $L^\infty(\Omega)\cap W_0^{1,2}(\Omega\cup\Gamma)$ into $W^{-1,2}(\Omega\cup\Gamma)$, in general.

The way out is to find a $\gamma\in(0,1)$, a Banach space $\mathcal{X}\hookrightarrow C^{0,\gamma}(\overline{\Omega})\cap W_0^{1,2}(\Omega\cup\Gamma)$ and a Banach space $\mathcal{Y}\hookrightarrow W^{-1,2}(\Omega\cup\Gamma)$ (which depend on u_0, ρ_0, f_0 and g_0, in general) such that the following holds:

(iii) There exist neighborhoods $\mathcal{U}_0$ of u_0 in $L^\infty(\Omega)\cap W_0^{1,2}(\Omega\cup\Gamma)$ and $\mathcal{V}_0$ of (ρ_0,f_0,g_0) in $L^\infty(\Omega)\times[L^p(\Omega)]^N\times L^{p-1}(\Gamma)$ such that, if $(u,\rho,f,g)\in\mathcal{U}_0\times\mathcal{V}_0$ is a solution to (1.7), then $u\in\mathcal{X}$.

(iv) The restriction of $\mathcal{F}$ on $\mathcal{U}_0 \cap \mathcal{X}$ maps $\mathcal{U}_0 \cap \mathcal{X}$ continuously differentiable into $\mathcal{Y}$, $\frac{\partial \mathcal{F}}{\partial u}(u_0, \rho_0, f_0, g_0)$ is a Fredholm operator from $\mathcal{X}$ into $\mathcal{Y}$, and $\frac{\partial \mathcal{F}}{\partial u}(\cdot, \rho_0, f_0, g_0)$ is locally Lipschitz continuous from $\mathcal{U}_0 \cap \mathcal{X}$ into $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.

Remark that $\mathcal{X}$ and $\mathcal{Y}$ are auxiliary spaces only, the assertions (i) and (ii) are independent on the choice of them.

In the case $N = 2$ the Sobolev spaces $\mathcal{X} = W_0^{1,q}(\Omega \cup \Gamma)$ and $\mathcal{Y} = W^{-1,q}(\Omega \cup \Gamma)$ with $q \in (2, p)$ sufficiently close to two satisfy (iii) and (iv). Using that, the assertion (i) has been proved in [18, Section 4.1], and assertion (ii) will be proved in Section 3 of the present paper. Moreover, in the case $N = 2$ we deal with general elliptic systems.

If $N > 2$, it seems not to be possible to find spaces $\mathcal{X}$ and $\mathcal{Y}$ with (iii) and (iv) in the scale of Sobolev spaces, in general. The reason is, that, on the one hand, one would have to take $q > N$ so that $\mathcal{F}(\cdot, \varepsilon, f, g)$ is well defined on $W_0^{1,q}(\Omega \cup \Gamma)$ and maps continuously differentiable into $W^{-1,q}(\Omega \cup \Gamma)$. On the other hand, the operator $\frac{\partial \mathcal{F}}{\partial u}(u_0, \varepsilon_0, f_0, g_0)$ (more exactly: the restriction to $W^{1,q}(\Omega \cup \Gamma)$ of the linear continuous extension of $\frac{\partial \mathcal{F}}{\partial u}(u_0, \varepsilon_0, f_0, g_0)$ onto $W_0^{1,2}(\Omega \cup \Gamma)$) is Fredholm from $W_0^{1,q}(\Omega \cup \Gamma)$ into $W^{-1,q}(\Omega \cup \Gamma)$ only for those $q > 2$ which are close to two, in general.

In the present paper we show that, for any space dimension N, the Sobolev-Campanato spaces $\mathcal{X} = W_0^{1,2,\omega}(\Omega \cup \Gamma)$ and $\mathcal{Y} = W^{-1,2,\omega}(\Omega \cup \Gamma)$ with $\omega > N - 2$ sufficiently close to $N - 2$, satisfy (iii) and (iv). Using that, we prove the assertions (i) and (ii). Moreover, we prove similar results for boundary value problems for quasilinear equations and systems much more general than (1.1). But remark that the boundary value problems we work with have the property that each weak solution is Hölder continuous. Hence, we cannot consider general elliptic systems in the case of space dimension $N > 2$. In that case we assume that the principal part of the appropriately linearized system is triangular.

Our results are a generalization of [8], where the applicability of the Implicit Function Theorem by means of Sobolev-Campanato spaces is shown for a special quasilinear elliptic system with non–smooth data from semiconductor theory. In [13] the Implicit Function Theorem as well as results from analytic bifurcation theory are applied to a semilinear elliptic boundary value problem with non–smooth data. There the semilinear structure of the equation is essentially used. In [18, Section 4.2] the Implicit Function Theorem is applied to a class of semilinear elliptic boundary value problems with non–smooth data, where the nonlinearities satisfy certain growth conditions.

There exists a large number of results of the type of the Implicit Function Theorem for nonlinear elliptic boundary value problems with smooth data, see, e.g., [5, Chapter 6], [15, Chapter 6.4], [19], [21, Chapter 61.12] and [22] for applications in elastostatics, [6] for applications in hydrodynamics, [14, Chapter 6.4] for applications in semiconductor device modeling, and [1, 2] for general elliptic problems. In all these papers it is supposed that the boundary $\partial\Omega$ and the dependence of the coefficient functions on the space variable are sufficiently smooth and that the boundary part Γ is open and closed on $\partial\Omega$ (i.e. "genuine" mixed boundary conditions are excluded). This is made in order to have the solutions sufficiently smooth and, hence, in order to work in function spaces of smooth functions and to use the property that superposition operators, generated by smooth functions, are smooth on such function spaces. However, such superposition operators are non–smooth, in general (if they are not affine), on "large" function spaces of non-smooth functions. This is the reason for the counter example [3], a second order fully nonlinear elliptic boundary value problem in one space dimension with smooth data, the solution behavior of which is of the type of the Implicit Function Theorem on "small", but not on "large" function spaces.

For second order fully nonlinear and for higher than second order nonlinear (even semilinear) elliptic boundary value problems with non–smooth data we do not know any results of the type of the Implicit Function Theorem.

2 Notation and Setting

Throughout this text Ω denotes a bounded open domain in $\mathbb{R}^N$. By $L^p(\Omega)$ and $\|\cdot\|_{L^p(\Omega)}$ we denote the usual Lebesgue spaces and their norms. For $1 \le p < \infty$ and $0 \le \omega \le p + N$ we denote by

$$L^{p,\omega}(\Omega) := \{u \in L^p(\Omega) : \ \|u\|_{L^{p,\omega}(\Omega)} < \infty\}$$

the Campanato space with its norm

$$\|u\|_{L^{p,\omega}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \sup_{\substack{x\in\Omega \\ r>0}} \left(r^{-\omega} \int\limits_{\Omega(x,r)} |u(y) - u_{\Omega(x,r)}|^p dy \right) \right)^{1/p},$$

where

$$\Omega(x,r) := \{y \in \Omega : \ \|y - x\| < r\}, \ \ u_{\Omega(x,r)} := \frac{1}{\text{meas } \Omega(x,r)} \int\limits_{\Omega(x,r)} u(y)dy.$$

Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$. The Campanato spaces $L^{p,\omega}(\Gamma)$ of functions defined on a Lipschitz submanifold Γ in $\mathbb{R}^N$ are defined analogously (cf., e.g., [9, 10]).

Further, by $W^{1,p}(\Omega)$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ we denote the usual Sobolev spaces and their norms, and we will work with Sobolev–Campanato spaces

$$W^{1,p,\omega}(\Omega) := \{u \in W^{1,p}(\Omega) : \ \frac{\partial u}{\partial x_j} \in L^{p,\omega}(\Omega) \text{ for } j = 1, \dots, N\}$$

and their norms

$$\|u\|_{W^{1,p,\omega}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \sum_{j=1}^N \left\| \frac{\partial u}{\partial x_j} \right\|^p_{L^{p,\omega}(\Omega)} \right)^{1/p}.$$

Throughout this text, for $\alpha = 1, \ldots n$, Γ_α are relatively open subsets of $\partial\Omega$. We shall assume that the following condition, concerning the boundary of Ω in $\mathbb{R}^N$ and the relative boundaries of the sets Γ_α in $\partial\Omega$, is satisfied:

$$\left.\begin{array}{l}\text{For all } \alpha = 1, \ldots, n \text{ and all } x \in \partial\Omega \text{ there exist an open} \\ \text{neighborhood } U \text{ of } x \text{ in } \mathbb{R}^N \text{ and a Lipschitz transformation } \Phi \\ \text{of } U \text{ into } \mathbb{R}^N \text{such that } \Phi\left(U \cap (\Omega \cup \Gamma_\alpha)\right) \in \{E_1, E_2, E_3\},\end{array}\right\} \quad (2.1)$$

where

$$\begin{array}{lcl} E_1 & := & \{x \in \mathbb{R}^N : \|x\| < 1, \, x_N < 0\}, \\ E_2 & := & \{x \in \mathbb{R}^N : \|x\| < 1, \, x_N \le 0\}, \\ E_3 & := & \{x \in E_2 : \, x_1 > 0 \text{ or } x_N < 0\}. \end{array}$$

If (2.1) is satisfied then Ω is a domain with Lipschitz boundary, and, hence, the continuous embeddings

$$W^{1,p,\omega}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega}) \text{ for } N - p < \omega \le N \text{ and } \gamma = 1 - \frac{N-\omega}{p}$$

are valid (cf, e.g., [7]). We write

$$W_0^{1,p}(\Omega \cup \Gamma_\alpha) := \mathrm{cl}_{W^{1,p}(\Omega)}\{u|_\Omega : \; u \in C_c^\infty(\mathbb{R}^N), (\partial\Omega \setminus \Gamma_\alpha) \cap \mathrm{supp}(u) = \emptyset\}$$

for the subspace of the Sobolev space $W^{1,p}(\Omega)$ of all functions vanishing on $\partial\Omega \setminus \Gamma_\alpha$ in the sense of trace, $W^{-1,p}(\Omega \cup \Gamma_\alpha)$ is the dual space of $W_0^{1,q}(\Omega \cup \Gamma_\alpha)$ with $1/p + 1/q = 1$, and the corresponding Sobolev–Campanato spaces are denoted by

$$W_0^{1,p,\omega}(\Omega \cup \Gamma_\alpha) := W_0^{1,p}(\Omega \cup \Gamma_\alpha) \cap W^{1,p,\omega}(\Omega).$$

We will also use the Sobolev–Campanato spaces of functionals (see, e.g., [9, 12, 16, 17])

$$W^{-1,p,\omega}(\Omega \cup \Gamma_\alpha) := \{F \in W^{-1,p}(\Omega \cup \Gamma_\alpha) : \; \|F\|_{W^{-1,p,\omega}(\Omega\cup\Gamma_\alpha)} < \infty\}.$$

The norm $\|F\|_{W^{-1,p,\omega}(\Omega\cup\Gamma_\alpha)}$ of $F \in W^{-1,p,\omega}(\Omega\cup\Gamma_\alpha)$ is defined as the supremum of the set of all $r^{-\omega/p}|\langle F, u\rangle_p|$, where $u \in W_0^{1,q}(\Omega \cup \Gamma_\alpha)$, $\|u\|_{W^{1,q}(\Omega)} \le 1$, $\mathrm{supp}(u) \subset \Omega(x, r)$, $x \in \Omega$ and $r > 0$. Here $\langle \cdot, \cdot \rangle_p : W^{-1,p}(\Omega \cup \Gamma_\alpha) \times W_0^{1,q}(\Omega \cup \Gamma_\alpha) \to \mathbb{R}$ is the duality pairing. For the sake of shortness we will use the following notation

$$\begin{array}{lcl} \mathcal{W}^{1,p,\omega} & := & W_0^{1,p,\omega}(\Omega \cup \Gamma_1) \times \ldots \times W_0^{1,p,\omega}(\Omega \cup \Gamma_n), \\ \mathcal{W}^{-1,p,\omega} & := & W^{-1,p,\omega}(\Omega \cup \Gamma_1) \times \ldots \times W^{-1,p,\omega}(\Omega \cup \Gamma_n). \end{array}$$

In place of $\mathcal{W}^{1,p,0}$ and $\mathcal{W}^{-1,p,0}$ we write more shortly $\mathcal{W}^{1,p}$ and $\mathcal{W}^{-1,p}$, respectively. We identify the space $\mathcal{W}^{-1,2}$ with the space $(\mathcal{W}^{1,2})^*$ in the usual way, and we denote by $\langle\cdot,\cdot\rangle : \mathcal{W}^{-1,2} \times \mathcal{W}^{1,2} \to \mathbb{R}$ the corresponding duality pairing.

Let $\mathcal{U}$ be an open subset in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$, Λ a normed vector space and $\mathcal{V}$ an open subset in Λ. We consider the variational equation

$$u \in \mathcal{U},\ \lambda \in \mathcal{V} : \int_\Omega A^{ij}_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial v^\beta}{\partial x_j}\,dx = \langle F_\alpha(u,\lambda), v^\alpha\rangle \text{ for all } v \in \mathcal{W}^{1,2}. \quad (2.2)$$

In (2.2) and in the sequel the summation over indices $\alpha,\beta = 1,\dots,n$ and $i,j = 1,\dots,N$ is understood if the indices appear pairwise, once "below" and once "above", but no summation is understood if they appear pairwise, but twice "below", (as, for example, in (3.4)) or twice "above". The free subscripts α,β vary from 1 to n and the free subscripts i,j from 1 to N. Concerning the nonlinearities we suppose:

$$A^{ij}_{\alpha\beta} \in C^1(\mathcal{U}\times\mathcal{V}; L^\infty(\Omega)), \quad (2.3)$$

$$F_\alpha \in C^1(\mathcal{U}\times\mathcal{V}; W^{-1,2}(\Omega)). \quad (2.4)$$

Besides of the nonlinear variational equation (2.2) we consider its linearization at $u = u_0 \in \mathcal{U}$ and $\lambda = \lambda_0 \in \mathcal{V}$

$$u \in \mathcal{W}^{1,2} : \int_\Omega \left(A^{ij}_{\alpha\beta}(u_0,\lambda_0)\frac{\partial u_\alpha}{\partial x_i} + \left[\frac{\partial A^{ij}_{\alpha\beta}}{\partial u}(u_0,\lambda_0)u\right]\frac{\partial u_0^\alpha}{\partial x_i}\right)\frac{\partial v_\beta}{\partial x_j}\,dx$$
$$= \left\langle \frac{\partial F_\alpha}{\partial u}(u_0,\lambda_0)u, v^\alpha\right\rangle \text{ for all } v \in \mathcal{W}^{1,2} \quad (2.5)$$

and the linear inhomogeneous variational equations determining Newton iterations $u_{l+1} \in \mathcal{U}$ for given $u_l \in \mathcal{U}$ by

$$\int_\Omega \left(A^{ij}_{\alpha\beta}(u_l,\lambda_0)\frac{\partial u^\alpha_{l+1}}{\partial x_i} + \left[\frac{\partial A^{ij}_{\alpha\beta}}{\partial u}(u_l,\lambda_0)\,(u_{l+1}-u_l)\right]\frac{\partial u_l^\alpha}{\partial x_i}\right)\frac{\partial v_\beta}{\partial x_j}\,dx$$
$$= \left\langle F_\alpha(u_l,\lambda_0) + \frac{\partial F_\alpha(u_l,\lambda_0)}{\partial u}\,(u_{l+1}-u_l)\,, v^\alpha\right\rangle \text{ for all } v \in \mathcal{W}^{1,2}. \quad (2.6)$$

3 The Case $N \geq 2$. Elliptic Systems With Triangular Main Part

The main result of this section is the following

Theorem 3.1 *Suppose (2.1) and (2.3) and that there exists an $\omega_0 > N-2$ such that*

$$F_\alpha \in C^1(\mathcal{U} \times \mathcal{V}; W^{-1,2,\omega_0}(\Omega)). \tag{3.1}$$

Further, let $u = u_0$, $\lambda = \lambda_0$ be a solution to (2.2) such that zero is the only solution to (2.5), that

$$A^{ij}_{\alpha\beta}(u_0, \lambda_0) = 0 \text{ for all } \alpha > \beta \tag{3.2}$$

and that

$$\frac{\partial F_\alpha}{\partial u}(u_0, \lambda_0) \text{ is completely continuous} \tag{3.3}$$

from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into $W^{-1,2,\omega_0}(\Omega \cup \Gamma_\alpha)$. Finally, suppose that there exists an $\varepsilon > 0$ such that

$$\left[A^{ij}_{\alpha\alpha}(u_0, \lambda_0)\right](x)\xi_i\xi_j \geq \varepsilon\|\xi\|^2 \text{ for almost all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N. \tag{3.4}$$

Then there exist an $\omega \in (N-2, \omega_0]$ and a neighborhood $\mathcal{U}_0$ of u_0 in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ with $\mathcal{U}_0 \subset \mathcal{U}$ such that the following is true:

(i) There exist a neighborhood $\mathcal{V}_0$ of λ_0 in Λ with $\mathcal{V}_0 \subset \mathcal{V}$ and a map $\Phi \in C^1(\mathcal{V}_0; \mathcal{W}^{1,2,\omega})$ such that $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ is a solution to (2.2) if and only if $u = \Phi(\lambda)$. In particular, for each solution $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ to (2.2) it holds $u \in \mathcal{W}^{1,2,\omega}$.

(ii) If the maps

$$\begin{aligned}
\frac{\partial A^{ij}_{\alpha\beta}}{\partial u}(\cdot, \lambda_0) : \mathcal{U} &\to \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, L^\infty(\Omega)\right), \\
\frac{\partial F_\alpha}{\partial u}(\cdot, \lambda_0) : \mathcal{U} &\to \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, W^{-1,2,\omega_0}(\Omega)\right)
\end{aligned}$$

are locally Lipschitz continuous, then, for each $u_1 \in \mathcal{U}_0$, (2.6) defines uniquely a sequence $u_2, u_3, \ldots \in \mathcal{U}$ such that $u_l \in \mathcal{W}^{1,2,\omega}$ and $u_l \to u_0$ in $\mathcal{W}^{1,2,\omega}$ as $l \to \infty$.

Proof For arbitrary $i, j = 1, \ldots, N$, $x \in \Omega$, $a \in L^\infty(\Omega)$, $r, \omega > 0$, $u \in W^{1,2,\omega}(\Omega)$ and $v \in W^{1,2}(\Omega)$ with $\operatorname{supp} v \subset \Omega(x,r)$ we have

$$\begin{aligned}
\left|\int_\Omega a \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx\right| &\leq \|a\|_{L^\infty(\Omega)} \sqrt{\int_{\Omega(x,r)} \left|\frac{\partial u}{\partial x_i}\right|^2 dx} \sqrt{\int_{\Omega(x,r)} \left|\frac{\partial v}{\partial x_j}\right|^2 dx} \\
&\leq r^\omega \|a\|_{L^\infty(\Omega)} \|u\|_{W^{1,2,\omega}(\Omega)} \|v\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

Hence, there exists a bilinear continuous map $\mathcal{B} : [L^\infty(\Omega)]^{n^2N^2} \times \mathcal{W}^{1,2} \to \mathcal{W}^{-1,2}$, which is defined for $A = [a^{ij}_{\alpha\beta}]^{i,j=1,\ldots,N}_{\alpha,\beta=1,\ldots,n} \in [L^\infty(\Omega)]^{n^2N^2}$ and $u, v \in \mathcal{W}^{1,2}$ by

$$\langle \mathcal{B}(A, u), v\rangle := \int_\Omega a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial v^\beta}{\partial x_j}\, dx,$$

such that the restriction of $\mathcal{B}(A, \cdot)$ to $\mathcal{W}^{1,2,\omega}$ maps $\mathcal{W}^{1,2,\omega}$ continuously into $\mathcal{W}^{-1,2,\omega}$. Because of assumption (2.1), the bilinear map $\mathcal{B}$ has the following properties (see [9, Theorem 4.12] and [10, Theorem 7.1]): For all $\delta \in (0,1)$ there exists an $\overline{\omega}(\delta) > N-2$ such that for all $\omega \in [0, \overline{\omega}(\delta)]$ and for all $A = [a^{ij}_{\alpha\beta}]^{i,j=1,\ldots,N}_{\alpha,\beta=1,\ldots,n} \in [L^\infty(\Omega)]^{n^2N^2}$ we have: If for almost all $x \in \Omega$

$$\delta\|\xi\|^2 \leq a^{ij}_{\alpha\alpha}(x)\xi_i\xi_j \leq \frac{1}{\delta}\|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^N$$

and if $a^{ij}_{\alpha\beta}(x) = 0$ for all $\alpha > \beta$, then

$$\mathcal{B}(A,\cdot) \text{ is a Fredholm operator (index zero) from } \mathcal{W}^{1,2,\omega} \text{ into } \mathcal{W}^{-1,2,\omega} \tag{3.5}$$

and

$$u \in \mathcal{W}^{1,2,\omega} \text{ for all } u \in \mathcal{W}^{1,2} \text{ with } \mathcal{B}(A,u) \in \mathcal{W}^{-1,2,\omega}. \tag{3.6}$$

Let us introduce the maps

$$\begin{array}{rclcrcl} \mathcal{A} & \in & C^1\left(\mathcal{U}\times\mathcal{V};[L^\infty(\Omega)]^{n^2N^2}\right): & \mathcal{A}(u,\lambda) & := & [A^{ij}_{\alpha\beta}(u,\lambda)]^{i,j=1,\ldots,N}_{\alpha,\beta=1,\ldots,n}, \\ \mathcal{F} & \in & C^1\left(\mathcal{U}\times\mathcal{V};\mathcal{W}^{-1,2,\omega_0}\right): & \mathcal{F}(u,\lambda) & := & (F_1(u,\lambda),\ldots,F_n(u,\lambda)), \end{array}$$

where $A^{ij}_{\alpha\beta}$ and F_α are the maps from (2.3) and (2.4). Then, obviously, the variational equation (2.2) is equivalent to the operator equation

$$u \in \mathcal{U}, \lambda \in \mathcal{V}: \ \mathcal{B}(\mathcal{A}(u,\lambda),u) = \mathcal{F}(u,\lambda). \tag{3.7}$$

Moreover, the linear variational equation (2.5) is equivalent to the linear operator equation

$$u \in \mathcal{W}^{1,2}: \ \mathcal{B}(\mathcal{A}(u_0,\lambda_0),u) + \mathcal{B}\left(\frac{\partial\mathcal{A}}{\partial u}(u_0,\lambda_0)u, u_0\right) = \frac{\partial\mathcal{F}}{\partial u}(u_0,\lambda_0)u. \tag{3.8}$$

Now, let us prove assertion (i) of Theorem 3.1. We have $\mathcal{B}(\mathcal{A}(u_0,\lambda_0),u_0) = \mathcal{F}(u_0,\lambda_0)$. Because of (2.3) and (3.4), there exists a $\delta \in (0,1)$ such that

$$\delta\|\xi\|^2 \leq \left[A^{ij}_{\alpha\alpha}(u,\lambda)\right](x)\xi_i\xi_j \leq \frac{1}{\delta}\|\xi\|^2$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, all u, which are close to u_0 in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ and all λ which are close to λ_0 in Λ. Hence, (3.6) yields that there exist $\omega \in (N-2,\omega_0]$ and neighborhoods $\mathcal{U}_0$ of u_0 in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ with $\mathcal{U}_0 \subset \mathcal{U}$ and $\mathcal{V}_0$ of λ_0 in Λ with $\mathcal{V}_0 \subset \mathcal{V}$ such that for all solutions $(u,\lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ we have $u \in \mathcal{W}^{1,2,\omega}$. In particular, $u_0 \in \mathcal{W}^{1,2,\omega}$. Hence, close to the solution (u_0,λ_0) equation (3.7) is equivalent to

$$u \in \mathcal{U}_0 \cap \mathcal{W}^{1,2,\omega}, \lambda \in \mathcal{V}_0: \ \mathcal{B}(\mathcal{A}(u,\lambda),u) - \mathcal{F}(u,\lambda) = 0, \tag{3.9}$$

and it remains to solve (3.9) by means of the classical Implicit Function Theorem.

Because of $\omega > N-2$ the space $W^{1,2,\omega}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$. Hence, $\mathcal{U}_0 \cap \mathcal{W}^{1,2,\omega}$ is open in $\mathcal{W}^{1,2,\omega}$. Moreover, the map $(u,\lambda) \mapsto \mathcal{B}(\mathcal{A}(u,\lambda),u)-$

$\mathcal{F}(u,\lambda)$ is a C^1-map from $(\mathcal{U}_0 \cap \mathcal{W}^{1,2,\omega}) \times \mathcal{V}_0$ into $\mathcal{W}^{-1,2,\omega}$. The partial derivative with respect to u of the left hand side of (3.9) at the solution $u = u_0$, $\lambda = \lambda_0$ is

$$\mathcal{B}(\mathcal{A}(u_0,\lambda_0),\cdot)+\mathcal{B}\left(\frac{\partial \mathcal{A}}{\partial u}(u_0,\lambda_0)\cdot, u_0\right) - \frac{\partial \mathcal{F}}{\partial u}(u_0,\lambda_0) \in \mathcal{L}(\mathcal{W}^{1,2,\omega};\mathcal{W}^{-1,2,\omega}). \quad (3.10)$$

It is injective because, by assumption, $u = 0$ is the only solution to (3.8). Supposing that it is Fredholm, then it is an isomorphism (because the Sobolev-Campanato spaces $W^{1,2,\omega}(\Omega)$ and $W^{-1,2,\omega}(\Omega)$ are complete), and the Implicit Function Theorem works.

Thus, it only remains to prove that (3.10) is Fredholm. To this end, it suffices to show that the operators

$$\mathcal{B}\left(\frac{\partial \mathcal{A}}{\partial u}(u_0,\lambda_0)\cdot, u_0\right) \text{ and } \frac{\partial \mathcal{F}}{\partial u}(u_0,\lambda_0) \quad (3.11)$$

are completely continuous from $\mathcal{W}^{1,2,\omega}$ into $\mathcal{W}^{-1,2,\omega}$ (because $\mathcal{B}(\mathcal{A}(u_0,\lambda_0),\cdot)$ is Fredholm by (3.5)). But the first operator in (3.11) is completely continuous because of the continuous embedding of $W^{1,2,\omega}(\Omega)$ into $C^{0,\gamma}(\overline{\Omega})$ with $\gamma = 1-(N-\omega)/2$ and of the completely continuous embedding of $C^{0,\gamma}(\overline{\Omega})$ into $L^\infty(\Omega)$, and the second operator is completely continuous by assumption (3.3).

Finally, let us prove assertion (ii) of Theorem 3.1. Newton iteration sequences for (3.9) are defined by

$$\begin{aligned}\mathcal{B}(\mathcal{A}(u_l,\lambda_0),u_{l+1}) &+ \mathcal{B}\left(\frac{\partial \mathcal{A}}{\partial u}(u_l,\lambda_0)u_{l+1}, u_l\right) - \frac{\partial \mathcal{F}}{\partial u}(u_l,\lambda_0)u_{l+1} \\ &= \mathcal{B}\left(\frac{\partial \mathcal{A}}{\partial u}(u_l,\lambda_0)u_l, u_l\right) - \frac{\partial \mathcal{F}}{\partial u}(u_l,\lambda_0)u_l + \mathcal{F}(u_l,\lambda_0). \quad (3.12)\end{aligned}$$

Let $u_1 \in \mathcal{U}$. Then the right hand side of (3.12) with $l = 1$ belongs to $\mathcal{W}^{-1,2,\omega_0}$ and, hence, to $\mathcal{W}^{-1,2,\omega}$. Because (3.10) is an isomorphism,

$$\mathcal{B}(\mathcal{A}(u,\lambda_0),\cdot)+\mathcal{B}\left(\frac{\partial \mathcal{A}}{\partial u}(u,\lambda_0)\cdot, u_0\right) - \frac{\partial \mathcal{F}}{\partial u}(u,\lambda_0) \in \mathcal{L}(\mathcal{W}^{1,2,\omega};\mathcal{W}^{-1,2,\omega}) \quad (3.13)$$

is also an isomorphism, if u is sufficiently close to u_0 in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$. Hence, if u_1 is sufficiently close to u_0 in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$, then u_2 is uniquely defined, belongs to $\mathcal{W}^{1,2,\omega}$ and is close to u_0 in $\mathcal{W}^{1,2,\omega}$. Now the classical Newton Iteration Procedure (cf. [20, Proposition 5.1]) for (3.9) works, because (3.13) depends Lipschitz continuously on u on a neighborhood of u_0 in $\mathcal{W}^{1,2,\omega}$. ■

The next theorem is a slightly weaker version of Theorem 3.1, which follows easily from Theorem 3.1 and which can be formulated without using Campanato spaces (similarly to assertions (i) and (ii) of Section 1). We don't know, however, any proof of this result without using Sobolev-Campanato spaces.

Theorem 3.2 *Suppose (2.1) and (2.3) and that there exists a* $p > N$ *such that* $F_\alpha \in C^1(\mathcal{U} \times \mathcal{V}; W^{-1,p}(\Omega))$. *Further, let* $u = u_0$, $\lambda = \lambda_0$ *be a solution to (2.2) such that zero is the only solution to (2.5), that (3.2) holds and that* $\frac{\partial F_\alpha}{\partial u}(u_0, \lambda_0)$ *is completely continuous from* $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ *into* $W^{-1,p}(\Omega \cup \Gamma_\alpha)$. *Finally, suppose that there exists an* $\varepsilon > 0$ *such that (3.4) holds.*

Then there exist $\gamma \in (0,1)$ *and a neighborhood* $\mathcal{U}_0$ *of* u_0 *in* $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ *with* $\mathcal{U}_0 \subset \mathcal{U}$ *such that the following is true:*

(i) There exist a neighborhood $\mathcal{V}_0$ *of* λ_0 *with* $\mathcal{V}_0 \subset \mathcal{V}$ *and a map* $\Phi \in C^1\left(\mathcal{V}_0; [C^{0,\gamma}(\overline{\Omega})]^n \cap \mathcal{W}^{1,2}\right)$ *such that* $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ *is a solution to (2.2) if and only if* $u = \Phi(\lambda)$. *In particular, for each solution* $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ *to (2.2) it holds* $u \in [C^{0,\gamma}(\overline{\Omega})]^n$.

(ii) If the maps

$$\begin{aligned}
\frac{\partial A^{ij}_{\alpha\beta}}{\partial u}(\cdot, \lambda_0) : \mathcal{U} \quad &\rightarrow \quad \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, L^\infty(\Omega)\right), \\
\frac{\partial F_\alpha}{\partial u}(\cdot, \lambda_0) : \mathcal{U} \quad &\rightarrow \quad \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, W^{-1,p}\right)
\end{aligned}$$

are locally Lipschitz continuous, then, for each $u_1 \in \mathcal{U}_0$, *(2.6) defines uniquely a sequence* $u_2, u_3, \ldots \in \mathcal{U}$ *such that* $u_j \in [C^{0,\gamma}(\overline{\Omega})]^n \cap \mathcal{W}^{1,2}$ *and* $u_l \to u_0$ *in* $[C^{0,\gamma}(\overline{\Omega})]^n \cap \mathcal{W}^{1,2}$ *as* $l \to \infty$.

Proof Let $1/p + 1/q = 1$. If $v \in W_0^{1,q}(\Omega)$ and supp $v \subset \Omega(x, r)$ for some $x \in \Omega$ and $r > 0$, then the Hölder inequality yields

$$\|v\|_{W^{1,q}(\Omega)} \leq \text{ const } r^{\omega_0/2} \|v\|_{W^{1,2}(\Omega)} \text{ with } \omega_0 = \frac{N(2-q)}{q} = \frac{N(p-2)}{p}.$$

Therefore, $W^{-1,p}(\Omega)$ is continuously embedded into $W^{-1,2,\omega_0}(\Omega)$. Hence, because of Theorem 3.1 we have an $\omega \in (N-2, \omega_0]$, a neighborhood $\mathcal{V}_0$ of λ_0 and a map $\Phi \in C^1\left(\mathcal{V}_0, \mathcal{W}^{1,2,\omega}\right)$, which parameterizes the set of solutions to (2.2) close to (u_0, λ_0), and we have the Newton iterations $u_l \to u_0$ for $l \to \infty$ in $\mathcal{W}^{1,2,\omega}$. But $\mathcal{W}^{1,2,\omega}$ is continuously embedded into $[C^{0,\gamma}(\overline{\Omega})]^n$ with $\gamma := 1 + (\omega - N)/2$, so Theorem 3.2 is proved. ∎

4 The Case $N = 2$. General Elliptic Systems

In this section, again, we consider the problem of local unique smooth continuation of a solution (u_0, λ_0) to the variational equation (2.2), but now in the case of space dimension $N = 2$. We replace the assumptions (3.2) and (3.4) by the assumption (4.1) below.

Theorem 4.1 *Let* $N = 2$. *Suppose (2.1) and (2.3) and that there exists a* $p_0 > 2$ *such that* $F_\alpha \in C^1(\mathcal{U} \times \mathcal{V}; W^{-1,p_0}(\Omega \cup \Gamma_\alpha))$. *Further, let* $u = u_0$, $\lambda = \lambda_0$ *be a solution to (2.2) such that zero is the only solution to (2.5) and that* $\frac{\partial F_\alpha}{\partial u}(u_0, \lambda_0)$

is completely continuous from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ *into* $W^{-1,p_0}(\Omega \cup \Gamma_\alpha)$. *Finally, suppose that there exists an* $\varepsilon > 0$ *such that*

$$\int_\Omega A^{ij}_{\alpha\beta}(u_0, \lambda_0) \frac{\partial v^\alpha}{\partial x_i} \frac{\partial v^\beta}{\partial x_j} \, dx \geq \varepsilon \sum_{\alpha=1}^n \|v^\alpha\|^2_{W^{1,2}(\Omega)} \quad \textit{for all } v \in \mathcal{W}^{1,2}. \tag{4.1}$$

Then there exist $p \in (2, p_0]$ *and a neighborhood* $\mathcal{U}_0$ *of* u_0 *in* $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ *with* $\mathcal{U}_0 \subset \mathcal{U}$ *such that the following holds:*

(i) There exist a neighborhood $\mathcal{V}_0$ *of* λ_0 *with* $\mathcal{V}_0 \subset \mathcal{V}$ *and a map* $\Phi \in C^1(\mathcal{V}_0; \mathcal{W}^{1,p})$ *such that* $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ *is a solution to (2.2) if and only if* $u = \Phi(\lambda)$. *In particular, for each solution* $(u, \lambda) \in \mathcal{U}_0 \times \mathcal{V}_0$ *to (2.2) it holds* $u \in \mathcal{W}^{1,p}$.

(ii) If the maps

$$\begin{aligned}
\frac{\partial A^{ij}_{\alpha\beta}}{\partial u}(\cdot, \lambda_0) : \mathcal{U} &\to \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, L^\infty(\Omega)\right), \\
\frac{\partial F_\alpha}{\partial u}(\cdot, \lambda_0) : \mathcal{U} &\to \mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, W^{-1,p_0}\right)
\end{aligned}$$

are locally Lipschitz continuous, then, for each $u_1 \in \mathcal{U}_0$, *(2.6) defines uniquely a sequence* $u_2, u_3, \ldots \in \mathcal{U}$ *such that* $u_l \in \mathcal{W}^{1,p}$ *and* $u_l \to u_0$ *in* $\mathcal{W}^{1,p}$ *as* $l \to \infty$.

Proof It follows from the Hölder inequality that the restriction of the map $\mathcal{B}(A, \cdot)$ to $\mathcal{W}^{1,p}$ maps $\mathcal{W}^{1,p}$ continuously into $\mathcal{W}^{-1,p}$ for all $p \in [2, \infty)$. Here $\mathcal{B} : [L^\infty(\Omega)]^{n^2N^2} \times \mathcal{W}^{1,2} \to \mathcal{W}^{-1,2}$ is the bilinear map introduced in the proof of Theorem 3.1. Because of assumption (2.1), the bilinear map $\mathcal{B}$ has the following properties (see [11]): For all $\delta \in (0,1)$ there exists a $\overline{p}(\delta) > 2$ such that for all $p \in (2, \overline{p}(\delta)]$ and for all $A = [a^{ij}_{\alpha\beta}]^{i,j=1,\ldots,N}_{\alpha,\beta=1,\ldots,n} \in [L^\infty(\Omega)]^{n^2N^2}$ we have: If

$$\delta \sum_{\alpha=1}^n \|v^\alpha\|^2_{W^{1,2}(\Omega)} \geq \int_\Omega a^{ij}_{\alpha\beta} \frac{\partial v^\alpha}{\partial x_i} \frac{\partial v^\beta}{\partial x_j} \, dx \geq \frac{1}{\delta} \sum_{\alpha=1}^n \|v^\alpha\|^2_{W^{1,2}(\Omega)} \quad \text{for all } v \in \mathcal{W}^{1,2}$$

then

$$\mathcal{B}(A, \cdot) \text{ is a Fredholm operator (index zero) from } \mathcal{W}^{1,p} \text{ into } \mathcal{W}^{-1,p}$$

and

$$u \in \mathcal{W}^{1,p} \text{ for all } u \in \mathcal{W}^{1,2} \text{ with } \mathcal{B}(A, u) \in \mathcal{W}^{-1,p}.$$

Now we proceed as in the proof of Theorem 3.1, replacing everywhere $\mathcal{W}^{1,2,\omega}$ and $\mathcal{W}^{-1,2,\omega}$ with $\omega > N - 2$ by $\mathcal{W}^{1,p}$ and $\mathcal{W}^{-1,p}$ with $p > 2$, respectively, and using the fact that, because of $N = 2$, the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded into the Hölder space $C^{0,\gamma}(\overline{\Omega})$ with $\gamma = 1 - 2/p$. ■

Remark 4.2 There exist various results concerning "pointwise" conditions (Legendre condition, systems of elasticity type) to the coefficient functions $A^{ij}_{\alpha\beta}(u_0, \lambda_0)$, which imply (4.1), see, e.g., [4, Chapter 6].

5 Examples of Coefficient Functions

In this section we indicate some classes of maps A which are candidates for the coefficient maps $A^{ij}_{\alpha\beta}$ of Sections 2–4. We consider superposition operators

$$[A(u,\lambda)](x) = a(x,u(x),\lambda) \text{ for almost all } x \in \Omega. \tag{5.1}$$

Here $a : \Omega \times U \times \mathcal{V} \to \mathbb{R}$ is the function, generating the superposition operator, U is an open subset in $\mathbb{R}^n$, and $\mathcal{V}$ is, as in Sections 2–4, an open subset of a normed vector space Λ. By $\mathcal{U}$ we denote the set of all $u \in [L^\infty(\Omega)]^n$ such that there exists a compact set $K \subset U$ with $u(x) \in K$ for almost all $x \in \Omega$. Obviously, $\mathcal{U}$ is open in $[L^\infty(\Omega)]^n$.

In Proposition 5.1 below we state conditions on the function a which imply that $A \in C^1(\mathcal{U} \times \mathcal{V}, L^\infty(\Omega))$. Hence, we describe a class of continuously differentiable superposition operators from $[L^\infty(\Omega)]^n \times \Lambda$ into $L^\infty(\Omega)$. Of course, this is only a subclass of the class of all continuously differentiable operators from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2} \times \Lambda$ into $L^\infty(\Omega)$ (in Theorems 3.1, 3.2 and 4.1 the coefficient maps $A^{ij}_{\alpha\beta}$ have to be in this class, cf. (2.3)), but most of the coefficient maps appearing in applications are in this subclass.

In what follows we use the notation $\|\cdot\|$ for the norm not only in $\mathbb{R}^n$ but also in Λ and in its dual space Λ^*.

Proposition 5.1 *Let the following conditions be fulfilled:*

(I) $a(x,\cdot,\cdot) \in C^1(U \times \mathcal{V})$ *for almost all* $x \in \Omega$, *and* $\frac{\partial a}{\partial u}(\cdot,u,\lambda)$ *and* $\frac{\partial a}{\partial \lambda}(\cdot,u,\lambda)$ *are measurable for all* $u \in U$ *and* $\lambda \in \mathcal{V}$.

(II) *For all* $\lambda \in \mathcal{V}$ *and compact* $K \subset U$ *there exists an* $M > 0$ *such that* $|a(x,u,\lambda)| + \|\frac{\partial a}{\partial u}(x,u,\lambda)\| + \|\frac{\partial a}{\partial \lambda}(x,u,\lambda)\| \le M$ *for almost all* $x \in \Omega$ *and all* $u \in K$.

(III) *For all* $\lambda \in \mathcal{V}$, *compact* $K \subset U$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|a(x,u,\lambda) - a(x,v,\mu)| < \varepsilon$, $\|\frac{\partial a}{\partial u}(x,u,\lambda) - \frac{\partial a}{\partial u}(x,v,\mu)\| < \varepsilon$ *and* $\|\frac{\partial a}{\partial \lambda}(x,u,\lambda) - \frac{\partial a}{\partial \lambda}(x,v,\mu)\| < \varepsilon$ *for almost all* $x \in \Omega$ *and all* $u \in K$, $v \in U$ *and* $\mu \in \mathcal{V}$ *such that* $\|u - v\| + \|\lambda - \mu\| < \delta$.

(IV) *For all* $\lambda \in \mathcal{V}$ *and compact* $K \subset U$ *there exists an* $L > 0$ *such that* $\|\frac{\partial a}{\partial u}(x,u,\lambda) - \frac{\partial a}{\partial u}(x,v,\lambda)\| \le L\|u - v\|$ *for almost all* $x \in \Omega$ *and all* $u, v \in K$.

Then the map A, *defined by (5.1), is continuously differentiable from* $\mathcal{U} \times \mathcal{V}$ *into* $L^\infty(\Omega)$,

$$\left[\frac{\partial A}{\partial u}(u,\lambda)v\right](x) = \frac{\partial a}{\partial u}(x,u(x),\lambda)v(x), \tag{5.2}$$

$$\left[\frac{\partial A}{\partial \lambda}(u,\lambda)\mu\right](x) = \frac{\partial a}{\partial \lambda}(x,u(x),\lambda)\mu, \tag{5.3}$$

and $\frac{\partial A}{\partial u}(\cdot,\lambda)$ *is locally Lipschitz continuous from* $\mathcal{U}$ *to* $\mathcal{L}([L^\infty(\Omega)]^n, L^\infty(\Omega))$ *for all* $\lambda \in \mathcal{V}$.

Proof It follows immediately from the definition of the set $\mathcal{U}$ and from assumptions (I) and (II) that A maps $\mathcal{U} \times \mathcal{V}$ into $L^\infty(\Omega)$.

In order to prove (5.2) let us fix $u \in \mathcal{U}$ and $\lambda \in \mathcal{V}$. Again, it follows from the definition of $\mathcal{U}$ and from (I) and (II) that

$$v \in L^\infty(\Omega)^n \mapsto \frac{\partial a}{\partial u}(\cdot, u(\cdot), \lambda)v(\cdot) \in L^\infty(\Omega)$$

is a linear bounded map. Now we show that $\frac{\partial A}{\partial u}(u, \lambda)$ exists and is this linear bounded map: There exist a compact set $K \subset U$ and a $\delta > 0$ such that for all $v \in L^\infty(\Omega)^n$ with $\|u - v\|_{L^\infty(\Omega)^n} < \delta$ it holds $u(x) \in K$ and $u(x) + v(x) \in K$ for almost all $x \in \Omega$. Taking δ small enough we can assume that it is the δ (corresponding to λ, K and ε) from (III). Hence, we have for all $v \in L^\infty(\Omega)^n$ with $\|u - v\|_{L^\infty(\Omega)^n} < \delta$ and almost all $x \in \Omega$

$$\left| a(x, u(x) + v(x), \lambda) - a(x, u(x), \lambda) - \frac{\partial a}{\partial u}(x, u(x), \lambda)v(x) \right| =$$
$$= \left| \int_0^1 \left(\frac{\partial a}{\partial u}(x, u(x) + tv(x), \lambda) - \frac{\partial a}{\partial u}(x, u(x), \lambda) \right) dt \; v(x) \right| \leq \varepsilon \|v\|_{L^\infty(\Omega)^n}.$$

In order to prove that $\frac{\partial A}{\partial u}$ is continuous we fix $u \in \mathcal{U}$, $\lambda \in \mathcal{V}$ and $\varepsilon > 0$ and take $K \subset U$ and a $\delta > 0$ as above. Moreover we assume δ to be small enough such that $\lambda + \mu \in \mathcal{V}$ for all $\mu \in \Lambda$ with $\|\lambda - \mu\| < \delta$. Then for all $v \in L^\infty(\Omega)^n$ and $\mu \in \Lambda$ with $\|u - v\|_{L^\infty(\Omega)^n} + \|\lambda - \mu\| < \delta$, for all $w \in L^\infty(\Omega)^n$ and for almost all $x \in \Omega$ we have

$$\left| \left[\left(\frac{\partial A}{\partial u}(u + v, \lambda + \mu) - \frac{\partial A}{\partial u}(u, \lambda) \right) w \right] (x) \right| =$$
$$= \left| \left(\frac{\partial a}{\partial u}(x, u(x) + v(x), \lambda + \mu) - \frac{\partial a}{\partial u}(x, u(x), \lambda) \right) w(x) \right| \leq \varepsilon \|w\|_{L^\infty(\Omega)^n}.$$

Analogously one shows that $\frac{\partial A}{\partial \lambda}$ exists, that it is continuous and that (5.3) holds. Hence, A is C^1.

Finally we show that $\frac{\partial A}{\partial u}(\cdot, \lambda)$ is locally Lipschitz continuous. Take arbitrarily fixed $u \in \mathcal{U}$ and $\lambda \in \mathcal{V}$. Take, again, a compact set $K \subset U$ and a $\delta > 0$ such that for all $v \in L^\infty(\Omega)^n$ with $\|u - v\|_{L^\infty(\Omega)^n} < \delta$ it holds $u(x) \in K$ and $u(x) + v(x) \in K$ for almost all $x \in \Omega$. Let L be the Lipschitz constant from (IV) corresponding to λ and K. Then for all $v \in L^\infty(\Omega)^n$ with $\|u - v\|_{L^\infty(\Omega)^n} < \delta$, for all $w \in L^\infty(\Omega)^n$ and for almost all $x \in \Omega$ we have

$$\left| \left[\left(\frac{\partial A}{\partial u}(u + v, \lambda) - \frac{\partial A}{\partial u}(u, \lambda) \right) w \right] (x) \right| =$$
$$= \left| \left(\frac{\partial a}{\partial u}(x, u(x) + v(x), \lambda) - \frac{\partial a}{\partial u}(x, u(x), \lambda) \right) w(x) \right| \leq L \|v\|_{L^\infty(\Omega)^n} \|w\|_{L^\infty(\Omega)^n}.$$

■

We close this section giving two examples of generating functions $a^{ij}_{\alpha\beta} : \Omega \times U \times \mathcal{V} \to \mathbb{R}$ with "large" infinite dimensional parameter spaces Λ, which satisfy the assumptions (I)–(IV) of Proposition 5.1 and for which the corresponding superposition operators (according to (5.1)) $A^{ij}_{\alpha\beta} : \mathcal{U} \times \mathcal{V} \to L^\infty(\Omega)$ satisfy assumptions (3.4) or (4.1) at each (u_0, λ_0):

Example 5.2 Take $m \in \mathbb{N}$, $\Lambda = [L^\infty(\Omega)]^{mn^2N^2}$, $\mathcal{V} \subset \Lambda$ open, and set for almost all $x \in \Omega$ and all $u \in U$ and $\lambda \in \mathcal{V}$

$$a^{ij}_{\alpha\beta}(x, u, \lambda) = \sum_{k=1}^{m} \lambda^{ijk}_{\alpha\beta}(x) a^{ijk}_{\alpha\beta}(u) \tag{5.4}$$

with $a^{ijk}_{\alpha\beta} \in C^1(U)$. Here the system of L^∞-functions $\lambda = \{\lambda^{ijk}_{\alpha\beta} : \alpha, \beta = 1, \dots, n;\ i, j = 1, \dots, N;\ k = 1, \dots, m\}$ is the control parameter. Then the assumptions (I)–(IV) of Proposition 5.1 are satisfied. In particular, from (5.2) and (5.3) it follows

$$\begin{aligned} \left[\frac{\partial A^{ijk}_{\alpha\beta}}{\partial u}(u, \lambda) v\right](x) &= \sum_{k=1}^{m} \lambda^{ijk}_{\alpha\beta}(x) \frac{d}{du} a^{ijk}_{\alpha\beta}(u(x)) v(x), \\ \left[\frac{\partial A^{ijk}_{\alpha\beta}}{\partial \lambda}(u, \lambda) \mu\right](x) &= \sum_{k=1}^{m} \mu^{ijk}_{\alpha\beta}(x) a^{ijk}_{\alpha\beta}(u(x)). \end{aligned}$$

Coefficient functions, which are piecewise constant with respect to the space variable x, are of type (5.4): In this case Ω is the disjoint union of measurable sets $\Omega_1, \dots, \Omega_m$, and the functions $\lambda^{ijk}_{\alpha\beta}$ are constant in Ω_k and zero in $\Omega \setminus \Omega_k$.

If, moreover, for all $u \in U$ it holds $a^{iik}_{\alpha\alpha}(u) > 0$ and $a^{ijk}_{\alpha\alpha}(u) = 0$ for $i \neq j$ and if $\mathcal{V}$ is the set of all $\lambda \in \Lambda$ such that ess inf $\lambda^{iik}_{\alpha\alpha} > 0$, then (3.4) is satisfied for all $u_0 \in \mathcal{U}$ and $\lambda_0 \in \mathcal{V}$.

Example 5.3 Let Λ be the vector space of all maps $\lambda : \Omega \times U \to \mathbb{R}^{n^2N^2}$ such that for all $\alpha, \beta = 1, \dots, n$ and $i, j = 1, \dots, N$ we have that $\lambda^{ij}_{\alpha\beta}(x, \cdot) \in C^2(U)$ for almost all $x \in \Omega$, that $\lambda^{ij}_{\alpha\beta}$, $\frac{\partial \lambda^{ij}_{\alpha\beta}}{\partial u}$ and $\frac{\partial^2 \lambda^{ij}_{\alpha\beta}}{\partial u^2}$ are measurable and that there exists an $M > 0$ such that for almost all $x \in \Omega$ and all $u \in U$

$$|\lambda^{ij}_{\alpha\beta}(x, u)| + \left\| \frac{\partial \lambda^{ij}_{\alpha\beta}}{\partial u}(x, u) \right\| + \left\| \frac{\partial^2 \lambda^{ij}_{\alpha\beta}}{\partial u^2}(x, u) \right\| < M. \tag{5.5}$$

As the norm of λ we take the infimum of all constants M in (5.5). If $\mathcal{V}$ is an arbitrary open set in Λ and

$$a^{ij}_{\alpha\beta}(x, u, \lambda) = \lambda^{ij}_{\alpha\beta}(x, u) \text{ for almost all } x \in \Omega \text{ and all } u \in U \text{ and } \lambda \in \mathcal{V},$$

then the assumptions (I)–(IV) of Proposition 5.1 are satisfied. Here the system of the Caratheodory functions $\lambda = \{\lambda^{ij}_{\alpha\beta} : \alpha, \beta = 1, \dots, n;\ i, j = 1, \dots, N\}$ is the control parameter. In particular, from (5.2) and (5.3) it follows

$$\left[\frac{\partial A^{ijk}_{\alpha\beta}}{\partial u}(u,\lambda)v\right](x) = \frac{\partial \lambda^{ijk}_{\alpha\beta}}{\partial u}(x,u(x))v(x), \quad \left[\frac{\partial A^{ij}_{\alpha\beta}}{\partial \lambda}(u,\lambda)\mu\right](x) = \mu^{ij}_{\alpha\beta}(x,u(x)).$$

If, moreover, $\mathcal{V}$ is the set of all $\lambda \in \Lambda$ such that for all compact $K \subset U$ there exists an $\varepsilon > 0$ such that $\lambda^{ij}_{\alpha\alpha}(x,u)\xi_i\xi_j \geq \varepsilon \|\xi\|^2$ for almost all $x \in \Omega$ and all $u \in K$ and $\xi \in \mathbb{R}^N$, then $\mathcal{V}$ is open, and (3.4) is satisfied for all $u_0 \in \mathcal{U}$ and $\lambda_0 \in \mathcal{V}$.

6 Examples of Right Hand Sides

In this section we indicate some classes of maps $F \in C^1\left(\mathcal{U} \times \mathcal{V}, W^{-1,2,\omega}(\Omega \cup \Gamma)\right)$ and $F \in C^1\left(\mathcal{U} \times \mathcal{V}, W^{-1,p}(\Omega \cup \Gamma)\right)$, which are candidates for the right hand sides F_α of Sections 3 and 4, respectively. Here Γ is a subset of $\partial\Omega$ such that

$$\left.\begin{array}{l}\text{for all } x \in \partial\Omega \text{ there exist an open neighborhood } U \text{ of } x \text{ in } \mathbb{R}^N \\ \text{and a Lipschitz transformation } \Phi \text{ of } U \text{ into } \mathbb{R}^N \text{ such that} \\ \Phi\left(U \cap (\Omega \cup \Gamma)\right) \in \{E_1, E_2, E_3\}\end{array}\right\} \qquad (6.1)$$

and, hence, a candidate for the "Neumann" boundary parts Γ_α of Section 2 (cf. (2.1)). The sets $\mathcal{U}$ and $\mathcal{V}$ are, as in Sections 2–4, open in $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ and in a normed vector space Λ, respectively.

We consider maps F which work as

$$\langle F(u,\lambda), v\rangle = \int_\Omega \left(B^i(u,\lambda)\frac{\partial v}{\partial x_i} + C(u,\lambda)v\right) dx + \int_\Gamma D(u,\lambda)v \, d\Gamma \qquad (6.2)$$

and such that, in a reasonable sense,

$$\left\langle \frac{\partial F}{\partial u}(u,\lambda)v, w\right\rangle = \int_\Omega \left(\left[\frac{\partial B^i}{\partial u}(u,\lambda)v\right]\frac{\partial w}{\partial x_i} + \left[\frac{\partial C}{\partial u}(u,\lambda)v\right] w\right) dx + \int_\Gamma \left[\frac{\partial D}{\partial u}(u,\lambda)w\right] v \, d\Gamma, \qquad (6.3)$$

$$\left\langle \frac{\partial F}{\partial \lambda}(u,\lambda)\mu, v\right\rangle = \int_\Omega \left(\left[\frac{\partial B^i}{\partial \lambda}(u,\lambda)\mu\right]\frac{\partial v}{\partial x_i} + \left[\frac{\partial C}{\partial \lambda}(u,\lambda)\mu\right] v\right) dx + \int_\Gamma \left[\frac{\partial D}{\partial \lambda}(u,\lambda)\mu\right] v \, d\Gamma. \qquad (6.4)$$

Proposition 6.1 *Suppose (6.1), and let*

$$\left.\begin{array}{rcl} B^i & \in & C^1(\mathcal{U} \times \mathcal{V}; L^{2,\omega}(\Omega)), \\ C & \in & C^1(\mathcal{U} \times \mathcal{V}; L^{2N/(N+2),\omega N/(N+2)}(\Omega)), \\ D & \in & C^1(\mathcal{U} \times \mathcal{V}; L^{2(N-1)/N,\omega(N-1)/N}(\Gamma)). \end{array}\right\} \qquad (6.5)$$

Then the following holds:

(i) The map F, defined by (6.2), is continuously differentiable from $\mathcal{U} \times \mathcal{V}$ into $W^{-1,2,\omega}(\Omega \cup \Gamma)$, and it holds (6.3) and (6.4).

(ii) If, for a certain $(u, \lambda) \in \mathcal{U} \times \mathcal{V}$, $\frac{\partial B^i}{\partial u}(u, \lambda)$, $\frac{\partial C}{\partial u}(u, \lambda)$ and $\frac{\partial D}{\partial u}(u, \lambda)$ are completely continuous (from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into the corresponding image spaces), then $\frac{\partial F}{\partial u}(u, \lambda)$ is completely continuous from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into $W^{-1,2,\omega}(\Omega \cup \Gamma)$.

(iii) If, for a certain $\lambda \in \mathcal{V}$, the maps $\frac{\partial B^i}{\partial u}(\cdot, \lambda)$, $\frac{\partial C}{\partial u}(\cdot, \lambda)$ and $\frac{\partial D}{\partial u}(\cdot, \lambda)$ are locally Lipschitz continuous (from $\mathcal{U}$ into the corresponding image space), then $\frac{\partial F}{\partial u}(\cdot, \lambda)$ is also locally Lipschitz continuous (from $\mathcal{U}$ into $\mathcal{L}([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, W^{-1,2,\omega}(\Omega)))$.

Proof (i) From [9, Theorem 3.9] follows that there exists a linear continuous map

$$\Phi : [L^{2,\omega}(\Omega)]^N \times L^{2N/(N+2),\omega N/(N+2)}(\Omega) \times L^{2(N-1)/N,\omega(N-1)/N}(\Gamma) \to W^{-1,2,\omega}(\Omega)$$

such that

$$\langle \Phi(f, g, h), v \rangle = \int_\Omega \left(f^i \frac{\partial v}{\partial x_i} + gv \right) dx + \int_\Gamma hv \, d\Gamma \tag{6.6}$$

for all $f \in [L^\omega(\Omega)]^N$, $g \in L^{2N/(N+2),\omega N/(N+2)}(\Omega)$, $h \in L^{2,2(N-1)/N,\omega(N-1)/N}(\Gamma)$ and $v \in W_0^{1,2}(\Omega \cup \Gamma)$. Because of (6.2) we get

$$F(u, \lambda) = \Phi\left(B(u, \lambda)), C(u, \lambda), D(u, \lambda)\right)$$

with $B(u, \lambda) := (B^1(u, \lambda), \ldots, B^N(u, \lambda))$. Thus, F is a superposition of C^1-maps, i.e. F is C^1. Moreover, the equation

$$\frac{\partial F}{\partial u}(u, \lambda)v = \Phi\left(\frac{\partial B}{\partial u}(u, \lambda)v, C(u, \lambda), D(u, \lambda)\right)$$
$$+\Phi\left(B(u, \lambda), \frac{\partial C}{\partial u}(u, \lambda)v, D(u, \lambda)\right) + \Phi\left(B(u, \lambda), C(u, \lambda), \frac{\partial D}{\partial u}(u, \lambda)v\right) \tag{6.7}$$

yields the differentiation rules (6.3) and (6.4).

(ii) The right hand side of (6.7) is completely continuous with respect to v (as a map from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into $W^{-1,2,\omega}(\Omega \cup \Gamma)$) by assumption.

(iii) The maps $\frac{\partial B}{\partial u}(\cdot, \lambda)$, $\frac{\partial C}{\partial u}(\cdot, \lambda)$ and $\frac{\partial D}{\partial u}(\cdot, \lambda)$ are locally Lipschitz continuous by assumption. Hence, the maps $B(\cdot, \lambda)$, $C(\cdot, \lambda)$ and $D(\cdot, \lambda)$ are locally Lipschitz continuous by the mean value theorem. Now it follows easily from (6.7) that $\frac{\partial F}{\partial u}(\cdot, \lambda)$ is locally Lipschitz continuous. ■

Proposition 6.2 *Suppose (6.1), and let*

$$\left.\begin{array}{rcl} B^i & \in & C^1(\mathcal{U} \times \mathcal{V}; L^p(\Omega)), \\ C & \in & C^1(\mathcal{U} \times \mathcal{V}; L^{pN/(p+N)}(\Omega)), \\ D & \in & C^1(\mathcal{U} \times \mathcal{V}; L^{p(N-1)/N}(\Gamma)). \end{array}\right\} \tag{6.8}$$

Then the following holds:

(i) The map F, defined by (6.2), is continuously differentiable from $\mathcal{U} \times \mathcal{V}$ into $W^{-1,p}(\Omega \cup \Gamma)$, and it holds (6.3) and (6.4).

(ii) If, for a certain $(u, \lambda) \in \mathcal{U} \times \mathcal{V}$, $\frac{\partial B^i}{\partial u}(u, \lambda)$, $\frac{\partial C}{\partial u}(u, \lambda)$ and $\frac{\partial D}{\partial u}(u, \lambda)$ are completely continuous (from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into the corresponding image spaces), then $\frac{\partial F}{\partial u}(u, \lambda)$ is completely continuous from $[L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}$ into $W^{-1,p}(\Omega \cup \Gamma)$.

(iii) If, for a certain $\lambda \in \mathcal{V}$, the maps $\frac{\partial B^i}{\partial u}(\cdot, \lambda)$, $\frac{\partial C}{\partial u}(\cdot, \lambda)$ and $\frac{\partial D}{\partial u}(\cdot, \lambda)$ are locally Lipschitz continuous (from $\mathcal{U}$ into the corresponding image space), then $\frac{\partial F}{\partial u}(\cdot, \lambda)$ is locally Lipschitz continuous from $\mathcal{U}$ into $\mathcal{L}\left([L^\infty(\Omega)]^n \cap \mathcal{W}^{1,2}, W^{-1,p}(\Omega)\right)$.

Proof The proof is analogous to that of Proposition 6.1, but now with

$$\Phi : [L^p(\Omega)]^N \times L^{pN/(p+N)}(\Omega) \times L^{p(N-1)/N}(\Gamma) \to W^{-1,p}(\Omega)$$

such that (6.6) holds for all $f \in [L^p(\Omega)]^N$, $g \in L^{pN/(p+N)}(\Omega)$, $h \in L^{p(N-1)/N}(\Gamma)$ and $v \in W_0^{1,2}(\Omega \cup \Gamma)$. The map Φ now is well-defined and continuous because of Hölder's inequality, the Sobolev embedding theorem and the trace theorem: If $v \in W^{1,q}(\Omega)$ with $1/p + 1/q = 1$, then $v \in L^{p(N-1)/(p(N-1)-N)}(\Gamma)$ and

$$\begin{aligned} \left| \int_\Gamma hv \, d\Gamma \right| &\leq \|h\|_{L^{p(N-1)/N}(\Gamma)} \|v\|_{L^{p(N-1)/(p(N-1)-N)}(\Gamma)} \\ &\leq \text{const } \|h\|_{L^{p(N-1)/N}(\Gamma)} \|v\|_{W^{1,q}(\Omega)}. \end{aligned}$$

■

Example 6.3 Let $b^i, c : \Omega \times \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be Caratheodory functions satisfying analogues of the assumptions (I)–(IV) of Proposition 5.1 (with $U = \mathbb{R}^n$ for simplicity) and, hence, generating continuously differentiable superposition operators from $[L^\infty(\Omega)]^n \times \mathcal{V}$ into $L^\infty(\Omega)$. Let $d : \Gamma \times \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be a Caratheodory function generating a continuously differentiable superposition operator from $[L^\infty(\Omega) \cap W^{1,2}(\Omega)]^n \times \mathcal{V}$ into $L^\infty(\Gamma)$. Remark that for $u \in [L^\infty(\Omega) \cap W^{1,2}(\Omega)]^n$ the trace on Γ is well-defined and belongs to $[L^\infty(\Gamma)]^n$. Hence, like in Proposition 5.1 one can easily formulate "pointwise" conditions on d which imply that d generates such a superposition operator. Let the operators B^i, C and D be defined for $u, \varphi \in [L^\infty(\Omega) \cap W^{1,2}(\Omega)]^n$, $\lambda \in \mathcal{V}$ and functions $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$, $g : \Omega \to \mathbb{R}$ and $h : \Gamma \to \mathbb{R}$ by

$$\left.\begin{aligned} [B^i(u, \varphi, f, \lambda)](x) &= b^i(x, u(x) - \varphi(x), \lambda) - f^i(x), \\ [C(u, \varphi, g, \lambda)](x) &= c(x, u(x) - \varphi(x), \lambda) - g(x) \end{aligned}\right\} \text{ for almost all } x \in \Omega$$

and

$$[D(u, \varphi, h, \lambda)](x) = d(x, u(x) - \varphi(x), \lambda) - h(x) \text{ for almost all } x \in \Gamma.$$

Then they satisfy the assumptions of Proposition 6.1 or Proposition 6.2 with control parameter $(\varphi, f, g, h, \lambda)$, $\varphi \in [L^\infty(\Omega) \cap W^{1,2}(\Omega)]^n$, $f \in [L^{2,\omega}(\Omega)]^N$ or

$f \in [L^p(\Omega)]^N$ respectively, $g \in L^{2N/(N+2),\omega N/(N+2)}(\Omega)$ or $g \in L^{pN/(p+N)}(\Omega)$ respectively, $h \in L^{2(N-1)/N,\omega(N-1)/N}(\Gamma)$ or $h \in L^{p(N-1)/N}(\Gamma)$ respectively, and $\lambda \in \mathcal{V}$. Obviously, taking in the variational equation (2.2) right hand sides of such type, one gets results on local uniqueness of weak solutions to boundary value problems of the type

$$\begin{array}{rcll} \frac{\partial}{\partial x_j}\left(a^{ij}_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i} + b^j_\beta(u,\lambda)\right) + c_\beta(u,\lambda) & = & \frac{\partial f^j_\beta}{\partial x_j} + g_\beta & \text{in } \Omega, \\ \left(a^{ij}_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i} + b^j_\beta(u,\lambda)\right)\nu_j + d_\beta(u,\lambda) & = & h_\beta & \text{on } \Gamma_\beta, \\ u^\beta & = & \varphi^\beta & \text{on } \partial\Omega\setminus\Gamma_\beta \end{array} \tag{6.9}$$

(with Caratheodory functions $a^{ij}_{\alpha\beta}, b^j_\beta, c_\beta : \Omega\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$ and $d_\beta : \Gamma_\beta\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$) as well as results on smooth dependence of the solutions on the boundary values φ^β, the right hand sides f^j_β, g_β and h_β and on the parameter λ.

Example 6.4 Let $C^i \in C^1(\mathcal{U}\times\mathcal{V};L^\infty(\Omega))$ for $i = 1,\ldots,N$. Then, obviously, the map C, which is defined on $\mathcal{U}\times\mathcal{V}$ by

$$C(u,\lambda) := C^i(u,\lambda)\frac{\partial u}{\partial x_i},$$

is continuously differentiable from $\mathcal{U}\times\mathcal{V}$ into $L^2(\Omega)$. But $L^{2N/(N+2),\omega N/(N+2)}(\Omega)$ is continuously embedded into $L^{pN/(p+N)}(\Omega)$ for $\omega \le 2$ and into $L^2(\Omega)$ provided $(N-2)p \le 2N$. Hence, if $N \le 3$, then $C \in C^1(\mathcal{U}\times\mathcal{V};L^{2N/(N+2),\omega N/(N+2)}(\Omega))$ for some $\omega > N-2$ and $C \in C^1(\mathcal{U}\times\mathcal{V};L^{pN/(p+N)}(\Omega))$ for some $p > 2$. In this way one gets, in the case $N \le 3$, results on local uniqueness of weak solutions to boundary value problems of the type

$$\begin{array}{rcl} \frac{\partial}{\partial x_j}\left(a^{ij}_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i} + b^j_\beta(u,\lambda)\right) + c^i_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i} + d_\beta(u,\lambda) & = & \frac{\partial f^j_\beta}{\partial x_j} + g_\beta \text{ in } \Omega, \\ \left(a^{ij}_{\alpha\beta}(u,\lambda)\frac{\partial u^\alpha}{\partial x_i} + b^j_\beta(u,\lambda)\right)\nu_j + e_\beta(u,\lambda) & = & h_\beta \text{ on } \Gamma_\beta, \\ u^\beta & = & \varphi^\beta \text{ on } \partial\Omega\setminus\Gamma_\beta \end{array}$$

(with Caratheodory functions $a^{ij}_{\alpha\beta}, b^j_\beta, c^i_{\alpha\beta}, d_\beta : \Omega\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$ and $e_\beta : \Gamma_\beta\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$) as well as results on smooth dependence of the solutions on the boundary values φ^β, the right hand sides f^j_β, g_β and h_β and on the parameter λ.

7 Van Roosbroecks Drift Diffusion System

As an example for a quasilinear elliptic system with non-smooth data, let us consider van Roosbroecks Drift Diffusion System (se, e.g., [14])

$$\left.\begin{array}{rcl} \nabla\cdot(\lambda_1(x)\nabla u_1) - e^{u_1}u_2 + e^{u_1}u_3 & = & f(x), \\ \nabla\cdot(\lambda_j(x)e^{u_1}\nabla u_j) - R(x,u_1,u_2,u_3) & = & 0, \quad j = 1,2, \end{array}\right\} \text{ in } \Omega. \tag{7.1}$$

System (7.1) describes the carrier distribution in nondegenerate (i.e. such that Boltzmann statistics can be applied) semiconductors. The unknown functions are the electrostatic potential u_1 and the Slotboom variables u_2 and u_3 ($-\ln u_2$ and $-\ln u_3$ are the electrochemical potentials of the electrons and holes, respectively). The coefficient function λ_1 is the dielectric permitivity, λ_2 and λ_3 are the mobilities of the electrons and holes, respectively, and f is the net impurity concentration. In applications these functions are discontinuous (piecewise constant) due to the heterostructure of the semiconductor device. Hence, $\lambda_j, f \in L^\infty(\Omega)$ and $\operatorname{ess\,inf} \lambda_j > 0$. Let us take the recombination generation term R in the Shockley-Read-Hall form

$$R(x, u_1, u_2, u_3) = \frac{u_2 u_3 - 1}{\tau_2 \left(e^{u_1} u_2 + \rho_2(x)\right) + \tau_3 \left(e^{u_1} u_3 + \rho_3(x)\right)}.$$

Here τ_j are positive constants (the life times of electrones and holes, respectively), and the functions $\rho_j \in L^\infty(\Omega)$ are (positive) reference densities. We have adopted a suitable system of units such that the intrinsic carrier density is one.

The domain Ω is a Lipschitz domain in $\mathbb{R}^N$ ($N = 1, 2$ or 3 in applications), describing the semiconductor device. We consider the boundary conditions

$$\nabla u_1 \cdot \nu - \gamma(x) u_1 = \nabla u_2 \cdot \nu = \nabla u_3 \cdot \nu = 0 \text{ on } \Gamma, \tag{7.2}$$

$$u_j = u_j^0 \text{ on } \partial\Omega \setminus \Gamma. \tag{7.3}$$

Here ν is the unit outward normal to $\partial\Omega$, $\gamma \in L^\infty(\Gamma)$ in nonnegative, and the functions u_j^0 are supposed to be traces on $\partial\Omega \setminus \Gamma$ of functions from $W^{1,p}(\Omega)$ with $p > N$. Condition (7.2) describes the isolated part of the device boundary and (7.3) Ohmic contacts. The boundary part Γ is relatively open in $\partial\Omega$ and satisfies (2.1) (with $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma$), and $\partial\Omega \setminus \Gamma$ has positive $N-1$ dimensional measure.

Obviously, the boundary value problem (7.1)-(7.3) is of the type (6.9) (if we take, for example, as control parameter λ the triple of coefficient functions $(\lambda_1, \lambda_2, \lambda_3)$, i.e. $\Lambda = [L^\infty(\Omega)]^3$ and $\mathcal{V} = \{(\lambda_1, \lambda_2, \lambda_3) \in \Lambda : \operatorname{ess\,inf} \lambda_j > 0\}$) with

$$\begin{aligned}
&a_{11}^{ij}(x, u, \lambda) = \lambda_1(x)\delta^{ij},\ c_1(x, u, \lambda) = -e^{u_1} u_2 + e^{u_1} u_3,\\
&a_{\alpha\alpha}^{ij}(x, u, \lambda) = \lambda_\alpha(x)\delta^{ij} e^{u_1},\ c_\alpha(x, u, \lambda) = R(x, u) \text{ for } \alpha = 2, 3,\\
&a_{\alpha\beta}^{ij} = 0 \text{ for } \alpha \neq \beta,\\
&b_\beta^j = 0,\ f_\beta^j = 0,\\
&g_1 = f,\ g_\alpha = 0 \text{ for } \alpha = 2, 3,\\
&d_1(x, u, \lambda) = -\tfrac{\gamma(x)}{\lambda_1(x)},\ d_\alpha = 0 \text{ for } \alpha = 2, 3,\\
&h_\beta = 0,\ u^\beta = u_\beta^0.
\end{aligned}$$

In particular, the functions $a_{\alpha\beta}^{ij}$ are of the type, described in Example 5.2 (with $m = 1, n = 3$).

Existence of weak solutions $u \in [L^\infty(\Omega) \cap W^{1,2}(\Omega)]^3$ to (7.1)-(7.3) can be shown by the Schauder fixed point theorem (see, e.g., [14]). For the sake of simplicity, let us consider the case, if

$$u_2^0 = \frac{1}{u_3^0} =: c$$

is a positive constant. Then the solution to (7.1)-(7.3) is known to be unique (the thermodynamic equilibrium), and

$$u_2(x) = \frac{1}{u_3(x)} = c \text{ for all } x \in \Omega.$$

Let us denote by v the electrostatic potential of the given thermodynamic equilibrium state. The linearization of (7.1)-(7.3) in this thermodynamic equilibrium state is

$$\left.\begin{array}{rcl} \nabla \cdot (\lambda_1 \nabla u_1) + e^v \left(-u_2 - c u_1 + u_3 - u_1/c\right) & = & 0, \\ \nabla \cdot (\lambda_j e^v \nabla u_j) - \frac{u_2 + u_3}{(\tau_2(e^v c + \rho_2(x)) + \tau_3(e^v/c + \rho_3(x)))^2} & = & 0, \quad j = 1, 2, \end{array}\right\} \text{ in } \Omega,$$

$$\nabla u_1 \cdot \nu - \gamma(x) u_1 = \nabla u_2 \cdot \nu = \nabla u_3 \cdot \nu = 0 \text{ on } \Gamma,$$

$$u_j = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

It is easy to check that this linear homogeneous boundary value problem does not have nontrivial solutions: We have

$$0 = \int_\Omega \left(e^v \left(\lambda_2 |\nabla u_2|^2 + \lambda_3 |\nabla u_3|^2\right) + \frac{(u_2 + u_3)^2}{\left(\tau_2 \left(e^v c + \rho_2(x)\right) + \tau_3 \left(e^v/c + \rho_3(x)\right)\right)^2} \right) dx.$$

This gives $u_2 = u_3 = 0$. Therefore

$$0 = \int_\Omega \left(\lambda_1 |\nabla u_1|^2 + e^v u_1^2 \left(c + \frac{1}{c} \right) \right) dx + \int_\Gamma \gamma u_1^2 d\Gamma,$$

i.e. $u_1 = 0$.

Thus, for data, which are close to those, creating thermodynamic equilibria, the weak solution to (7.1)-(7.3) depends C^1-smoothly (in fact analytically) in the sense of $[L^\infty(\Omega) \cap W^{1,2}(\Omega)]^3$ on all data $\lambda_1, \lambda_2, \lambda_3, f, \rho_2, \rho_3 \in L^\infty(\Omega)$, $\gamma \in L^\infty(\Gamma)$, $u_1^0, u_2^0, u_3^0 \in W^{1,p}(\Omega)$ and $\tau_2, \tau_3 \in \mathbb{R}$.

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