Applications of a new G-invariant Implicit Function Theorem

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Abstract

We formulate a result of the type of the Implicit Function Theorem for abstract equivariant equations, and we demonstrate by two examples (problems for ordinary and partial differential equations) how the assumptions can be verified and how the assertions can be interpreted.

Let U and V be Banach spaces, Λ a normed vector space, $k \geq 2$ a natural number, $F : U \times \Lambda \to V$ a C^k-map and $u_0 \in U$ a point such that $F(u_0, 0) = 0$ and that

$$L := \partial_u F(u_0, 0) \text{ is a Fredholm operator (index zero) from } U \text{ into } V.$$

Further, let G be a compact Lie group which works linearly on U and V, respectively, such that the maps $u \in U \mapsto \gamma \cdot u \in U$ and $v \in V \mapsto \gamma \cdot v \in V$ are continuous for all $\gamma \in G$, that the map $\gamma \in G \mapsto (\gamma \cdot u, \gamma \cdot v) \in U \times V$ is continuous for all $(u, v) \in U \times V$ and that

$$F(\gamma \cdot u, \lambda) = \gamma \cdot F(u, \lambda) \text{ for all } u \in U,\ \lambda \in \Lambda \text{ and } \gamma \in G.$$

Then the group orbit $\mathcal{O}(u_0) := \{\gamma \cdot u_0 : \gamma \in G\}$ is a C^k-submanifold in U (cf. [2]), and the tangential space $T_{u_0}\mathcal{O}(u_0)$ is a subspace of $\ker L$. We assume that $\ker L$ is as small as possible under these assumptions, i.e. that

$$\ker L = T_{u_0}\mathcal{O}(u_0). \tag{1}$$

Let $G_0 := \{\gamma \in \Gamma : \gamma \cdot u_0 = u_0\}$ be the isotropy subgroup of u_0 and $U_0 := \{u \in U : \gamma \cdot u = u \text{ for all } \gamma \in G_0\}$ and $V_0 := \{v \in V : \gamma \cdot v = v \text{ for all } \gamma \in G_0\}$ the corresponding fixed point subspaces. Then LU_0 is a closed subspace of finite codimension in V_0, and we denote its codimension in V_0 by $\text{codim}_{V_0} LU_0$.

The following theorem was proved in [3]. It describes the solution behavior near $\mathcal{O}(u_0) \times \{0\}$ of equation

$$F(u, \lambda) = 0. \tag{2}$$

Theorem *Let Λ_2 be a subspace in Λ such that*

$$\dim \Lambda_2 = \text{codim}_{V_0} LU_0 \text{ and } V_0 = LU_0 \oplus \partial_\lambda F(u_0, 0)\Lambda_2. \tag{3}$$

Further, let Λ_1 be a closed complement of Λ_2 in Λ.

Then there exist neighbourhoods $\mathcal{W} \subseteq U$ of $\mathcal{O}(u_0)$ and $\mathcal{W}_j \subseteq \Lambda_j$ of the zero's $(j = 1, 2)$ and C^k–maps $\hat{u} : \mathcal{W}_1 \to U_0$ and $\hat{\lambda}_2 : \mathcal{W}_1 \to \Lambda_2$ with $\hat{u}(0) = u_0$ and $\hat{\lambda}_2(0) = 0$ such that for $u \in \mathcal{W}$ and $(\lambda_1, \lambda_2) \in \mathcal{W}_1 \times \mathcal{W}_2$ it holds $F(u, \lambda_1 + \lambda_2) = 0$ if and only if $\lambda_2 = \hat{\lambda}_2(\lambda_1)$ and $u = \gamma \cdot \hat{u}(\lambda_1)$ for some $\gamma \in \Gamma$.

The so-called G-invariant Implicit Function Theorem of E. DANCER [2] is a special case of the theorem above, namely that of $\mathrm{codim}_{V_0} LU_0 = 0$. In that case the theorem states that for all small λ there exists exactly one orbit of solutions to (2) near $\mathcal{O}(u_0)$. But if $\mathrm{codim}_{V_0} LU_0 > 0$ ($\mathrm{codim}_{V_0} LU_0$ can be any nonnegative integer up to $\dim G - \dim G_0$), then (2) is solvable near $\mathcal{O}(u_0) \times \{0\}$ if and only if $\lambda_2 = \hat{\lambda}_2(\lambda_1)$, i.e if and only if λ belongs to a C^k-submanifold in Λ of codimension $\mathrm{codim}_{V_0} LU_0$. In other words: In order to have solutions $u \approx \mathcal{O}(u_0)$ to (2), one can choose $\lambda_1 \approx 0$ arbitrarily, but then λ_2 is determined by λ_1. In this spirit λ_1 is a "control" parameter and λ_2 a "state" parameter. In applications the role of condition (3) is to show how to split Λ into subspaces of "control" and "state" parameters. Of course, such a splitting is not unique, in general.

Let us consider two examples. The results produced in the setting of these examples, are already known, of course. So the aim is to show how the assumptions (1) and (3) can be verified in simple, but typical for more complicated applications, examples, which type of assertions follows and that quite different "dynamic" and "static" problems fit in the abstract setting above.

Periodic traveling waves The problem of periodic traveling wave solutions to reaction-diffusion systems $\partial_t w = A\partial_x^2 w + f(w)$, $x \in \mathbb{R}$, $w \in \mathbb{R}^n$, leads, via the ansatz $w(t,x) = u(dx - ct)$, to the problem of 2π-periodic solutions to the system of ordinary differential equations $d^2Au'' + cu' + f(u) = 0$. Here $f : \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be smooth, and A is a positive definite $n \times n$-matrix. Thus, we have the setting above with $U := C^2_{2\pi}(\mathbb{R}^n)$, $V := C_{2\pi}(\mathbb{R}^n)$, $\Lambda := \mathbb{R}^2$, $F(u,c,d) := d^2Au'' + cu' + f(u)$, $G := \mathbf{SO}(2) \approx \mathbb{R}/2\pi$ and $(\gamma \cdot u)(t) := u(t+\gamma)$. Suppose that $F(u_0, c_0, d_0) = 0$, that $d_0 > 0$ and that 2π is the minimal period of u_0. Then $U_0 = U$ and $V_0 = V$, and (1) is satisfied with $T_{u_0}\mathcal{O}(u_0) = \mathrm{span}\,\{u_0'\}$ iff the eigenvalue zero of $\partial_u F(u_0, c_0, d_0)$ is geometrically simple. Further, we have $\partial_c F(u_0, c_0, d_0) = u_0' \notin \mathrm{im}\ \partial_u F(u_0, c_0, d_0)$, and hence (3) with $\Lambda_2 = \{(c,d) : d = 0\}$, iff the eigenvalue zero of $\partial_u F(u_0, c_0, d_0)$ is algebraically simple. Therefore, in this case the wave speed c is locally determined by the spatial period $2\pi/d$.

On the other hand, we have $\partial_d F(u_0, c_0, d_0) = 2d_0Au_0'' \notin \mathrm{im}\ \partial_u F(u_0, c_0, d_0)$ iff it holds $\int_0^{2\pi} \langle Au_0'', v\rangle dt \neq 0$ for all nonzero 2π-periodic solutions to the adjoint linearized equation $d_0^2Av'' - c_0v' + f'(u_0)^T v = 0$. Hence, in that case the spatial period is locally determined by the wave speed.

Now, let us consider solutions with $c = 0$, i.e. periodic "frozen" or "standing" waves. In this case we have $\Lambda := \mathbb{R}$ and $G := \mathbf{O}(2)$, because $F(\cdot, 0, d)$ is $\mathbf{O}(2)$-equivariant (with the reflection $(\delta \cdot u)(t) := u(-t)$). Suppose again that 2π is the minimal period of u_0, but now, additionally, assume that there exists a $\gamma \in \mathbf{SO}(2)$ such that $\delta\gamma \cdot u = \gamma \cdot u$. Then $V_0 = \{v \in V : \delta\gamma \cdot v = \gamma \cdot v\}$ and $U_0 = U \cap V_0$. If the eigenvalue zero of $\partial_u F(u_0, 0, d_0)$ is simple, then $\partial_u F(u_0, 0, d_0)$ is a Fredholm operator from U_0 into V_0 which is injective (because $u_0' \notin U_0$). Hence, (3) is satisfied with $\Lambda_2 = \Lambda$, and for all $d \approx d_0$ there exists exactly one orbit of symmetric periodic standing wave solutions near $\mathcal{O}(u_0)$.

Symmetric elliptic boundary value problems Consider the elliptic boundary value problem

$$A\Delta u + f(u,\mu) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \ u \in \mathbb{R}^n \tag{4}$$

in a symmetric bounded domain $\Omega \subset \mathbb{R}^2$. Here $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is supposed to be smooth, again, and A belongs to the space $\mathbb{M}_n$ of $n \times n$-matrices. We have the setting above with $U := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $V := L^2(\Omega)$, $\Lambda := \mathbb{M}_2 \times \mathbb{R}$, $F(u, A, \mu) := A\Delta u + f(u, \mu)$, $G := \mathbf{O}(2)$, $(\gamma \cdot u)(r, \varphi) := u(r, \varphi + \gamma)$ for all $\gamma \in \mathbf{SO}(2)$ and $(\delta \cdot u)(r, \varphi) := u(r, -\varphi)$. Suppose $F(u_0, A_0, 0) = 0$ with a positive definite A_0, and let the eigenvalue zero of $\partial_u F(u_0, A_0, 0)$ be simple. Then (1) is satisfied with $T_{u_0}\mathcal{O}(u_0) = \text{span}\,\{\partial_\varphi u_0\}$. Further, let $\frac{2\pi}{n}$ be the minimal period of $u_0(r, \cdot)$ (with $n \in \mathbb{N}$). Then $G_0 = \mathbf{D}_n$ (the dihedral group) or $G_0 = \mathbf{Z}_n$ (the cyclic group) if there exists a $\gamma \in \mathbf{SO}(2)$ such that $\delta\gamma \cdot u_0 = \gamma \cdot u_0$ or if not. In the first case we have, as in the example above, $\partial_u F(u_0, A_0, 0)U_0 = V_0$ (because $\partial_\varphi u_0 \notin U_0$). Hence, in that case (3) is satisfied with $\Lambda_2 = \Lambda$, and for all $A \approx A_0$ and $\mu \approx 0$ there exists exactly one orbit of symmetric solutions near the orbit of u_0. In the second case we have $\partial_\mu F(u_0, A_0, 0) \in V_0 \setminus \partial_u F(u_0, A_0, 0)U_0$ iff $\int_\Omega \langle \partial_\mu f(u_0, 0), v\rangle dx \neq 0$ for all nonzero solutions to the adjoint linearized boundary value problem $A\Delta v + \partial_u f(u_0, 0)^T v = 0$ in Ω, $v = 0$ on $\partial\Omega$. Hence, in that case (3) is satisfied with $\Lambda_2 = \{(A, \mu) : A = 0\}$, and only for all $A \approx A_0$, which satisfy an equation $A = \hat{A}(\mu)$, there exists an orbit of solutions near $\mathcal{O}(u_0)$.

In other words: If a solution to (4) is not radially symmetric, but has a reflection symmetry, then generically its orbit survives under all small perturbations of A and μ as an orbit of solutions. In [1] a problem from elastostatics is analyzed in a similar setting, where all the radially nonsymmetric solutions (up to eight for appropriate parameters) have a reflection symmetry.

If a solution to (4) doesn't have a reflection symmetry, then generically it survives as an orbit of solutions only under quite special perturbations. If in that case (4) is the stationary problem of an evolution problem, for example of $\partial_t u = A\Delta u + f(u, \mu) = 0$, it is natural to ask whether or not the orbit of u_0 survives under all small perturbations as an invariant manifold. But that's another question, of course.

Rotating and modulated waves In [3] it is shown how the abstract theorem above can be applied in order to describe the parameter dependence of rotating and of modulated wave solutions (especially of the wave frequencies and the modulation frequencies) to equivariant ordinary differential equations.

References

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