WELL-POSEDNESS, SMOOTH DEPENDENCE AND CENTER MANIFOLD REDUCTION FOR A SEMILINEAR HYPERBOLIC SYSTEM FROM LASER DYNAMICS

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ABSTRACT. We prove existence, uniqueness, regularity and smooth dependence of the weak solution on the initial data for a semilinear, first order, dissipative hyperbolic system with discontinuous coefficients. Such hyperbolic systems have succesfully been used to model the dynamics of distributed feedback multisection semiconductor lasers. We show that in a function space of continuous functions the weak solutions generate a smooth skew product semiflow. Using slow fast structure and dissipativity we prove the existence of smooth exponentially attracting invariant center manifolds for the nonautonomous model.

AMS Subject Classification: 35L50, 35L60, 35B30, 35Q60, 35R05, 37D10, 37N20

Keywords: Existence, uniqueness, regularity of weak solutions, smooth dependence on data, smooth semiflow property, existence of smooth invariant center manifolds, nonautonomous system, discontinuous coefficients, laser dynamics

1. INTRODUCTION

This paper concerns a model for the dynamical behavior of distributed feedback multisection semiconductor lasers. The model consists of an initial boundary value problem for a dissipative semilinear hyperbolic system of first order PDEs (two coupled traveling wave equations describing the forward and backward propagating complex amplitudes of the light) coupled to a spatially extended ordinary differential equation (carrier rate equation). The model describes the longitudinal dynamics of edge emitting lasers, so the dimension of the space variable is one. The coefficients in the equations and the boundary conditions are allowed to be discontinuous with respect to the space and time variables, but have to be smooth with respect to the unknown functions. The spatial discontinuities appear in the equations due to the significantly different electrical and optical properties of each section in a multi-section laser. Discontinuities with respect to time occur in the equations and/or boundary conditions when the laser is subject to step-like forcings through electrical or optical injection. Models of this kind exhibit very rich

[1 2] Supported by DFG Research Center MATHEON "Mathematics for key technologies" in Berlin

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dynamics, even in the autonomous case, and have received remarkable attention in physics literature, see e.g. [5, 6, 20, 26].

The focus of our paper is to show existence, uniqueness of global weak solutions as well as their smooth dependence on the initial data in a suitable function space setting, and then apply the results to smooth invariant center manifold reduction. We do this via an abstract variation of constants formula setting of the type

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(s, u(s))\, ds, \tag{1}$$

where A is the generator of a C_0 semigroup on $L^p\left(]0, L[\,; \mathbb{C}^2 \times \mathbb{R}\right) \times L^p_\eta\left(]0, \infty[\,; \mathbb{C}\right)$ for $p \in [1, \infty[$ (L^p_η is a weighted L^p space, see section 3). The operator A generates also a semigroup on $L^\infty\left(]0, \infty[\,; \mathbb{C}^2 \times \mathbb{R}\right) \times L^\infty\left(]0, \infty[\,; \mathbb{C}\right)$, but this semigroup is not measurable in the sense of Bochner[1]. On the other hand, the nonlinear Nemytskij operator f maps $L^p\left(]0, L[\,, \mathbb{C}^2 \times \mathbb{R}\right) \times L^p_\eta\left(]0, \infty[\,, \mathbb{C}\right)$ smoothly into itself for $p = \infty$[2], but not for $p < \infty$.

We circumvent this technical difficulty by showing that (1) generates a smooth equation in $L^\infty\left(]0, T[\, \times\,]0, L[\,; \mathbb{C}^2 \times \mathbb{R}\right) \times L^\infty\left(]0, \infty[\,; \mathbb{C}\right)$ for any $T > 0$. After truncation, this equation can be solved by means of Banachs fixed point theorem. Then a priori estimates show that the solutions of the truncated equation solve also the untruncated equation. And finally, applying the implicit function theorem, we show that the solution depends smoothly in the sense of $L^\infty\left(]0, T[\, \times\,]0, L[\,; \mathbb{C}^2 \times \mathbb{R}\right) \times L^\infty\left(]0, \infty[\,; \mathbb{C}\right)$ on the initial data.

In Theorems 2.7 and 2.8 we state that in a function space of continuous functions the weak solutions form a smooth semiflow in the autonomous case and that our system can be viewed as a smooth process after a suitable boundary homogenization in the nonautonomous case. Hence, if we restrict to continuous solutions, we have both the C_0 property, i.e. continuous evolution in time, and smooth dependence on the initial data. Both properties are required for applying persistence theorems for invariant manifolds [3, 4] to (1) which we will do in section 6. Theorems 2.4 and 2.6 state basic regularity results for the weak solutions, and Theorem 2.5 contains a priori estimates. Remark that these results hold true locally in time for more general hyperbolic systems with arbitrary nonlinearities appearing in many other applications [12, 13]. In Theorem 6.2 we state the existence of smooth exponentially attracting (with respect to the supremum norm) invariant center manifolds for the nonautonomous laser model.

The semiconductor lasers, described by our models, exhibit very rich dynamics, including bifurcations, selfpulsations, hystheresis, excitability, frequency synchronization etc., and so do the models also. A lot of such behaviour is described numerically, see e.g. [2, 5, 19, 22, 26], but only a few of these results are mathematically rigorously founded [18, 21, 23, 24]. The reason is that for applying, for example, abstract dynamical bifurcation theory, one needs a smooth dynamical system, existence and persistence of smooth invariant manifolds, that the linearized semigroup has a spectrum determined exponential dichotomy or a spectral mapping property etc., and within the class of models, we are dealing with, such properties are proved

[1] it can be shown that a C_0 semigroup on L^∞ has a bounded generator [15]
[2] for $p = \infty$ put $L^p_\eta := L^\infty$

in some exceptional cases only. This has been resolved recently in [12, 13, 14] for a general class of semilinear hyperbolic systems.

On the other hand, it turns out the following: If the coefficients are sufficiently smooth with respect to time then all the interesting dynamics, which is observable in numerical and real world experiments, can be rigorously described by our general models. It occurs on exponentially attracting invariant manifolds of continuous functions. Such solutions do not possess jumps in space due to discontinuities of the inital data or to incompatible boundary data as discussed, e.g., in [17].

Let us shortly discuss some related results:

Jochmann/Recke [11] got existence and uniqueness of weak solutions under the assumption that the coupled traveling wave equations are linear with respect to the light amplitudes. They did not deal with smooth dependence of the solutions on the initial data.

Peterhof/Sandstede [18] and Sieber [22, 23] also assumed the coupled traveling wave equation to be linear, and, moreover, they considered a Galerkin projected version of the carrier rate equation. In this setting the equations are linear with respect to the infinite dimensional state parameter (the space dependent light amplitudes) and really nonlinear only with respect to the remaining finite dimensional state parameter (the carrier densities, which are piecewise constant in space). Hence, the state space for the light amplitudes could be chosen as a "large" L^2 space, and, nevertheless, the authors rigorously got smooth semiflows and a rich bifurcation behaviour. Remark that in this setting the spectrum determined exponential dichotomy of the linearized semiflow is known due to a result of Neves/Ribeiro/Lopes [16].

Renardy [21] and Haken/Renardy [9] considered not edge emitting, but ring lasers. Thus the spatial domain is not an interval, but a circle, and the Nemytskij operators map the "small" space of continuously differentiable functions on the circle into itself.

Similarly Illner/Reed [10] and Vanderbauwhede/Iooss [25, Section 4, Example 3] considered semilinear hyperbolic initial boundary value problems (not related to laser dynamics), where the nonlinearities are compatible with the boundary conditions.

We have divided this work into five sections. First we state our assumptions and results in Section 2. Then in Section 3 we establish the variation of constants formula for the weak solutions defined in Section 2 and prove the results for the problem with truncated nonlinearities. In Section 4 we show a priori estimates which are independent of the truncation parameter. Thus all results hold for the non-truncated problem. In Section 5 we present a concrete model and related numerical results. Finally, in Section 6 we state the invariant manifold theorem.

2. Assumptions and Results

The system we consider is of the following form:

$$
\left.
\begin{array}{rcl}
\partial_t \psi(t,x) & = & (-\partial_x \psi_1(t,x), \partial_x \psi_2(t,x)) + G\left(x, \psi(t,x), n(t,x)\right) \\
\partial_t n(t,x) & = & I(t,x) + H\left(x, \psi(t,x), n(t,x)\right) \\
 & & + \sum_{k=1}^{m} b_k \chi_{S_k}(x) \left(⨍_{S_k} n(t,y)\, dy - n(t,x) \right)
\end{array}
\right\} \tag{2}
$$

with boundary conditions

$$\left.\begin{array}{rcl}\psi_1(t,0) & = & r_0\psi_2(t,0)+\alpha(t)\\ \psi_2(t,L) & = & r_L\psi_1(t,L)\end{array}\right\} \tag{3}$$

and initial conditions

$$\psi(0,x)=\psi^0(x), \quad n(0,x)=n^0(x). \tag{4}$$

The function n is real valued, $\psi=(\psi_1,\psi_2)$ is $\mathbb{C}^2$ valued. They depend on the time $t\in\mathbb{R}$ and space variable $x\in[0,L]$. The interval $[0,L]=\cup_{k=1}^m\overline{S_k}$ is divided into m subsectional intervals $S_k:=\,]x_{k-1},x_k[,\ x_{k-1}<x_k,\ k=1,...,m$. By χ_{S_k} we denote the characteristic function of S_k, that is $\chi_{S_k}(x):=1$ for $x\in S_k$, $\chi_{S_k}(x):=0$ if $x\notin S_k$. The symbol $\fint_{S_k}:=\frac{1}{x_k-x_{k-1}}\int_{S_k}$ denotes the integral average on the subinterval S_k. The nonlinearities $G:\,]0,L[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{C}^2$ and $H:\,]0,L[\times\mathbb{C}^2\times\mathbb{R}\to\mathbb{R}$ are differentiable with respect to the phase variables (ψ,n), but only measurable and bounded with respect to the spatial variable $x\in[0,L]$. We now list the assumptions and refer to Section 5 for an example from semiconductor laser dynamics fulfilling all our assumptions:

We assume that $T>0$ is arbitrarily chosen but fixed. The abbreviation "*a.a.*" stands for "almost all" in the sense of Lebesgue's measure, $\mathfrak{Re}$ denotes the real part of a complex number, $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{C}^2$ and $\|\cdot\|$ its corresponding norm.

- **(I)** The functions G and H are C^k-Carathéodory functions (see Def. 2.12) on $]0,L[$ from $\mathbb{C}^2\times\mathbb{R}$ into $\mathbb{C}^2$ and $\mathbb{R}$, respectively.
- **(II)** There exist constants $0<\nu_1<\nu_2$ and $c_1,c_2,d_1,d_2>0$ such that for all $\psi\in\mathbb{C}^2$ and *a.a.* $x\in]0,L[$ the relations
  $H(x,\psi,n)\geq -c_1n$, if $n\leq\nu_1$,
  $H(x,\psi,n)\leq -c_2n$, if $n\geq\nu_2$,
  $H(x,\psi,n)+d_1\mathfrak{Re}\,\langle G(x,\psi,n),\psi\rangle\leq -d_2\left(n+\|\psi\|^2\right)$ for all $n\in\mathbb{R}$
  hold.
- **(III)** For every compact $K\subset\mathbb{R}$ there exists $M>0$ such that for all $n\in K$, $\psi\in\mathbb{C}^2$ and *a.a* $x\in\,]0,L[$ we have $\|G(x,\psi,n)\|\leq M\left(\|\psi\|+1\right)$.
- **(IV)** $I\in L^\infty\left(]0,T[\times]0,L[,\mathbb{R}\right)$, $I(t,x)\geq 0$ for *a.a.* $(t,x)\in\,]0,T[\times]0,L[$.
- **(V)** $\alpha\in L^\infty\left(]0,T[;\mathbb{C}\right)$.
- **(VI)** $r_0,r_L\in\mathbb{C},\ |r_0|<1,\ |r_L|\leq 1$.
- **(VII)** $n^0\in L^\infty(]0,L[;\mathbb{R}),\ n^0(x)\geq 0$ for *a.a.* $x\in]0,L[,\ \psi^0\in L^\infty(]0,L[;\mathbb{C}^2)$.
- **(VIII)** $b_k\in\mathbb{R},b_k\geq 0$ for $1\leq k\leq m$.

Remark 2.1. *The third relation in condition* (II) *implies the apriori estimate* (8) *which allows to treat the nonlocal term appearing in the carrier rate equation. When the nonlocal term vanishes, as in* [11, 18]*, this condition can be dropped.*

Definition 2.2. A pair $(\psi, n) \in L^\infty\left(]0,T[\times]0,L[;\mathbb{C}^2\times\mathbb{R}\right)$ is a weak solution to (2), (3), (4) if

$$\begin{aligned}(5)\qquad &\int_0^L \left\langle \psi(t,x)-\psi^0(x), \varphi(x)\right\rangle dx \\ &= \int_0^t \Bigg\{ \int_0^L \Big[\psi_1(s,x)\overline{(\partial_x\varphi_1)(x)} - \psi_2(s,x)\overline{(\partial_x\varphi_2)(x)} \\ &\quad + \langle G(x,\psi(s,x),n(s,x)),\varphi(x)\rangle \Big] dx + \alpha(s)\overline{\varphi_1(0)} \Bigg\} ds\end{aligned}$$

for all $t \in [0,T]$ and all $\varphi \in W^{1,2}\left(]0,L[,\mathbb{C}^2\right)$ with $\varphi_2(0) = \overline{r_0}\varphi_1(0)$ and $\varphi_1(L) = \overline{r_L}\varphi_2(L)$ and if

$$\begin{aligned}(6)\qquad n(t,x) &= n^0(x) + \int_0^t \Bigg\{ I(s,x) + H(x,\psi(s,x),n(s,x)) \\ &\quad + \sum_{k=1}^m b_k\chi_{S_k}(x)\left[⨍_{S_k} n(s,y)\,dy - n(s,x)\right] \Bigg\} ds\end{aligned}$$

for all $t \in [0,T]$ and a.a. $x \in]0,L[$.

Theorem 2.3 (Existence, Uniqueness and smooth Dependence)**.** *Assume* (I) – (VIII). *Then there exists a unique weak solution* (ψ, n) *to* (2), (3), (4). *Moreover, the map*

$$\begin{aligned}&(\psi_0, n_0, I, \alpha) \in L^\infty\left(]0,L[;\mathbb{C}^2\times\mathbb{R}\right) \times L^\infty\left(]0,T[\times]0,L[,\mathbb{R}\right) \times L^\infty\left(]0,T[;\mathbb{C}\right) \\ &\mapsto (\psi,n) \in L^\infty\left(]0,T[\times]0,L[;\mathbb{C}^2\times\mathbb{R}\right)\end{aligned}$$

is C^k-smooth.

We denote the closed subspace in $L^\infty(]0,L[,\mathbb{R})$ of section-wise uniformly continuous functions by

$$C_P := \left\{ n \in L^\infty(]0,L[;\mathbb{R}) \mid n_{|S_k} \text{ uniformly continuous for } k=1,2,\dots,m \right\}.$$

Theorem 2.4 (Solution Regularity I)**.** *Assume* (I) – (VIII) *and let* (ψ, n) *be the weak solution. Then the following holds:*
i) $\psi \in C\left([0,T];L^2\left(]0,L[;\mathbb{C}^2\right)\right)$, $n \in W^{1,\infty}\left(]0,T[;L^\infty\left(]0,L[;\mathbb{R}\right)\right)$.
ii) For $t \in [0,T]$ *denote* $\tilde\psi(t) := \int_0^t \psi(s)\,ds$. *Then for all* $t \in [0,T]$ *we have* $\tilde\psi(t) \in W^{1,2}\left(]0,L[;\mathbb{C}^2\right)$ *and*

$$\tilde\psi_1(t)(0) = r_0\tilde\psi_2(t)(0) + \int_0^t \alpha(s)ds,\ \tilde\psi_2(t)(L) = r_L\tilde\psi_1(t)(L).$$

iii) Let $\alpha \in W^{1,2}\left(]0,T[;\mathbb{C}\right)$, $\psi^0 \in W^{1,2}(]0,L[;\mathbb{C}^2)$ *and suppose*

$$(7)\qquad \psi_1^0(0) = r_0\psi_2^0(0) + \alpha(0),\ \psi_2^0(L) = r_L\psi_1^0(L).$$

Then $\psi \in C\left([0,T];W^{1,2}\left(]0,L[;\mathbb{C}^2\right)\right) \cap C^1\left([0,T];L^2\left(]0,L[;\mathbb{C}^2\right)\right)$ *and* (2), (3) *hold for* $t \in [0,T]$ *in the classical sense. Moreover, if* $I \in C\left([0,T];L^\infty(]0,T[;\mathbb{R})\right)$ *then* $n \in C^1([0,T];L^\infty(]0,L[;\mathbb{R}))$.
iv) Suppose $\psi^0 \in C\left([0,L];\mathbb{C}^2\right)$, $\alpha \in C\left([0,T];\mathbb{C}\right)$ *and* (7). *Then*

$$\psi \in C\left([0,T]\times[0,L];\mathbb{C}^2\right) \text{ and (3) is satisfied pointwise.}$$

Further assume $n^0 \in C_P$, $I(t) \in C_P$ *for a.a.* $t \in [0,T]$ *and*

(IX) $$H(\cdot,\psi,n) \in C_P \text{ for all } \psi \in \mathbb{C}^2 \text{ and } n \in \mathbb{R}.$$

Then $n \in C([0,T];C_P)$. *Moreover, if* $I \in C([0,T];C_P)$, *then* $n \in C^1([0,T];C_P)$.

Theorem 2.5 (A priori estimates)**.** *Suppose* (I) – (VIII) *and let* (ψ,n) *be the weak solution. Then for all* $t \in [0,T]$ *we have*

$$\begin{aligned} &\int_0^L n(t,x)dx + \frac{d_1}{2}\left\|\psi(t)\right\|_{L^2}^2 \\ &\quad\le \mu + \max\left\{\int_0^L n^0(x)dx + \frac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2 - \mu, 0\right\} e^{-ct} \end{aligned} \tag{8}$$

with

$$c := \min\left\{d_2, \frac{2d_2}{d_1}\right\} \quad \text{and} \quad \mu := c^{-1}\left(\frac{d_1}{2(1-|r_0|^2)}\left\|\alpha\right\|_{L^\infty}^2 + L\left\|I\right\|_{L^\infty}\right).$$

Moreover, for all $t \in [0,T]$ *and a.a.* $x \in]0,L[$

$$\min\left\{n^0(x),\nu_1\right\}e^{-(c_1+b)t} \le n(t,x) \le N + \max\left\{n^0(x)-N,0\right\}e^{-c_2 t}, \tag{9}$$

where $b := \max\{b_k \mid 1 \le k \le m\}$ *and*

$$N := \max\left\{\nu_2, \tfrac{1}{c_2}\left(\left\|I\right\|_{L^\infty} + \max_{1\le k\le m}\left(\tfrac{b_k}{|S_k|}\right)\cdot\max\left\{\mu, \textstyle\int_0^L n^0(x)dx + \tfrac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2\right\}\right)\right\}.$$

If the data ψ^0 and α are $W^{1,2}$-smooth, then Theorem 2.4, iii), states that the weak solution ψ will be $W^{1,2}$-smooth with respect to the spatial variable x. Of course, under natural assumptions of piecewise smoothness for the data entering the carrier rate equation, this smoothness of ψ carries over to n via the coupling of ψ and n in (2). Theorem 2.6 states this precisely. Let

$$W_P^{1,2} := \left\{n \in L^\infty(]0,L[;\mathbb{R}) \mid n_{|S_k} \in W^{1,2}\left(S_k;\mathbb{R}\right)\ k=1,2,\dots,m\right\}$$

denote the space of piecewise $W^{1,2}$ functions.

Theorem 2.6 (Solution Regularity II)**.** *Suppose* (I) – (VIII) *and the following conditions:*

- **(X)** $H_{|S_k\times\mathbb{C}^2\times\mathbb{R}} \in C^1\left(\overline{S_k}\times\mathbb{C}^2\times\mathbb{R};\mathbb{R}\right)$ *for* $1 \le k \le m$.
- **(XI)** *For all compact* $K \subset \mathbb{R}$ *there exists* $\Lambda > 0$ *such that for* $x \in S_k$, $\psi \in \mathbb{C}$ *and* $n_1, n_2 \in K$ *we have* $\|DH(x,\psi,n_1) - DH(x,\psi,n_2)\| \le \Lambda|n_1-n_2|$.
- **(XII)** *There exists a constant* $\tau > 0$ *such that for all compact* $K \subset \mathbb{R}$ *there exists* $R > 0$ *with*

$$\begin{aligned} &\partial_x H(x,\psi,n)\tilde{n} + \partial_n H(x,\psi,n)\tilde{n}^2 + \partial_\psi H(x,\psi,n)\tilde{\psi}\tilde{n} \\ &\qquad \le R\left(1 + |\tilde{n}| + \left\|\tilde{\psi}\right\| + \left\|\tilde{\psi}\right\||\tilde{n}| + \left\|\tilde{\psi}\right\|^2\right) - \tau\tilde{n}^2 \end{aligned}$$

for all $x \in S_k$, $1 \le k \le m$, $\psi \in \mathbb{C}^2$, $\tilde{\psi} \in \mathbb{C}^2$, $n \in K$ *and* $\tilde{n} \in \mathbb{R}$.

Moreover, suppose $\alpha \in W^{1,2}(]0,T[;\mathbb{C})$, $\psi^0 \in W^{1,2}(]0,L[;\mathbb{C}^2)$, (7), $n^0 \in W_P^{1,2}$ *and* $I \in C\left([0,T];W_P^{1,2}\right)$. *Then, if* (ψ,n) *is the weak solution, we have*

$$n \in C^1([0,T];W_P^{1,2}).$$

In (XI) the symbol DH denotes the total derivative of H with respect to all variables (x, ψ, n), and in (XII) $\partial_x H$, $\partial_n H$ and $\partial_\psi H$ are the corresponding partial derivatives. Remark that $\mathbb{C}^2$ is always considered as a four dimensional real vector space, and the derivatives have to be understood correspondingly. We note that all assumptions (I) – (XII) are fulfilled in applications, see Section 5.

Define the phase space

$$\mathfrak{P} := \{\psi \in C([0,L], \mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0), \psi_2(L) = r_L\psi_1(L)\} \times C_P.$$

The following Theorems 2.7-2.8 are a direct consequence of Theorems 2.3-2.5:

Theorem 2.7 (C^k-Semiflow property)**.** *Suppose* (I) – (IX)*,* $\alpha = 0$ *and let* I *be constant with respect to time. Then the weak solutions generate a smooth semiflow in the function space* $\mathfrak{P}$*. Then the operator* $S^t : \mathfrak{P} \to \mathfrak{P}$*, defined through*

$$S^t\left(\psi^0, n^0\right) := (\psi(t), n(t))$$

for $t \geq 0$ *and* $\left(\psi^0, n^0\right) \in \mathfrak{P}$*, where* $(\psi(t), n(t))$ *denotes the weak solution corresponding to the initial values* $\left(\psi^0, n^0\right)$*, has the following properties*

i) $(t, \psi, n) \mapsto S^t(\psi, n)$ *is continuous from* $[0, \infty[\times\mathfrak{P}$ *into* $\mathfrak{P}$*,*
ii) $S^t : \mathfrak{P} \to \mathfrak{P}$ *is* C^k *smooth,*
iii) $S^{t+s} = S^t \circ S^s,\ t, s \in \mathbb{R},\ t, s \geq 0,$
iv) S^0 *is the identity operator on* $\mathfrak{P}$*.*

Finally consider the nonautonomous case: Assume

$$\textbf{(XIII)} \quad \alpha \in C(\mathbb{R}; \mathbb{C}) \quad \text{and} \quad I \in L^\infty(\mathbb{R}; C_P).$$

Let $g \in C(\mathbb{R}_+ \times C_p; C([0,L], \mathbb{C}^2))$ be such that g satisfies the inhomogeneous boundary condition

$$g_1(t,n)(0) = r_0 g_2(t,n)(0) + \alpha(t) \quad \text{and} \quad g_2(t,n)(L) = r_L g_1(t,n)(L)$$

for $t \geq 0$. For $t \geq s$ define $X\left(t, s, \left(\psi^0, n^0\right)\right) := (\psi(t-s), n(t-s))$, where (ψ, n) is the weak solution in the sense of Definition 2.2 to the initial data ψ^0, n^0 and $\alpha(s + \cdot)$. Then $X\left(t, s, \left(\psi^0, n^0\right)\right)$, $s \leq t$, can be interpreted as the weak solution at time $t + s$ corresponding to the initial condition $\psi(s, x) = \psi^0(x)$, $n(s, x) = n^0(x)$ for *a.a.* $x \in]0, L[$ at time s. Define the operator $Y(t,s) : \mathfrak{P} \to \mathfrak{P}$, through

$$\begin{aligned} Y(t,s)\left(\psi^0, n^0\right) :=& X\left(t, s, \left(\psi^0 + g(s, n^0), n^0\right)\right) \\ &- \begin{pmatrix} g\left(t, \Pi_n X\left(t, s, \left(\psi^0 + g(s,n^0), n^0\right)\right)\right) \\ 0 \end{pmatrix} \end{aligned}$$

for $t \geq s$ and $\left(\psi^0, n^0\right) \in \mathfrak{P}$. Here $\Pi_n X$ denotes the n-component of X. The operator $Y(t,s)$ maps $\mathfrak{P}$ into itself, hence the function g homogenizes the boundary condition (3). From the definition of Y one verifies that Y has the process property stated in Theorem 2.8.

Theorem 2.8. *Suppose* (I) – (IX) *and* (XIII)*. Further suppose that for* $t \geq 0$ *the map*

$$n \in C_p \mapsto g(t,n) \in C([0,L], \mathbb{C}^2) \quad \text{is } C^k \text{ smooth.}$$

Then the operator $Y(t,s)$ *is a* C^k *smooth two parameter nonautonomous process satisfying*

ı) for $t \geq s$ the map $p \in \mathfrak{P} \mapsto Y(t,s,p) \in \mathfrak{P}$ is C^k smooth,
ıı) the map $(t,s,p) \mapsto Y(t,s,p)$ is continuous from $\{(t,s) \in \mathbb{R}^2 \mid s \leq t\} \times \mathfrak{P}$ into $\mathfrak{P}$,
ııı) $Y(s,s,\cdot)$ is the identity operator on $\mathfrak{P}$,
ıv) for $t \geq s \geq r$ the process property $Y(t,s,Y(s,r,p)) = Y(t,r,p)$ holds.

Example 2.9. *In applications one has to choose an appropriate homogenization g. We give two examples for choices of g:*
(i) $g(t,n)(x) = \frac{L-x}{L}\begin{pmatrix}\alpha(t)\\ 0\end{pmatrix}$,
(ii) For each $n \in C_p$ $g(t,n)$ solves

$$\left\{\begin{array}{rcl} \partial_t g(t,n) & = & \left(\operatorname{diag}\left(-\partial_x, \partial_x\right) + \partial_\psi G(\cdot,0,n)\right) g(t,n), \\ g_1(t,n)_{|x=0} & = & r_0 g_2(t,n)_{|x=0} + \alpha(t), \\ g_2(t,n)_{|x=L} & = & r_L g_1(t,n)_{|x=L} \end{array}\right.$$

with suitable initial data.
The simple example (i) has been used by Sandstede and Peterhof in [18]*. We choose (ii) in section 6 to perform a center manifold reduction for the nonautonomous ($\alpha \neq 0$ in* (3)*) traveling wave equation* (23)*.*

The process Y can be equivalently written as a skew product semiflow Z^t on the trivial Banach bundle $\mathfrak{P} \times [0,\infty[$ if one defines for $(p,\theta) \in \mathfrak{P} \times [0,\infty[$

$$Z^t\,(p,\theta) := (Y(\theta+t,\theta,p), \theta+t)\,,\ p \in \mathfrak{P},\ (\theta, t \geq 0).$$

We extend Z^t onto the Banach space $\mathfrak{P}_e := \mathfrak{P} \times \mathbb{R}$ by setting

$$Z^t(p,\theta) = \left\{\begin{array}{ll} Z^t\,(p,\theta) & ,\theta \geq 0 \\ \left(\Pi_p Z^{t+\theta}\,(p,0)\,, \theta+t\right) & ,\theta < 0, \theta+t \geq 0 \\ (p,\theta+t) & ,\theta < 0, \theta+t < 0. \end{array}\right.$$

Then we can state the following

Corollary 2.10. *If $\alpha \in C^k([0,\infty[\,,\mathbb{R})$ and $g(t,n)$ is of class C^k in both variables (t,n), then the operator Z^t is a C^k smooth semiflow on $\mathfrak{P}_e$.*

After introducing the concrete model in section 5 we will consider the local existence of smooth exponentially attracting[3] invariant center manifolds in section 6. For this we first write the model in suitable dimensionless variables and find the following slow fast structure

$$\left\{\begin{array}{rcl} \partial_t \psi(t) & = & \mathfrak{A}(n)\psi(t) + \epsilon\mathfrak{K}(n(t),\psi(t)) \\ \partial_t n(t) & = & \epsilon\mathfrak{F}\left(t, n(t), \psi(t)\right), \end{array}\right. \tag{10}$$

where ψ satisfies (3),

$$\mathfrak{A}(n)\psi := \left(\begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix} + L(\cdot, n(\cdot))\right)\psi, \tag{11}$$

$L(x,n)$ is matrix valued, $\mathfrak{K}$ is a nonlinear Nemytskij operator generated by a vector valued C^k Carathéodory function, $\mathfrak{F}$ is composed of Carathéodory functions and a nonlocal term, ϵ is small and all other parameters of the PDE are of order one.

[3]with respect to the topology of $\mathfrak{P}_e$

Under a spectral gap assumption for $\mathfrak{A}(n)$, using Corollary 2.10, the persistence theory for invariant manifolds [4] and the spectral gap mapping theorem obtained in [12] we show the following Theorem which is a brief summary of Theorem 6.2

Theorem 2.11. *For sufficiently small $\epsilon > 0$ the process Y has local smooth exponentially attracting invariant center manifolds. The PDE locally reduces to nonautonomous ODEs on these manifolds.*

The flow on this manifold can be expanded in powers of ϵ, the unknown graph of this manifold only enters terms of order ϵ^2. If one drops these terms one gets an approximated flow on this manifold called mode approximation. Such mode approximations have been used succesfully for numerical bifurcation analysis in the autonomous case ($\alpha = 0$) [19, 22].

In assumption (I) we require that both G and H are C^k-Carathéodory functions, which we define next

Definition 2.12 (C^k-Carathéodory functions)**.** Let V, W be finite dimensional vector spaces and $k \in \mathbb{N}$. A function $S :]0, L[\times V \to W$, $S = S(x, v)$, $x \in]0, L[$, $v \in V$, is called a C^k Carathéodory function iff S satisfies the following three conditions:

$\imath$) For $a.a.$ $x \in]0, L[$ $S(x, \cdot) \in C^k(V;W)$ and $S(\cdot, v)$ is measurable for all $v \in V$.
$\imath\imath$) For all compact $K \subset V$ there exists a constant $M > 0$ such that $\left\| \frac{\partial^i S(x,v)}{\partial v^i} \right\| \leq M$ for $0 \leq i \leq k$, all $v \in K$ and $a.a.$ $x \in]0, L[$.
$\imath\imath\imath$) For all compact $K \subset V$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $v_1 \in K$, $v_2 \in V$ with $\|v_1 - v_2\| < \delta$ and $a.a.$ $x \in]0, L[$ we have $\left\| \frac{\partial^k S(x,v_1)}{\partial v^k} - \frac{\partial^k S(x,v_2)}{\partial v^k} \right\| < \epsilon$.

Obviously, any C^k Carathéodory function $S :]0, L[\times V \to W$ generates a superposition operator

$$\mathfrak{S} : \mathcal{M}(]0, L[\,; V) \to \mathcal{M}(]0, L[\,; W),\ \mathfrak{S}(v)(x) := S(x, v(x)),\ \text{for } a.a.\, x \in]0, L[\,, \tag{12}$$

where $\mathcal{M}(]0, L[\,; V)$ denotes the linear space of measurable functions defined almost everywhere on $]0, L[$ with values in V. We need the following easy to prove differentiability property of $\mathfrak{S}$.

Proposition 2.13. *(see [8]) The superposition operator $\mathfrak{S}$ maps $L^\infty(]0, L[\,; V)$ C^k-smoothly into $L^\infty(]0, L[\,; W)$.*

3. Variation of constants formula and proofs for the truncated problem

In the following we will frequently make use of the superposition operators

$$\mathfrak{G} \in C^k(L^\infty(]0, L[\,; \mathbb{C}^2 \times \mathbb{R}), L^\infty(]0, L[\,; \mathbb{C}^2))$$
$$\mathfrak{H} \in C^k(L^\infty(]0, L[\,; \mathbb{C}^2 \times \mathbb{R}), L^\infty(]0, L[\,; \mathbb{R}))$$

generated by G and H through (12). Also the following operator

$$\mathfrak{B} \in \mathcal{L}\left(L^\infty(]0, L[\,; \mathbb{R})\right)$$

will appear which is defined through

$$\mathfrak{B}(n)(x) := \sum_{k=1}^{m} b_k \chi_{S_k}(x) \left(\fint_{S_k} n(t,y)\,dy - n(t,x) \right) \quad \text{for } a.a.\ x \in \,]0,L[\,.$$

Here $\mathcal{L}(L^\infty(]0,L[;\mathbb{R}))$ denotes the space of bounded linear mappings of $L^\infty(]0,L[;\mathbb{R})$ into itself. Finally, for sake of simple writing, we denote

$$\mathfrak{I}(t) := I(t,\cdot) \quad \text{for } a.a.\ t \in [0,T]\,.$$

The map $\mathfrak{I}$ is not Bochner measurable from $[0,T]$ into $L^\infty(]0,L[;\mathbb{R})$, in general. But, because of assumption (IV), it is Bochner integrable as a map from $[0,T]$ into $L^p\,(]0,L[\,,\mathbb{R})$ for any $p \in [1,\infty[$, and

$$\left\| \int_0^t \mathfrak{I}(s)\,ds \right\|_{L^p} \le \operatorname{ess} \sup_{\substack{0<t<T \\ 0<x<L}} I.$$

Hence, the map $(t,x) \in \,]0,T[\, \times \,]0,L[\, \mapsto \left(\int_0^t \mathfrak{I}(s)\,ds\right)(x) = \int_0^t I(s,x)\,ds$ is in $L^\infty\,(]0,T[\, \times \,]0,L[\,;\mathbb{R})$. This will be used in what follows.

For establishing the variation of constants formula for our notion of weak solution we first need some definitions:

For $\eta \in \mathbb{R}$ let

$$L^2_\eta(]0,\infty[,\mathbb{C}) := \left\{ f :]0,\infty[\to \mathbb{C} \mid f \text{ measurable } \int_0^\infty |f(x)|^2(1+x^2)^\eta dx < \infty \right\}$$

denote the Hilbert space of complex valued weighted square integrable functions on $]0,\infty[$ with weight $(1+x^2)^\eta$ with respect to the Lebesque measure on $]0,\infty[$. We denote its scalar product by $\langle f,g\rangle_{L^2_\eta} := \int_0^\infty f(x)\overline{g(x)}(1+x^2)^\eta dx$. Let $W^{1,2}_\eta$ denote the corresponding Sobolev space of functions $f \in L^2_\eta(]0,\infty[,\mathbb{C})$ with distributional derivative in $L^2_\eta(]0,\infty[,\mathbb{C})$. Define the extended phase space

$$X_e := L^2(]0,L[;\mathbb{C}^2) \times L^2(]0,L[;\mathbb{R}) \times L^2_\eta(]0,\infty[;\mathbb{C}) \tag{13}$$

with some fixed $\eta < -0.5$. This choice of η guarantees that $L^\infty(]0,\infty[;\mathbb{C})$ is continuously embedded in $L^2_\eta(]0,\infty[;\mathbb{C})$. Put

$$T_e(t)\left(\psi_1^0,\psi_2^0,n^0,a\right) := \left(\psi_1(t),\psi_2(t),n^0,\tau_t a\right),$$

where $\tau_t a(x) := a(t+x)$ denotes the left translation of a by t, and ψ_1,ψ_2 are given by

$$\begin{aligned} \psi_1(t,x) &:= \begin{cases} \psi_1^0(x-t) & , \text{ for } a.a.\ x \in \,]t,L[\\ r_0\psi_2^0(t-x) + a(t-x) & , \text{ for } a.a.\ x \in \,]0,t[\end{cases} \\ \psi_2(t,x) &:= \begin{cases} \psi_2^0(x+t) & , \text{ for } a.a.\ x \in \,]0,L-t[\\ r_L\psi_1^0(2L-x-t) & , \text{ for } a.a.\ x \in \,]L-t,L[\,. \end{cases} \end{aligned} \tag{14}$$

Extend $T_e(t), t \in [0,L]$ to the whole positive axis $[0,\infty[$ by defining for $t > L$ inductively $T_e(t) := T_e(t-L)T_e(L)$. Then it is easy to verify that $T_e(\cdot)$ is a C_0 semigroup of bounded operators in X_e with infinitesimal generator

$$A_e := \operatorname{diag}(-\partial_{\mathrm{x}},\partial_{\mathrm{x}},0,\partial_{\mathrm{x}})$$

having the domain

$$\mathfrak{D}(A_e) := \{(\psi, n, a) \in W^{1,2}(]0, L[; \mathbb{C}^2) \times L^2(]0, L[; \mathbb{R}) \times W^{1,2}_\eta(]0, \infty[; \mathbb{C}) \mid \psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(L) = r_L\psi_1(L)\}.$$

Set

$$T(t)(\psi^0) := \Pi_\psi T_e(t)(\psi^0, 0, 0)$$

for $t \geq 0$ and $\psi^0 \in L^2(]0, L[\,, \mathbb{C}^2)$, where Π_ψ is the projection onto the first variable ψ. Then $T(t)$ is a C_0 semigroup of contractions in $L^2(]0, L[\,, \mathbb{C}^2)$ with infinitesimal generator

$$A := \operatorname{diag}(-\partial_{\mathrm{x}}, \partial_{\mathrm{x}})$$

and domain

$$\mathfrak{D}(A) := \left\{\psi \in W^{1,2}(]0, L[\,; \mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0),\ \psi_2(L) = r_L\psi_1(L)\right\}.$$

Let $\prod_{(\psi,n)}$ denote the projection of X_e onto $L^2(]0, L[; \mathbb{C}^2 \times \mathbb{R})$ by dropping the trivial last component. Then the following Lemma holds

Lemma 3.1. *The pair (ψ, n) is a weak solution to* (2), (3), (4) *iff for all $t \in [0, T]$ the functions $\psi(t) := \psi(t, \cdot)$ and $n(t) := n(t, \cdot)$ satisfy*

$$\begin{pmatrix} \psi(t) \\ n(t) \end{pmatrix} = \prod\nolimits_{(\psi,n)} T_e(t) \begin{pmatrix} \psi^0 \\ n^0 \\ \alpha \end{pmatrix} + \int_0^t \begin{pmatrix} T(t-s)\mathfrak{G}(\psi(s), n(s)) \\ \mathfrak{I}(s) + \mathfrak{B}n(s) + \mathfrak{H}(\psi(s), n(s)) \end{pmatrix} ds. \tag{15}$$

Proof. Straightforward calculations yield that the adjoint A_e^* of A_e is the closed densely defined operator

$$A_e^*(\psi, n, a) = (\partial_x\psi_1, -\partial_x\psi_2, 0, -(1+x^2)^{-\eta}\partial_x(a(x)\cdot(1+x^2)^\eta)) =: (A^*\psi, 0, B^*a)$$

with the domain

$$\begin{aligned} \mathfrak{D}(A_e^*) &= \Big\{(\psi, n, a) \in W^{1,2}(]0, L[; \mathbb{C}^2) \times L^2(]0, L[; \mathbb{R}) \\ &\qquad \times W^{1,2}_\eta(]0, \infty[; \mathbb{C}) \mid \psi_2(0) = \overline{r_0}\psi_1(0), \psi_1(L) = \overline{r_L}\psi_2(L), a(0) = \psi_1(0)\Big\}. \end{aligned}$$

We trivially extend α on the whole axis $[0, \infty[$ by setting α to zero on $[T, \infty[$. Then define $a \in C([0, \infty[; L^2_\eta([0, \infty[; \mathbb{C}))$, $a(t) := \tau_t\alpha, t \in [0, \infty[$. By definition (ψ, n) is a weak solution iff $(\psi, n) \in L^\infty\big(]0, T[\times]0, L[; \mathbb{C}^2 \times \mathbb{R}\big)$ and for all $(\varphi, 0, \varphi_a) \in D(A_e^*)$ the equation

$$\begin{aligned} &\langle\psi(t) - \psi^0, \varphi\rangle_{L^2} + \langle a(t) - a(0), \varphi_a\rangle_{L^2_\eta} \\ = \ &\lim_{\rho\to 0}\left\{\int_0^t \Big(\langle\psi(s), A^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s), n(s)), \varphi\rangle_{L^2} + \alpha_\rho(s)\overline{\varphi_1(0)}\Big)\, ds \right. \\ &\left. + \int_0^t \langle(\partial_x\alpha_\rho)(s+\cdot), \varphi_a\rangle_{L^2_\eta} ds\right\} \\ = \ &\lim_{\rho\to 0}\left\{\int_0^t \Big(\langle\psi(s), A^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s), n(s)), \varphi\rangle_{L^2} + \langle\alpha_\rho(s+\cdot), B^*\varphi_a\rangle_{L^2_\eta}\Big) ds\right\} \\ = \ &\int_0^t \Big(\langle\psi(s), A^*\varphi\rangle_{L^2} + \langle\mathfrak{G}(\psi(s), n(s)), \varphi\rangle_{L^2} + \langle a(s), B^*\varphi_a\rangle_{L^2_\eta}\Big) ds \end{aligned}$$

holds and (6) is satisfied for n. Here

$$\alpha_\rho(x) := \int_0^T m_\rho(x-y)\alpha(y)dy, \quad m_\rho(y) := \frac{m_0(\rho y)}{\rho} \quad (x, y \in \mathbb{R})$$

denotes the mollification of α with parameter $\rho > 0$ with respect to some mollifier $m_0 \in C^\infty(\mathbb{R})$, $m_0 \geq 0$, $\operatorname{supp} m_0 \subset B_1$, $\int_{-\infty}^{\infty} m_0(y)dy = 1$. It was used above in order to perform partial integration. For the first equality one should note that for $\alpha \in L^2_\eta, \alpha_\rho \in W^{1,2}_\eta$ and $\lim_{x\to\infty} \alpha_\rho(x)\left(1+x^2\right)^\eta = 0$. The above calculations together with [1] proves: (ψ, n) is a weak solution iff (15) holds for $t \in [0, T]$. $\square$

Remark 3.2. *The map $s \mapsto T(t-s)f(s)$, where $f(s) := \mathfrak{G}(\psi(s), n(s))$ is not Bochner measurable from $\mathbb{R}$ into $L^\infty(]0, L[; \mathbb{C}^2)$, but it is integrable as a map into $L^p\left(]0, L[; \mathbb{C}^2\right)$ for any $p \in [1, \infty[$. Because the integral satisfies*

$$\left\| \int_0^t T(t-s)f(s)\,ds \right\|_{L^p} \leq c \int_0^t \|f(s)\|_{L^p}\, ds,$$

where the constant c does not depend on f and p, we get, using Lebesgue's convergence Theorem, that

$$\left\| \int_0^t T(t-s)f(s)\,ds \right\|_{L^\infty} \leq c \int_0^t \|f(s)\|_{L^\infty}\, ds.$$

We now define the truncated problem to (2)-(4):

Definition 3.3. Let $\delta \in]0, \infty[$ be arbitrary. Let $T_1^\delta : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $T_1^\delta(n) = n$ for $|n| \leq \delta^{-1}$ and $T_1^\delta(n) = 2\delta^{-1}|n|^{-1}n$ for $|n| \geq 2\delta^{-1}$. Similarly let $T_2^\delta : \mathbb{C}^2 \to \mathbb{C}^2$ be C^∞ with $T_2^\delta(v) = v$ for $\|v\| \leq \delta^{-1}$ and $T_2^\delta(v) = 2\delta^{-1}\|v\|^{-1} v$ for $\|v\| \geq 2\delta^{-1}$. Define the truncated nonlinearities

$$G^\delta :]0, L[\times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2, \quad G^\delta(x, \psi, n) := G(x, T_2^\delta(\psi), T_1^\delta(n)),$$
$$H^\delta :]0, L[\times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}, \quad H^\delta(x, \psi, n) := H(x, T_2^\delta(\psi), T_1^\delta(n)).$$

Then G^δ, H^δ are C^k-smooth Carathéodory functions generating the smooth superposition operators $\mathfrak{G}^\delta, \mathfrak{H}^\delta$. The truncated problem reads:

$$(16) \qquad \left\{ \begin{array}{lcl} \partial_t \psi^\delta(t,x) &=& \left(-\partial_x \psi_1^\delta(t,x), \partial_x \psi_2^\delta(t,x)\right) + G^\delta(x, \psi^\delta(t,x), n^\delta(t,x)) \\ \partial_t n^\delta(t,x) &=& I(t,x) + H^\delta(x, \psi^\delta(t,x), n^\delta(t,x)) \\ && + \sum_{k=1}^m b_k \chi_{S_k}(x) \left(⨍_{S_k} n^\delta(t,y)\,dy - n^\delta(t,x)\right) \end{array} \right.$$

with the same boundary conditions and initial values:

$$(17) \qquad \psi_1^\delta(t,0) = r_0 \psi_2^\delta(t,0) + \alpha(t), \ \psi_2^\delta(t,L) = r_L \psi_1^\delta(t,L)$$

$$(18) \qquad \psi^\delta(0,x) = \psi^0(x), \ n^\delta(0,x) = n^0(x).$$

Weak solutions to (16)-(18) are defined analogously to Def. 2.2.

Remark 3.4. *After truncation G^δ and H^δ satisfy condition ii) of Definition 2.12 globally. In particular G^δ and H^δ become globally Lipschitz uniformly with respect to $x \in]0, L[$, that is for each $\delta > 0$ there exists a constant Λ such that for all $\psi_1, \psi_2 \in \mathbb{C}^2$, $n_1, n_2 \in \mathbb{R}$ and a.a. $x \in]0, L[$*

$$\left\| G^\delta(x, \psi_1, n_1) - G^\delta(x, \psi_2, n_2) \right\| + |H^\delta(x, \psi_1, n_1) - H^\delta(x, \psi_2, n_2)|$$
$$\leq \Lambda \left(\|\psi_1 - \psi_2\| + |n_1 - n_2| \right).$$

The superposition operators $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ become globally Lipschitz from $L^p(]0, L[; \mathbb{C}^2 \times \mathbb{R})$ into $L^p(]0, L[; \mathbb{C}^2)$ and $L^p(]0, L[; \mathbb{R})$, respectively, for any $p \in [1, \infty]$.

Lemma 3.5. *For each $\delta > 0$ the Theorems 2.3 and 2.4 hold for the weak solution $\left(\psi^\delta, n^\delta\right)$ to the truncated problem* (16)-(18)

Proof. Denote the weak solution space

$$\mathfrak{X} := L^\infty\left(]0,T[\times]0,L[;\mathbb{C}^2\times\mathbb{R}\right).$$

Extend it to

$$\mathfrak{X}_e := \mathfrak{X}\times L^\infty(]0,L[;\mathbb{C}^2\times\mathbb{R})\times L^\infty(]0,T[;\mathbb{C})\times L^\infty(]0,T[\times]0,L[;\mathbb{R})$$

by attaching the corresponding spaces of the initial data ψ^0, n^0 and the dynamic data α, I. Both $\mathfrak{X}$ and $\mathfrak{X}_e$ are equipped with the corresponding L^∞ norms. Define the operator $\mathfrak{F}:\mathfrak{X}_e\to\mathfrak{X}$,

$$\mathfrak{F}\begin{pmatrix}\psi\\ n\\ \psi^0\\ n^0\\ \alpha\\ I\end{pmatrix}(t) := \begin{pmatrix}\psi(t)\\ n(t)\end{pmatrix} - \prod\nolimits_{(\psi,n)}\left\{T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}\right.$$

$$\left.+\int_0^t T_e(t-s)\begin{pmatrix}\mathfrak{G}^\delta(\psi(s),n(s))\\ \mathfrak{I}(s)+\mathfrak{B}n(s)+\mathfrak{H}^\delta(\psi(s),n(s))\\ 0\end{pmatrix}ds\right\}.$$

Note that the image of $\mathfrak{F}$ is measurable on the product space $]0,T[\times]0,L[$ and hence $\mathfrak{F}$ maps into the Banach space $\mathfrak{X}$. For fixed ψ^0,n^0,α,I denote $\mathfrak{F}_0:\mathfrak{X}\to\mathfrak{X}$,

$$\mathfrak{F}_0(\psi,n)(t) := (\psi(t),n(t)) - (\mathfrak{F}(\psi,n,\psi^0,n^0,\alpha,I))(t).$$

By Lemma 3.1 the truncated problem (16)-(18) has a unique weak solution (ψ^δ,n^δ) corresponding to the data ψ^0,n^0,α,I iff $\mathfrak{F}_0$ has a unique fixed point in $\mathfrak{X}$. By Remark 3.4 $\mathfrak{G}^\delta$ and $\mathfrak{H}^\delta$ are globally Lipschitz from $L^\infty\left(]0,L[;\mathbb{C}^2\times\mathbb{R}\right)$ into $L^\infty\left(]0,L[,\mathbb{C}^2\right)$ and $L^\infty(]0,L[,\mathbb{R})$, respectively, with some Lipschitz constant Λ depending on the truncation parameter δ. Thus from the explicit formula (14) for the semigroup $T_e(t)$ it follows by induction that for $l\in\mathbb{N}$, $(\psi_a,n_a),(\psi_b,n_b)\in\mathfrak{X}$

$$\left\|\mathfrak{F}_0^l(\psi_a,n_a)-\mathfrak{F}_0^l(\psi_b,n_b)\right\|_{\mathfrak{X}} \le \frac{(\Lambda T)^l}{l!}\left\|(\psi_a,n_a)-(\psi_b,n_b)\right\|_{\mathfrak{X}}.$$

Hence, for l sufficiently large $\mathfrak{F}_0^l$ is a contraction in the Banach space $\mathfrak{X}$. By a generalization of Banachs fixed point theorem $\mathfrak{F}_0$ has a unique fixed point (ψ^δ,n^δ) in $\mathfrak{X}$. This proves the existence and uniqueness part of Theorem 2.3.

From the assumptions that G,H are C^k Carathéodory functions (Definition 2.12) and Proposition 2.13 we get that $\mathfrak{F}$ maps $\mathfrak{X}_e$ C^k-smoothly into $\mathfrak{X}$ (this follows by Taylor expansion). The existence and uniqueness of the weak solutions just proved is equivalent to saying that for any ψ^0,n^0,α,I there exists a unique $(\psi,n)\in\mathfrak{X}$ such that $\mathfrak{F}(\psi,n,\psi^0,n^0,\alpha,I)=0$. The partial derivative of $\mathfrak{F}$ with respect to (ψ,n) operating on $v=(v_\psi,v_n)\in\mathfrak{X}$ satisfies the formula

$$\left(\frac{\partial\mathfrak{F}}{\partial(\psi,n)}(\psi,n,\psi^0,n^0,\alpha,I)\begin{pmatrix}v_\psi\\ v_n\end{pmatrix}\right)(t) = \begin{pmatrix}v_\psi(t)\\ v_n(t)\end{pmatrix}$$

$$-\prod_{(\psi,n)}\int_0^t T_e(t-s)\begin{pmatrix}\left(\partial\mathfrak{G}^\delta(\psi(s),n(s))\right)v(s)\\ \mathfrak{B}v_n(s)+\partial\mathfrak{H}^\delta(\psi(s),n(s))v(s)\\ 0\end{pmatrix}ds.$$

Again it follows by Banachs fixed point theorem that for any $w\in\mathfrak{X}$ there exists a unique $v\in\mathfrak{X}$ such that

$$v(t)=\prod_{(\psi,n)}\int_0^t T_e(t-s)\begin{pmatrix}\left(\partial\mathfrak{G}^\delta\left(\psi(s),n(s)\right)\right)v(s)\\ \mathfrak{B}v_n(s)+\partial\mathfrak{H}^\delta\left(\psi(s),n(s)\right)v(s)\\ 0\end{pmatrix}ds+w(t).$$

Banachs open mapping theorem implies that $\partial_{(\psi,n)}\mathfrak{F}\left(\psi,n,\psi^0,n^0,\alpha,I\right)$ is an isomorphism from $\mathfrak{X}$ onto $\mathfrak{X}$. Hence Theorem 2.3 is a consequence of the implicit function theorem.

Statement $\imath$) of Theorem 2.4 follows directly from Definition 2.2 and the variation of constants formula.

We now prove $\imath\imath$): As in the proof of Lemma 3.1 trivially extend α to the whole $[0,\infty[$ by setting α almost everywhere to zero on $[T,\infty[$ and define

$$a\in C\left(]0,\infty[\,;L^2_\eta(]0,\infty[\,;\mathbb{C})\right),\ a(s)(x):=\tau_s\alpha(x),$$

for $s\geq 0$ and $a.a.$ $x\in\,]0,\infty[$, where τ_s denotes the left translation of α again. Integrating the variation of constants formula (15) with respect to time yields

$$\begin{aligned}\int_0^t\begin{pmatrix}\psi(s)\\ n(s)\\ a(s)\end{pmatrix}ds &= \int_0^t T_e(s)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}ds\\ &+\int_0^t\int_0^s T_e(s-r)\begin{pmatrix}\mathfrak{G}^\delta(\psi(r),n(r))\\ \mathfrak{I}(r)+\mathfrak{B}n(r)+\mathfrak{H}^\delta(\psi(r),n(r))\\ 0\end{pmatrix}drds.\end{aligned}$$

From this formula and the uniform continuity $(t,p)\mapsto T_e(t)p$ of the C_0 semigroup T_e one easily proves that the limit

$$\lim_{h\downarrow 0}\frac{T_e(h)-I}{h}\int_0^t(\psi(s),n(s),a(s))ds$$

exists in X_e (see (13)) for each $t\in[0,T]$. This is equivalent to $\int_0^t(\psi(s),n(s),a(s))ds\in\mathfrak{D}(A_e)$ or statement $\imath\imath$).

Now assume $\alpha\in W^{1,2}(]0,T[\,;\mathbb{C})$, $\psi^0\in W^{1,2}(]0,L[;\mathbb{C}^2)$ and (7). Extend α to the whole $]0,\infty[$ such that the extension lies in $W^{1,2}_\eta(]0,\infty[\,;\mathbb{C})$. Then (ψ^0,n^0,α) belongs to $\mathfrak{D}(A_e)$. Since X_e is reflexive it follows from Proposition 4.3.9 in [7] that

$$(\psi,n,\tau_t\,\alpha)\in C([0,T];\mathfrak{D}(A_e))\cap C^1([0,T];X_e),$$

which proves $\imath\imath\imath$).

We prove Theorem 2.4, $\imath v$). Choose sequences $\psi_i^0\in W^{1,2}(]0,L[;\mathbb{C}^2)$, $\alpha_i\in W^{1,2}(]0,T[\,;\mathbb{C})$, $i\in\mathbb{N}$, which satisfy the boundary condition $\psi^0_{i\,1}(0)=r_0\psi^0_{i\,2}(0)+\alpha_i(0)$ and $\psi^0_{i\,2}(L)=r_L\psi^0_{i\,1}(L)$, and have the property that $\psi_i^0\to\psi^0$ in $L^\infty(]0,L[\,;\mathbb{C}^2)$ and $\alpha_i\to\alpha$ in $L^\infty(]0,T[\,;\mathbb{C})$. By Theorem 2.4 $\imath\imath\imath$) $\psi_i\in C([0,T]\times[0,L];\mathbb{C}^2)$, and by Theorem 2.3 the solution sequences (ψ_i,n) converge to (ψ,n) in $\mathfrak{X}$. Thus

$\psi \in C([0,T]\times[0,L];\mathbb{C}^2))$ and ψ satisfies (3) pointwise in $[0,T]$. By assumption (IX) on H the superposition operator $\mathfrak{H}^\delta$ keep the space C_P invariant. The ψ-part of the fixed point (ψ,n) of the operator $\mathfrak{F}_0$ is uniformly continuous on $[0,T]\times[0,L]$. Since $n^0 \in C_P$ and the part n can be obtained by a fixed point iteration in the space $C([0,T];C_P)$ alone, keeping ψ unchanged, we obtain that $n \in C([0,T];C_P)$. The relation $n \in C^1([0,T],C_P)$ follows directly from (6) if $I \in C([0,T];C_P)$. □

Remark 3.6. *(Lipschitz dependence of solutions with respect to L^2) Because of Remark 3.4 Gronwall's Lemma applied to* (15) *easily shows that there exists a constant $C = C(\delta,T)$ such that*

$$\left\|(\psi,n)-(\tilde{\psi},\tilde{n})\right\|_{C([0,T];L^2(]0,L[;\mathbb{C}^2\times\mathbb{R})} \le$$

$$C\left(\left\|(\psi^0,n^0)-(\tilde{\psi^0},\tilde{n}^0)\right\|_{L^2(]0,L[;\mathbb{C}^2\times\mathbb{R})}+\|\alpha-\tilde{\alpha}\|_{L^2(]0,T[;\mathbb{C})}\right)$$

where (ψ,n) and $(\tilde{\psi},\tilde{n})$ denote the weak solution with initial data $\left(\psi^0,n^0,\alpha\right)$ and $\left(\tilde{\psi}^0,\tilde{n}^0,\tilde{\alpha}\right)$, respectively.

4. A priori estimates

We will use the following elementary inequality:

Proposition 4.1. *Let $u:[0,b]\to\mathbb{R}$ be absolutely continuous and $u^*\in\mathbb{R}$. Suppose there are constants $r_1,r_2>0$ such that $u'(t)\le -r_1u(t)+r_2$ for a.a. $t\in[0,b]$ with $u(t)\ge u^*$. Then $u(t)\le \bar{u}+\max\{u(0)-\bar{u},0\}\,e^{-r_1t}$ for $t\in[0,b]$ with $\bar{u}:=\max\left\{\frac{r_2}{r_1},u^*\right\}$.*

Proof. Define $h:\mathbb{R}\to\mathbb{R}, h(x):=(\max\{x-\bar{u},0\})^2$. Set $f(t):=h(u(t))$. Then f is absolutely continuous and

$$f'(t)=h'(u(t))u'(t)\le -h'(u(t))r_1\left(u(t)-\frac{r_2}{r_1}\right)\le -2r_1f(t)$$

for *a.a.* $t\in[0,b]$. Therefore $f(t)\le e^{-2r_1t}f(0)$ for $t\in[0,b]$ and taking the square root yields the inequality. □

Lemma 4.2. *Let (ψ^δ,n^δ) be the weak solution to the truncated problem (16), (17), (18). There exists $\delta_0>0$ such that for all $0<\delta<\delta_0$ estimate* (8) *holds for $t\in[0,T]$ and the bounds* (9) *are satisfied for $t\in[0,T]$ and a.a. $x\in\,]0,L[$. Moreover, there exists a constant B not depending on $\delta>0$ such that*

$$\left\|\psi^\delta(t)\right\|_{L^\infty}\le B \quad \text{for all } t\in[0,T]. \tag{19}$$

Proof. Let $t_0\in[0,T]$ be arbitrary and assume first that $⨍_{S_k} n^\delta(t,y)\,dy\ge 0$ for all $t\in[0,t_0]$ and all $1\le k\le m$. Let $k\in\mathbb{N}$, $1\le k\le m$. Suppose $0<\delta\le\nu_1^{-1}$. Then for *a.a.* $x\in S_k$ assumptions (II), (IV), (VIII) imply that for *a.a.* $t\in[0,t_0]$ which satisfy $n^\delta(t,x)\le\nu_1$ the inequality

$$\frac{d}{dt}n^\delta(t,x)\ge(-c_1-b_k)\,n^\delta(t,x)$$

holds. Put

$$h(t,x):=\min\left\{n^\delta(t,x),\nu_1\right\}\quad\text{and}\quad \tau_k(n):=\begin{cases}1 & ,n\le\nu_1\\ 0 & ,n>\nu_1\end{cases}.$$

Then for $a.a.$ $x \in S_k$ and $a.a.$ $t \in [0, t_0]$

$$\begin{aligned}\frac{d}{dt} h(t,x) &= \tau_k\left(n^\delta(t,x)\right)\frac{d}{dt} n^\delta(t,x)\\ &\geq (-c_1 - b_k)\,\tau_k\left(n^\delta(t,x)\right) n^\delta(t,x)\\ &\geq (-c_1 - b_k)\, h(t,x).\end{aligned}$$

Therefore for $a.a.$ $x \in S_k$ and all $t \in [0, t_0]$

$$n^\delta(t,x) \geq h(t,x) \geq h(0,x)e^{-(c_1+b_k)t} = \min\left\{n^0(x), \nu_1\right\} e^{-(c_1+b_k)t} \quad (\geq 0). \tag{20}$$

Now we show that $⨍_{S_k} n^\delta(t,y)\,dy \geq 0$ for all $t \in [0,T]$ and all $1 \leq k \leq m$. Assume the contrary. Then there exists a $k \in \mathbb{N}$, $1 \leq k \leq m$, such that

$$t_0 := \sup\left\{t \in [0,T] \mid ⨍_{S_k} n^\delta(s,y)dy \geq 0 \text{ for } s \in [0,t]\right\} < T. \tag{21}$$

By (20) we have $n^\delta(t_0,x) \geq 0$ for $a.a.$ $x \in]0,L[$ and by (21) $\int_{S_k} n^\delta(t_0,y)\,dy = 0$. Therefore $n^\delta(t_0,x) = 0$ for $a.a$ $x \in S_k$. Hence, by continuity, there exists $0 < \epsilon < T - t_0$ such that for all $t \in [t_0, t_0+\epsilon[$ and $a.a.$ $x \in S_k$ we have $n^\delta(t,x) \leq \nu_1$. Thus from the assumptions (II) and (IV), definition of H^δ and due to the choice $\delta \leq \nu_1^{-1}$ we have for $a.a$ $t \in [t_0, t_0+\epsilon[$

$$\frac{d}{dt}⨍_{S_k} n^\delta(t,y)dy = ⨍_{S_k}\left(I(t,y) + H(y, \psi^\delta(t,y), n^\delta(t,y))\right)dy \geq -c_1 ⨍_{S_k} n^\delta(t,y)dy.$$

This yields $⨍_{S_k} n^\delta(t,y)dy \geq ⨍_{S_k} n^\delta(t_0,y)\,dy \cdot e^{-c_1(t-t_0)} = 0$ for $t \in [t_0, t_0+\epsilon[$ which contradicts the choice of t_0 from which there exist infinitely many points $s \in]t_0, t_0+\epsilon[$ with $⨍_{S_k} n^\delta(s,y)dy < 0$ accumulating in t_0. This proves (20) for all $t \in [0,T]$ and the lower bound for n^δ in (9).

Now define

$$T_\delta := \sup\left\{t \in [0,T] \mid \left\|\psi^\delta(s)\right\|_{L^\infty} \leq \delta^{-1} \text{ and } \left\|n^\delta(s)\right\|_{L^\infty} \leq \delta^{-1} \text{ for } s \in [0,t]\right\}.$$

Suppose $\delta > 0$ is sufficiently small such that $T_\delta > 0$. Assume $\alpha \in W^{1,2}(]0,T[;\mathbb{C})$ and $\psi_0 \in W^{1,2}(]0,L[;\mathbb{C}^2)$ together with (7). Denote

$$h(t) := \int_0^L n^\delta(t,x)\,dx + \frac{d_1}{2}\int_0^L \left\|\psi^\delta(t,x)\right\|^2\,dx.$$

From (I), (II), (VI) and Theorem 2.4 *iii*), proved for the truncated problem in Lemma (3.5), it follows by partial integration that for $a.a\ t \in [0, T_\delta]$

$$\begin{aligned}
\frac{d}{dt}h(t) &= d_1\,\mathfrak{Re}\int_0^L \Big[-\partial_x\psi_1^\delta(t,x)\overline{\psi_1^\delta(t,x)} + \partial_x\psi_2^\delta(t,x)\overline{\psi_2^\delta(t,x)}\Big]\,dx \\
&\quad + \int_0^L \Big[I(t,x) + H(x,\psi^\delta(t,x),n^\delta(t,x)) \\
&\quad + d_1\mathfrak{Re}\left\langle G(x,\psi^\delta(t,x),n^\delta(t,x)),\psi^\delta(t,x)\right\rangle\Big]dx \\
&\le \frac{d_1}{2}\left(-\left|\psi_1^\delta(t,L)\right|^2 + \left|\psi_1^\delta(t,0)\right|^2 + \left|\psi_2^\delta(t,L)\right|^2 - \left|\psi_2^\delta(t,0)\right|^2\right) \\
&\quad + \int_0^L I(t,x)\,dx - d_2\left(\int_0^L n^\delta(t,x)\,dx + \int_0^L \left\|\psi^\delta(t,x)\right\|^2\,dx\right) \\
&\le \frac{d_1}{2}\Big(\left(|r_0|^2-1\right)|\psi_2^\delta(t,0)|^2 + |\alpha(t)|^2 + 2|r_0||\psi_2^\delta(t,0)||\alpha(t)| \\
&\quad + \left(|r_L|^2-1\right)|\psi_1^\delta(t,L)|^2\Big) + L\left\|I\right\|_{L^\infty} - c\cdot h(t) \\
&\le L\left\|I\right\|_{L^\infty} + \frac{d_1}{2}\left\|\alpha\right\|_{L^\infty}^2 + d_1\max_{\rho\in\mathbb{R}}\left(\tfrac{|r_0|^2-1}{2}\rho^2 + |r_0|\left\|\alpha\right\|_{L^\infty}\rho\right) - c\cdot h(t) \\
&= \frac{d_1}{2(1-|r_0|^2)}\left\|\alpha\right\|_{L^\infty}^2 + L\left\|I\right\|_{L^\infty} - c\cdot h(t).
\end{aligned}$$

Therefore the δ-independend estimate (8) for $\left(\psi^\delta, n^\delta\right)$ and $t\in[0,T_\delta]$ follows from Proposition 4.1. Because of Remark 3.6 this remains valid by density if $\alpha \in L^\infty(]0,T[\,;\mathbb{C})\setminus W^{1,2}(]0,T[\,;\mathbb{C})$ or $\psi_0 \in L^\infty(]0,L[\,;\mathbb{C}^2)\setminus W^{1,2}(]0,L[\,;\mathbb{C}^2)$. By Definition 2.2 $n^\delta(\cdot,x)$ is absolutely continuous on $[0,T]$ for $a.a\ x\in]0,L[$. From assumption (II) it follows that for $a.a\ t\in[0,T_\delta]$ with $n^\delta(t,x)\ge\nu_2$ the inequality

$$\frac{d}{dt}n^\delta(t,x) \le \left\|I\right\|_{L^\infty} + \max_{1\le k\le m}\left(\tfrac{b_k}{|S_k|}\right)\cdot\max\left\{\mu, \int_0^L n^0(x)dx + \tfrac{d_1}{2}\left\|\psi^0\right\|_{L^2}^2\right\} - c_2 n^\delta(t,x)$$

holds. Proposition 4.1 yields the δ-independend upper bound for n^δ and $t\in[0,T_\delta]$ in (9).

From the explicit formula (14) we have the following decay rates for the semigroups T and T_e: For $t\ge 0$

$$\left\|\Pi_\psi T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}\right\|_{L^\infty} \le D_0 e^{-\gamma t}\left\|\psi^0\right\|_{L^\infty} + 2\left(1-|r_0 r_l|\right)^{-1}\left\|\alpha\right\|_{L^\infty}, \tag{22}$$

where $D_0 := \begin{cases}|r_0 r_L|^{-1} & ,r_0 r_L \ne 0\\ e & ,r_0 r_L = 0\end{cases}$ and $\gamma := \begin{cases}-(2L)^{-1}\log|r_0 r_L| & ,r_0 r_L\ne 0\\ (2L)^{-1} & ,r_0 r_L = 0\end{cases}$.

Let M_0 be a constant in assumption (III) for $K = \left[0, N + \left\|n^0\right\|_{L^\infty}\right]$. From (22), (15), (9) and (III) we get for $t\in[0,T_\delta]$

$$\begin{aligned}
\left\|\psi^\delta(t)\right\|_{L^\infty} &\le \left\|\Pi_\psi T_e(t)\begin{pmatrix}\psi^0\\ n^0\\ \alpha\end{pmatrix}\right\|_{L^\infty} + \int_0^t \left\|T(t-s)\mathfrak{G}^\delta(\psi^\delta(s),n^\delta(s))\right\|_{L^\infty}ds \\
&\le D_0 e^{-\gamma t}\left\|\psi^0\right\|_{L^\infty} + \frac{2\left\|\alpha\right\|_{L^\infty}}{1-|r_0 r_l|} + M_0 T + \int_0^t M_0\left\|\psi^\delta(s)\right\|_{L^\infty}ds.
\end{aligned}$$

Gronwall's Lemma yields the existence of a constant B independent on $\delta > 0$ such that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta]$.
Moreover, since assumption (III) is valid also for the truncated nonlinearity G^δ and n^δ is continuous from $[0,T]$ to L^∞ by choosing a possibly larger M_0 corresponding to a larger set K than above we can find a constant B independent of $\delta > 0$ such that for each $\delta > 0$ there exists a neighborhood U_δ of T_δ so that $\left\|\psi^\delta(t)\right\|_{L^\infty} \leq B$ for $t \in [0, T_\delta] \cup U_\delta$. This proves that $T_\delta = T$ if δ is chosen sufficiently small. $\square$

We have shown that for sufficiently small $\delta > 0$ the weak solutions of the truncated problem coincide with the original weak solutions of the nontruncated problem. Hence the proof of Theorems 2.3-2.5 is complete. We are left with the proofs of Theorem 2.6 and Corollary 2.10.

Proof. (Corollary 2.10) From the assumption that α and g are of class C^k it follows that the map

$$\begin{pmatrix} \psi^0 \\ n^0 \\ \theta \end{pmatrix} \in \mathfrak{P}_e \mapsto \begin{pmatrix} \psi^0 + g(\theta, n^0) \\ n^0 \\ \alpha(\theta + \cdot) \end{pmatrix} \in L^\infty(]0,l[\,, \mathbb{C}^2 \times \mathbb{R}) \times L^\infty(]0,T[\,, \mathbb{C})$$

is C^k. Hence Theorems 2.3 and 2.4 imply that

$$\begin{pmatrix} \psi^0 \\ n^0 \\ \theta \end{pmatrix} \in \mathfrak{P}_e \mapsto X(\theta + t, \theta, \begin{pmatrix} \psi^0 + g(\theta, n^0) \\ n^0 \end{pmatrix}) \in C([0,l], \mathbb{C}^2) \times C_P$$

is C^k. This shows that for $t \geq 0$ the map $(p, \theta) \in \mathfrak{P}_e \mapsto Y(\theta + t, \theta, p) \in \mathfrak{P}$ is of class C^k. Hence Z^t is a C^k smooth semiflow on $\mathfrak{P}_e$. $\square$

Proof. (Theorem 2.6) Let (ψ, n) be the weak solution. From the differentiability assumption (X) on H the map $w \mapsto \mathfrak{H}(\psi(s), w)$ is well defined from $W_P^{1,2}$ into itself for $s \in [0,T]$ since $\psi \in C([0,T], W^{1,2})$. Furthermore condition (XI) implies that this map is Lipschitz on bounded subsets of $W_P^{1,2}$ uniformly in $s \in [0,T]$. By truncation we can make it globally Lipschitz: for $\eta > 0$ let $T_\eta : W_P^{1,2} \to W_P^{1,2}$ be globally Lipschitz with $T_\eta(w) = w$, if $\|w\|_{W_P^{1,2}} \leq \eta^{-1}$, $T_\eta(w) = 2\eta^{-1} w \|w\|_{W_P^{1,2}}^{-1}$, if $\|w\|_{W_P^{1,2}} \geq 2\eta^{-1}$. Define the following truncated operators

$$\mathfrak{H}_\eta(p, w) := \mathfrak{H}(p, T_\eta(w)) \text{ for } p \in W^{1,2} \text{ and } w \in W_P^{1,2}.$$

Then for all $p \in W^{1,2}$ the map $w \mapsto \mathfrak{H}_\eta(p, w)$ is globally Lipschitz in $W_P^{1,2}$ where the Lipschitz constant depends only on η and $\|p\|_{W^{1,2}}$.

Define $\mathfrak{F} : C([0,T], W_P^{1,2}) \to C([0,T], W_P^{1,2})$,

$$(\mathfrak{F}m)(t) := n^0 + \int_0^t (\mathfrak{I}(s) + \mathfrak{B}m(s) + \mathfrak{H}_\eta(\psi(s), m(s)))\, ds \quad (t \in [0,T]).$$

Then $\mathfrak{F}$ has a unique fixed point n_η in $C([0,T], W_P^{1,2})$ by a generalization of Banachs fixed point theorem since sufficient high iterates of $\mathfrak{F}$ become contractive. In particular $n_\eta \in C^1([0,T], W_P^{1,2})$.

Set $T_\eta := \sup\Big\{t \in [0,T] \mid \|n_\eta(s)\|_{W_P^{1,2}} \leq \eta^{-1} \text{ for } 0 \leq s \leq t\Big\}$. By (XII) and the Hölder-Young inequalities we have for all $t \in [0, T_\eta]$

$$
\begin{aligned}
& \partial_t \frac{1}{2} \|\partial_x n_\eta(t)\|^2_{L^2(S_k)} \\
= \;& \int_{S_k} \partial_x \left(I(t,x) - b_k n_\eta(t,x) + H(x, \psi(t,x), n_\eta(t,x)) \right) \partial_x n_\eta(t,x)\, dx \\
\leq \;& \int_{S_k} |\partial_x I(t,x) \partial_x n_\eta(t,x)|\, dx + \int_{S_k} \Big(\partial_x H(x, \psi(t,x), n_\eta(t,x)) \partial_x n_\eta(t,x) \\
& + \partial_\psi H(x, \psi(t,x), n_\eta(t,x)) \partial_x \psi(t,x) \partial_x n_\eta(t,x) \\
& + \partial_n H(x, \psi(t,x), n_\eta(t,x)) \left(\partial_x n_\eta(t,x) \right)^2 \Big)\, dx \\
\leq \;& \frac{3}{2\tau} \|\partial_x I(t)\|^2_{L^2(S_k)} - \tau \frac{5}{6} \|\partial_x n_\eta(t)\|^2_{L^2(S_k)} + R_0 \Big(\|1\|_{L^1(S_k)} + \|\partial_x n_\eta(t)\|_{L^1(S_k)} \\
& + \|\partial_x n_\eta(t)\|_{L^2(S_k)} \|\partial_x \psi(t)\|_{L^2(S_k)} + \|\partial_x \psi(t)\|^2_{L^2(S_k)} \Big) \\
\leq \;& \frac{3}{2\tau} \sup_{t \in [0,T]} \|\partial_x I(t)\|^2_{L^2} + R_0 L + \frac{3}{2\tau} R_0^2 L + \left(\frac{3R_0^2}{2\tau} + 1 \right) \|\partial_x \psi(t)\|^2_{L^2(S_k)} \\
& - \tau \tfrac{1}{2} \|\partial_x n_\eta(t)\|^2_{L^2}.
\end{aligned}
$$

Hence (see Prop. 4.1) we get the following η independent bound

$$
\begin{aligned}
\|\partial_x n_\eta(t)\|^2_{L^2(S_k)} \quad \leq \quad & \frac{3}{2\tau^2} \sup_{t \in [0,T]} \|\partial_x I(t)\|^2_{L^2(S_k)} + \frac{R_0 L}{\tau} + \frac{3R_0^2 L}{2\tau^2} \\
& + \left(\frac{3R_0^2}{2\tau^2} + \frac{1}{\tau} \right) \sup_{s \in [0, T_\eta]} \|\partial_x \psi(s)\|^2_{L^2}
\end{aligned}
$$

which is valid for $t \in [0, T_\eta]$.

Since the a priori estimates of Theorem 2.5 must hold for n_η as long as $t \in [0, T_\eta]$ we see that $T_\eta = T$ and $n_\eta = n$ if η is chosen sufficiently small. □

5. Example

The system of equations (2)-(4) is a general form of the traveling wave model used to simulate temporal-longitudinal behaviour of slowly varying complex amplitudes of counterpropagating optical fields and carriers in multisection semiconductor lasers [5, 6, 22, 26]. Different dynamical behaviour of properly designed lasers can be effectively used in different technological applications. Examples are wavelength tuning, chirp reduction, enhanced modulation bandwidths, mode-locking of short pulses, and frequency-tunable self-pulsations for high-speed data transmission in optical communication systems (see, e.g., technology references in [5, 6]).

In the nonnormalized form the model equations can be written as follows:

$$
(23) \quad \left\{
\begin{array}{lcl}
\partial_t \psi(t,x) & = & v_{gr} \left[(-\partial_x \psi_1, \partial_x \psi_2) + (\beta \psi_1 + i\kappa(x) \psi_2, i\kappa(x) \psi_1 + \beta \psi_2) \right] \\
\beta(x, \psi, n) & := & -i \left(\delta(x) + \beta_{th}(x) I(x) \right) - \frac{\alpha_0(x)}{2} + \frac{(1 - i\alpha_H) \tilde{g}(x,n)}{2(1 + \epsilon_G(x) \|\psi\|^2)} \\
\partial_t n(t,x) & = & \frac{I(x) + I_M(t,x)}{e \sum_{k=1}^m \chi_{S_k}(x) |S_k|} + H(x, \psi, n) \\
& & + \sum_{k=1}^m \frac{\chi_{S_k}(x)}{e |S_k| R_k} \left(⨍_{S_k} n(t,y)\, dy - n(t,x) \right) \\
H(x, \psi, n) & := & - \left[A(x) n + B(x) n^2 + C(x) n^3 \right] - \frac{v_{gr} \tilde{g}(x,n) \|\psi\|^2}{1 + \epsilon_G(x) \|\psi\|^2}.
\end{array}
\right.
$$

Moreover, field function ψ satisfies boundary conditions (3) and (ψ, n) satisfy the initial value condition (4).

The group velocity v_{gr} is assumed to be positive and constant within all laser. It can be easily eliminated from the above equations by simple scaling of time or space. Due to this elimination the equations (23) are a particular case of the eqs. (2).

Functions $\|\psi(t,x)\|^2 = \langle\psi,\psi\rangle$ and $n(t,x)$ are local photon and carrier densities. When multiplying $|\psi_j(t,x)|^2$ by factor $v_{gr}\frac{\hbar c_0}{\lambda_0}$ and by local crossection area of active zone one gets the local power of the forward $(j=1)$ or backward $(j=2)$ propagating field. The function $\tilde{g}(x,n)$ denotes the gain function. It is increasing in n, that is

$$\text{for } a.a.\ x \in\,]0,L[\quad \text{and all} \quad n \in \mathbb{R} \quad \partial_n \tilde{g}(x,n) \geq 0.$$

In the following simulations we assume a frequently used linear in n approximation of gain function: $\tilde{g}(x,n) \simeq g^d(x)\,(n - n_{tr})$.

In the equations above the used physical constants e, c_0 and $\hbar$ denote electron charge, speed of light in vacuum and Planck's constant, respectively. The remaining parameters specifying the considered laser are described below in Table 1.

After elimination of v_{gr} and taking into account the dependence of the used functions and parameters to the functional spaces indicated in Table 1 one can easily check the validity of all assumptions (I)-(XII) taken in Section 2. The operators β and H are physically meaningless for $n < 0$. In particular assumption (II) may not be satisfied for $n < 0$. However, since our a priori estimates guarantee that n will always stay positive, we are free to extend the definitions of the operators G (or β) and H so that for all $n \in \mathbb{R}$ our required assumptions are satisfied. In our example condition (II) holds with $d_1 = 2$ and $d_2 = \operatorname{ess\,inf}_{\mathrm{x}\in]0,\mathrm{L}[} \min\{\alpha_0(\mathrm{x}), \mathrm{A}(\mathrm{x})\} > 0$, since the internal absorption and inverse linear carrier life time are both positive. Assumptions (XI) and (XII) are satisfied due to nonlinear gain compression $\mathrm{essinf}_{\mathrm{x}\in]0,\mathrm{L}[}\epsilon_{\mathrm{G}}(\mathrm{x}) > 0$. Hence, the results described in previous sections fits our system (23), (3), (4) originated from the real world applications.

In the four right columns of Table 1 we specify typical parameters of a 3 section (i.e. $m = 3$) distributed feedback (DFB) laser schematically depicted in Fig. 1 and considered in more details in, e.g., [6]. The symbol

$$C_P^1 \quad := \quad \left\{ n \in L^\infty(]0,L[;\mathbb{R}) \mid \forall k \ n_{|S_k} \in C^1\left(\overline{S_k}, \mathbb{R}\right) \right\}$$

appearing in the first column denotes the space of on each laser section S_k, $1 \leq k \leq m$, C^1 functions.

The first section (DFB_1) of this laser contains Bragg grating which couples counterpropagating fields (nonzero $\kappa(x)_{|S_1}$), is active (sufficiently large positive $I(x)_{|S_1}$ and strictly positive $g^d(x)_{|S_1}$), generating optical field output from this section and from the full laser. The third section (DFB_2) is similar, but here the applied current is low. This section operates mainly as a wavelength dependent reflector. The middle section is passive ($g^d(x)_{|S_2} = 0$), has no coupling ($\kappa(x)_{|S_2} = 0$) and provides an additional possibility of control in experiments via the thermal detuning term $\beta_{th}(x)I(x)_{|S_2}$. We have performed a test simulation of the laser described above using the software LDSL-tool (abbreviation for "longitudinal dynamics in semiconductor lasers"). Fig. 2 shows an induced transition of the simulated solution from the quasi-stationary state to quasi-periodic state after a kick of the current injection in the middle section. Using the language of equivariant dynamics, these states are rotating waves (relative equilibria) or modulated waves (relative periodic orbits). In Fig. 3 we draw spatial-temporal distributions of some functions in already established quasi-periodic state. These figures indicate continuity of optical fields and discontinuity of the carrier densities at the interfaces between the sections. We also point out a strong nonuniformity (spatial hole burning) of the carrier densities within the first section, while in other sections its variation is less pronounced.

These observed quasi-periodic, or "pulsating" states with ~ 7 GHz repetition frequency are of particular interest for high speed data transmission in optical communication systems.

A rich dynamics of the considered system is represented in Fig. 4. Here we show orbit changes by varying current injection in the middle section S_2. Our simulations were made as follows. After fixing the parameter we have performed a numerical integration of the model over some transient time interval, during which the trajectory could approach the attractor. Extremas of the outgoing field intensity computed in the consequent time interval are depicted in Fig. 4. Next, we change the parameter I with a small step and repeat the procedure described above.

In the considered parameter interval we distinguish a few different dynamical regimes. The rotating and periodically modulated waves already discussed above are found in regions A and D, respectively. In region C we have quasi-periodically modulated waves. Besides the rotational (optical) frequency, here the optical fields posses two other characteristic frequencies. In region B more complex chaotic behaviour can be found.

All these dynamical regimes are typical for different multisection semiconductor lasers. These states with the same sequence of bifurcations were observed experimentally and theoretically in [5, 20, 26]. The identification and numerical continuation of bifurcations of such states for a slightly simpler version of our model (23) was performed in [5, 20, 22].

6. Center manifold for the traveling wave model with nonautonomous optical injection

In this section we show existence of smooth center manifolds for the traveling wave model (23). First we write (23) in dimensionless variables using the following scaling

$$\begin{aligned} x &\mapsto \frac{x}{|S_1|} =: \tilde{x}, \qquad t \mapsto \tfrac{v_{gr}}{|S_1|} t =: \tilde{t}, \\ n &\mapsto \frac{n}{n_{tr,1}} =: \tilde{n}, \quad \psi \mapsto (n_{tr,1}\epsilon)^{-\frac{1}{2}}\, \psi =: \tilde{\psi}, \end{aligned} \tag{24}$$

where $n_{tr,1}$ is the transparent density in section S_1 and $\epsilon > 0$ is an arbitrary scaling parameter. The model equations then become (we omit tildas for better readability) of the form (10). If we choose $\epsilon = 10^{-2}$ then all remaining parameters entering (10) are of order one so that we have a slow fast structure. We assume ϵ is a variable parameter and all remaining parameters are constant. For sufficiently small $\epsilon > 0$ we show that there exists a smooth exponentially attracting local center manifold for (10) under a common spectral gap assumption.

Assume $\alpha \in C^k\left([0,\infty[\,;\mathbb{C}\right)$. Let

$$g \in C^k\left([0,\infty[\, \times L^\infty\left(]0,l[\,,\mathbb{R}\right); C\left([0,l]\,;\mathbb{C}^2\right)\right)$$

solve for all $n \in L^\infty\left(]0,l[\,,\mathbb{R}\right)$

$$\begin{cases} \partial_t g(t,n) &=& \mathfrak{A}(n)g(t,n), \\ g_1(t,n)_{|x=0} &=& r_0 g_2(t,n)_{|x=0} + \alpha(t), \\ g_2(t,n)_{|x=l} &=& r_l g_1(t,n)_{|x=l}, \end{cases}$$

where $\mathfrak{A}(n)$ was defined in (11). Let Z^t be the smooth skew product semiflow on $\mathfrak{P}_e$ defined in section 2. Then Z^t is generated by the equations

$$\begin{cases} \partial_t\psi(t) &=& \mathfrak{A}(n(t)) + \epsilon\Big[\mathfrak{K}\left(n(t),\psi(t)+g(s,n(t))\right) \\ & & -\partial_n g(s,n(t))\mathfrak{F}(s,n(t),\psi(t))\Big] \\ \partial_t n(t) &=& \epsilon\,\mathfrak{F}\left(s,n(t),\psi(t)+g(s,n(t))\right) \\ \partial_t s &=& 1, \end{cases} \tag{25}$$

with boundary condition

$$\begin{cases} \psi(0) &=& \psi^0 - g(0,n^0), \\ n(0) &=& n^0, \\ s(0) &=& \theta, \\ \psi_1(t,0) &=& r_0\psi_2(t,0), \\ \psi_2(t,l) &=& r_l\psi_1(t,l). \end{cases}$$

Note that the boundary conditions are homogeneous and time independent now, the nonautonomous time dependence now appear through the variable s in the terms $g(s,n(t))$ and $\partial_n g(s,n(t))$ in both equations for ψ and n. Next we note spectral properties of the infinitesimal generator $\mathfrak{A}(n)$ (see [12]):

Lemma 6.1. *The spectrum of $\mathfrak{A}(n)$ only consists of eigenvalues with finite algebraic multiplicity. There exist $\eta_0 > 0$ and $\Gamma, \Delta \in \mathbb{R}$, depending on the reflection coefficients r_0, r_L and the propagation constant $\beta(n)$, so that for $0 < \eta < \eta_0$ all but finitely many eigenvalues of $\mathfrak{A}(n)$ lie in $\cup_{z\in\mathbb{Z}}\{\lambda \in \mathbb{C} \mid |\lambda - (\Gamma + i\Delta z)| < \eta\}$ and have algebraic multiplicity one.*

Under physical realistic parameters the laser model is dissipative so that $\Gamma < 0$. Moreover, $\mathfrak{A}(n)$ only possesses few critical eigenvalues which are close to the imaginary axis (usually one to four). Hence there exist $\Gamma < \Gamma_* < 0$ and $\delta > 0$ so that

$$\textbf{(A)} \qquad \{\lambda \in \mathbb{C} \mid -\delta + \Gamma_* < \mathfrak{Re}\,\lambda < \Gamma_* + \delta\} \subset \rho(\mathfrak{A}(n)).$$

Since the asymptotics of the eigenvalues can be controlled in terms of the coefficients entering $\mathfrak{A}(n)$ the spectral gap property (A) holds for sufficient large open subsets

$\mathcal{U} \subset C_P$. For $n \in \mathcal{U}$ let $\mathfrak{P}(n)$ be the spectral projection for the critical eigenvalues and $\mathfrak{Q} := I - \mathfrak{P}$. We perform a change of coordinates,

$$\psi = \mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s, \tag{26}$$

where $\mathfrak{B}$ and $\mathfrak{C}$ are smooth bases,

$$\mathfrak{B} : \mathcal{U} \to \mathcal{L}(\mathbb{C}^q, L^2) \quad \text{and} \quad \mathfrak{C} : \mathcal{U} \to \mathcal{L}(\mathcal{Y}, L^2).$$

The image of $\mathfrak{B}(n)$ is equal to the image of $\mathfrak{P}(n)$, q denotes the sum of the algebraic multiplicities of the critical eigenvalues, $\mathcal{Y} \subset L^2$ is a codimension q subspace of $L^2 := L^2([0,l],\mathbb{C}^2)$, the image of $\mathfrak{C}$ equals the kernel (image) of $\mathfrak{P}(n)$ ($\mathfrak{Q}(n)$), $\mathcal{Y}$ is a Banach coordinate system which can be chosen as the kernel of $\mathfrak{P}(n_0)$ for some fixed $n_0 \in \mathcal{U}$. Let $\mathcal{W} := \{\psi = (\psi_1,\psi_2) \in C([0,l],\mathbb{C}^2) \mid \psi_1(0) = r_0\psi_2(0), \psi_2(l) = r_l\psi_1(l)\}$, $\mathcal{Y}_{\mathcal{W}} := \mathcal{Y} \cap \mathcal{W}$ (equipped with the norm of $\mathcal{W}$) and $\mathfrak{C}_{\mathcal{W}}$ denote the restriction of $\mathfrak{C}$ to $\mathcal{Y}_{\mathcal{W}}$, $\mathfrak{C}_{\mathcal{W}}(n)(y) := \mathfrak{C}(n)(y)$ for $y \in \mathcal{Y}_{\mathcal{W}}$. We have $\mathfrak{C}_{\mathcal{W}} \in C^k\left(\mathcal{U}, \mathcal{L}\left(\mathcal{Y}_{\mathcal{W}}, \mathcal{W}\right)\right)$. We are interested in solutions with $n(t) \in \mathcal{U}$ for $0 \le t < \infty$, such as periodic or quasiperiodic solutions. Using (26) we arrive at the following set of equations

$$\left\{\begin{array}{rcl}
\partial_t x_c &=& \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c(s,n,x_c,x_s) \\
\partial_t x_s &=& \mathfrak{A}_s(n)x_s + \epsilon\mathfrak{G}_s(s,n,x_c,x_s) \\
\partial_t n &=& \epsilon\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + g(s,n)) \\
\partial_t s &=& 1 \\
x_c(0) &=& \mathfrak{B}(n^0)^{-1}\mathfrak{P}(n^0)\left(\psi^0 - g(\theta,n^0)\right), \\
x_s(0) &=& \mathfrak{C}(n^0)^{-1}\mathfrak{Q}(n^0)\left(\psi^0 - g(\theta,n^0)\right), \\
n(0) &=& n^0, \\
s(0) &=& \theta,
\end{array}\right.$$

where

$$\begin{array}{rcl}
\mathfrak{A}_c(n) &:=& (\mathfrak{B}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{B}(n), \\
\mathfrak{A}_s(n) &:=& (\mathfrak{C}(n))^{-1}\,\mathfrak{A}(n)\mathfrak{C}(n),
\end{array}$$

$$\begin{array}{rcl}
\mathfrak{G}_c(s,n,x_c,x_s) &:=& (\mathfrak{B}(n))^{-1}\,\mathfrak{P}(n)\mathfrak{G}(s,n,x_c,x_s), \\
\mathfrak{G}_s(s,n,x_c,x_s) &:=& (\mathfrak{C}(n))^{-1}\,\mathfrak{Q}(n)\mathfrak{G}(s,n,x_c,x_s),
\end{array}$$

and

$$\begin{array}{rcl}
\mathfrak{G}(s,n,x_c,x_s) &:=& \mathfrak{K}(n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + g(s,n)) \\
&& -\left(\partial\mathfrak{B}(n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + g(s,n))\right)x_c \\
&& -\left(\partial\mathfrak{C}(n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + g(s,n))\right)x_s \\
&& -\partial_n g(s,n)\mathfrak{F}(s,n,\mathfrak{B}(n)x_c + \mathfrak{C}(n)x_s + g(s,n)).
\end{array}$$

From Corollary 2.10 Z^t is a smooth semiflow in the Banach space $\mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P \times \mathbb{R}$. Our spectral mapping results [12, 13, 14] together with (A) yield that for $\epsilon = 0$ Z^t has the normally hyperbolic invariant manifold $\mathbb{C}^q \times \{0\} \times \mathcal{U} \times \mathbb{R} \subset \mathbb{C}^q \times \mathcal{Y}_{\mathcal{W}} \times C_P \times \mathbb{R}$. Using a cut off modification we can construct overflowing manifolds

$$\mathrm{IM}_0^{\mathrm{r}} := \{\psi_{\mathrm{c}} \in \mathbb{C}^{\mathrm{q}} \mid |\psi_{\mathrm{c}}| < \mathrm{r}\} \times \{0\} \times \mathcal{U} \times \{\mathrm{s} \in \mathbb{R} \mid |\mathrm{s}| < \mathrm{r}\}$$

for any given $r > 0$, so that the modified equation coincides with the original one within a radius of $\frac{r}{2}$. By applying persistence theory for semiflows in Banach spaces [4] we get

Theorem 6.2. *For any $r > 0$ there exists an $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$ the manifold $\mathrm{IM}_0^{\mathrm{r}}$ persists as a nonlinear exponentially attracting smooth invariant manifold $\mathrm{IM}_\epsilon^{\mathrm{r}}$, which can be represented as a C^k smooth graph $x_s = \gamma(x_c, n, s, \epsilon)$,*

$$\gamma : \mathrm{IM}_0^{\mathrm{r}} \times]0, \epsilon_0[\to \mathcal{Y}_{\mathcal{W}}.$$

The flow on $\mathrm{IM}_\epsilon^{\mathrm{r}}$ is given by the equations

$$\begin{cases} \partial_t x_c &= \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c\,(s, n, x_c, \gamma(s, x_c, n, \epsilon)) \\ \partial_t n &= \epsilon\mathfrak{F}(s, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(s, x_c, n, \epsilon) + g(s, n)) \\ \partial_t s &= 1. \end{cases}$$

If $z : I \to \mathfrak{P}_e$ is a trajectory on $\mathrm{IM}_\epsilon^{\mathrm{r}}$ then $z \in C^k(I, \mathfrak{P}_e)$.

Rewriting the equations without the time substitute variable s we arrive to the following C^k-smooth ordinary nonautonomous differential equation in the Banach space $\mathbb{C}^q \times \mathcal{U}$:

$$\begin{cases} \partial_t x_c &= \mathfrak{A}_c(n)x_c + \epsilon\mathfrak{G}_c\,(t, n, x_c, \gamma(t, x_c, n, \epsilon)) \\ \partial_t n &= \epsilon\mathfrak{F}(t, n, \mathfrak{B}(n)x_c + \mathfrak{C}(n)\gamma(t, x_c, n, \epsilon) + g(t, n)). \end{cases} \tag{27}$$

Since the graph γ is smooth and $\gamma(t, x_c, n, 0) = 0$ we have that γ is of order ϵ,

$$\gamma(t, x_c, n, \epsilon) = \epsilon\overline{\gamma}(t, x_c, n, \epsilon),$$

where $\overline{\gamma}$ is smooth. If we expand (27) in powers of ϵ then we see that the unknown graph γ only appears in terms of order ϵ^2. By dropping ϵ^2 terms we achieve an approximation to (27) which does not depend on γ and which can be used for numerical bifurcation analysis, see [19, 22].

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TABLE 1. Parameters used in simulations.

symbol	allowed spaces for (I)-(XII)	explanation	values S_1	S_2	S_3	units
$\kappa(x)$	$L^\infty(]0,L[,\mathbb{R})$	field coupling coefficients	15	0	5	$10^3/\mathrm{m}$
$\delta(x)$	$L^\infty(]0,L[,\mathbb{R})$	static detuning	35	0	0	$10^3/\mathrm{m}$
$\beta_{th}(x)$	$L^\infty(]0,L[,\mathbb{R})$	thermal detuning	0	1	0	$10^{-7}\mathrm{m/A}$
$\alpha_0(x)$	$L^\infty(]0,L[,\mathbb{R})$ $\mathrm{essinf}_{x\in]0,L[}\alpha_0>0$	internal absorption	3	1	2	$10^3/\mathrm{m}$
$\alpha_H(x)$	$L^\infty(]0,L[,\mathbb{R})$	Henry factor	-4	0	-4	
$\epsilon_G(x)$	C_P^1, >0	nonlinear gain saturation	3	3	3	$10^{-24}\mathrm{m}^3$
$A(x)$	C_P^1, >0	inverse carrier life time	3	5	3	$10^8/\mathrm{s}$
$B(x)$	C_P^1, ≥ 0	bimolecular recombination	1	0	1	$10^{-16}\mathrm{m}^3/\mathrm{s}$
$C(x)$	C_P^1, ≥ 0	auger recombination	1	0	1	$10^{-40}\mathrm{m}^6/\mathrm{s}$
$g^d(x)$	C_P^1, ≥ 0	differential gain	1	0	1	$10^{-20}\mathrm{m}^2$
$n_{tr}(x)$	C_P^1, ≥ 0	transparency density	1	1	1	$10^{24}/\mathrm{m}^3$
$I(x)$	$W_P^{1,2}$	current injection density	12		3	$10^{10}\mathrm{A/m}^2$
$I_M(t,x)$	$C([0,T],W_P^{1,2})$	modulated current density	0	0	0	$\mathrm{A/m}^2$
R_k	$\mathbb{R}$, >0	series resistance factor in section S_k	5	5	5	$10^{-39}/\mathrm{Am}$
$\lvert S_k\rvert$	$\mathbb{R}$, >0	length of the section S_k	3	2	1	$10^{-4}\mathrm{m}$
λ_0	$\mathbb{R}$, >0	central wavelength	1.54			$10^{-6}\mathrm{m}$
c_0/v_{gr}	$\mathbb{R}$, >0	group velocity factor	3.6			
(r_0,r_L)	$\mathbb{C}^2$	facet reflectivities	0,0			

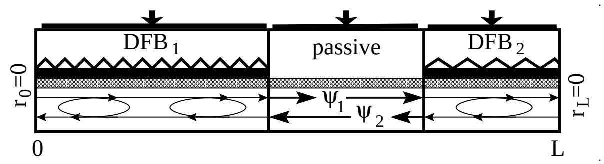

FIGURE 1. Scheme of 3-section DFB laser

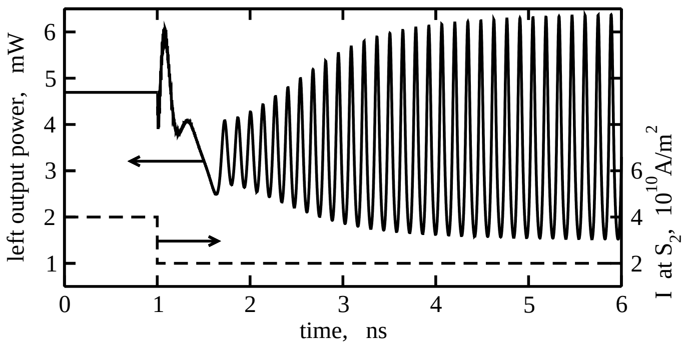

FIGURE 2. Simulated response of the laser (function $|\psi_2(t,0)|^2$) to the change of current.

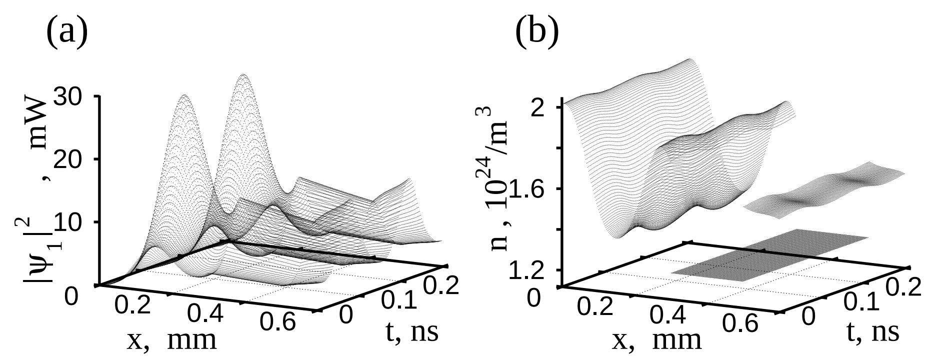

FIGURE 3. Spatial-temporal distributions of functions $|\psi_2(t,x)|^2$ (a) and $n(t,x)$ (b).

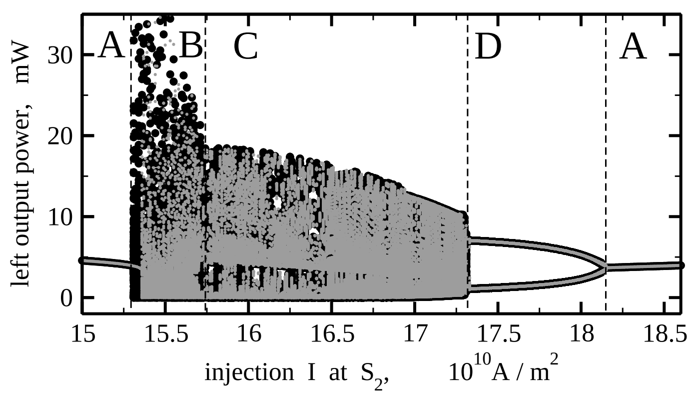

FIGURE 4. Changes of orbits when tuning current injection. Dots for fixed I represent extrema of the simulated field intensity at the left facet of the laser. Grey and black: increased and decreased I.