On the String-Theoretic Euler Number of 3-dimensional A-D-E Singularities

[*]**Dimitrios I. Dais** and [◇]**Marko Roczen**

[*]Mathematics Department, Section of Algebra and Geometry,
University of Ioannina, GR-45110 Ioannina, Greece
e-mail: ddais@cc.uoi.gr

[◇] Institut für Mathematik, Humboldt Universität zu Berlin,
Rudower Chaussee 25, D-10099 Berlin, Germany
e-mail: roczen@mathematik.hu-berlin.de

Abstract

The string-theoretic E-functions $E_{\mathrm{str}}(X;u,v)$ of normal complex varieties X having at most log-terminal singularities are defined by means of snc-resolutions. We give a direct computation of them in the case in which X is the underlying space of the three-dimensional **A**-**D**-**E** singularities by making use of a canonical resolution process. Moreover, we compute the string-theoretic Euler number for several compact complex threefolds with prescribed **A**-**D**-**E** singularities.

1 Introduction

The *string-theoretic* (or *stringy*) *Hodge numbers* $h_{\mathrm{str}}^{p,q}(X)$ of normal, projective complex varieties X with at most Gorenstein quotient or toroidal singularities were introduced in [7] in an attempt to determine a suitable mathematical formulation (and generalization) for the numbers which are encoded into the Poincaré polynomial of the chiral and antichiral rings of the physical "integer charge orbifold theory", due to the LG/CY-correspondence of Vafa, Witten, Zaslow and others. (See [47], [49, §3-5], [50, §4]). These numbers are generated by the so-called E_{str}-*polynomials* and, as it was shown in [7] and [6], they are the right quantities to establish several *mirror-symmetry identities* for Calabi-Yau varieties. In fact, as long as a stratification (separating singularity types) for such an X is available, the key-point is how one defines the E_{str}-polynomial *locally* at these special Gorenstein singular points (by "measuring", in a sense, how far they are from admitting of crepant resolutions).

Recently Batyrev [4] generalized this definition and made it work also for the case in which one allows X to have at most log-terminal singularities. In this general framework, ones has to introduce appropriate E_{str}-*functions* $E_{\mathrm{str}}(X;u,v)$ instead which may be not even rational. The treatment of varieties X with $e_{\mathrm{str}}(X) = \lim_{u,v\to 1} E_{\mathrm{str}}(X;u,v) \notin \mathbb{Z}$ is therefore unavoidable. Nevertheless, as it turned out, this new language is a very important tool as it unifies the considerations of

[*]GSRT-fellow supported by the E.U. and the Greek Ministry of Research and Technology.
2000 MS-Classification. Primary 14Q15, 32S35, 32S45; Secondary 14B05, 14E15, 32S05, 32S25.

certain invariants associated to a wide palette of "MMP-singularities" and leads to the use of more flexible manipulations, as for example in the study of the behaviour of *log-flips*, and in the proof of cohomological *Mckay correspondence* - both on the level of counting dimensions and on the level of determining the motivic Gorenstein volume. (See [5, 1.6, 4.11 and 8.4] and [13, Thm. 5.1]).

In the present paper we deal with the evaluation of the E_{str}-functions and string-theoretic Euler numbers for the three-dimensional **A**-**D**-**E** singularities, and emphasize some distinctive features of the computational methodology.

(a) **Log-terminal singularities.** Let X be a normal complex variety, i.e., a normal, integral, separated scheme of finite type over $\mathbb{C}$. Suppose that X is $\mathbb{Q}$-Gorenstein, i.e., that a positive integer multiple of its canonical Weil divisor K_X is a Cartier divisor. X is said to have at most *log-terminal* (respectively, *canonical / terminal*) singularities if there exists an *snc*-desingularization $\varphi : \widetilde{X} \longrightarrow X$, i.e., a desingularization of X whose exceptional locus $\mathfrak{Exc}(\varphi) = \cup_{i=1}^{r} D_i$ consists of **s**mooth prime divisors $D_1, D_2, \ldots, D_r$ with only **n**ormal **c**rossings, such that the "discrepancy" w.r.t. φ, which is the difference between the canonical divisor of $\widetilde{X}$ and the pull-back of the canonical divisor of X, is of the form

$$K_{\widetilde{X}} - \varphi^*(K_X) = \sum_{i=1}^{r} a_i \ D_i$$

with all the a_i's > -1 (≥ 0 / > 0).

Examples 1.1 (i) The quotients $\mathbb{C}^2/G$, for G a linearly acting finite subgroup of $\mathrm{GL}(2,\mathbb{C})$ (resp. of $\mathrm{SL}(2,\mathbb{C})$), have at most log-terminal (resp. canonical) isolated singularities.
(ii) All $\mathbb{Q}$-Gorenstein toric varieties have at most log-terminal (but not necessarily isolated) singularities.

(b) E**-polynomials.** As it was shown by Deligne in [12, §8], the cohomology groups $H^i(X,\mathbb{Q})$ of any complex variety X are equipped with a functorial *mixed Hodge structure* (MHS). The same remains true if one works with cohomologies $H_c^i(X,\mathbb{Q})$ with compact supports. There exist namely an increasing *weight-filtration*

$$\mathcal{W}_\bullet : \quad 0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{2i-1} \subset W_{2i} = H_c^i(X,\mathbb{Q})$$

and a decreasing *Hodge-filtration*

$$\mathcal{F}^\bullet : \quad H_c^i(X,\mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^i \supset F^{i+1} = 0,$$

such that $\mathcal{F}^\bullet$ induces a natural filtration

$$F^p\left(Gr_k^{\mathcal{W}_\bullet}(H_c^i(X,\mathbb{C}))\right) =$$
$$\left(W_k\left(H_c^i(X,\mathbb{C})\right) \cap F^p\left(H_c^i(X,\mathbb{C})\right) + W_{k-1}\left(H_c^i(X,\mathbb{C})\right)\right) / \ W_{k-1}\left(H_c^i(X,\mathbb{C})\right)$$

(denoted again by $\mathcal{F}^\bullet$) on the complexification of the graded pieces

$$Gr_k^{\mathcal{W}_\bullet}(H_c^i(X,\mathbb{Q})) = W_k/W_{k-1}.$$

Let now

$$h^{p,q}\left(H_c^i(X,\mathbb{C})\right) := \dim_{\mathbb{C}} Gr^p_{\mathcal{F}^\bullet} Gr^{\mathcal{W}_\bullet}_{p+q}\left(H_c^i(X,\mathbb{C})\right)$$

denote the corresponding *Hodge numbers* by means of which one defines the so-called *E-polynomial* of X:

$$E(X;u,v) := \sum_{p,q} e^{p,q}(X)\ u^p v^q \in \mathbb{Z}[u,v],$$

where

$$e^{p,q}(X) := \sum_{i\geq 0} (-1)^i\ h^{p,q}\left(H_c^i(X,\mathbb{C})\right).$$

The E-polynomials are to be viewed as “generating functions” encoding our invariants. For instance, the topological Euler characteristic $e(X)$ is $E(X;1,1)$. In fact, the E-polynomial behaves similarly; e.g, for locally closed subvarieties Y, Y_1, Y_2 of X,

$$E(X\smallsetminus Y;u,v) = E(X;u,v) - E(Y;u,v), \tag{1.1}$$

$$E(Y_1\cup Y_2;u,v) = E(Y_1;u,v) + E(Y_2;u,v) - E(Y_1\cap Y_2;u,v) \tag{1.2}$$

and

$$E(X;u,v) = E(F;u,v)\cdot E(Z;u,v) \tag{1.3}$$

whenever F denotes the fiber of a Zariski locally trivial fibration $X \longrightarrow Z$.

Example 1.2 If $Y \longrightarrow X$ is the blow-up of a d-dimensional complex manifold X at a point $x\in X$ and $D\cong \mathbb{P}^{d-1}_{\mathbb{C}}$ the exceptional divisor, then $E(Y;u,v)$ equals

$$E(X\smallsetminus\{x\};u,v) + E(D;u,v) = E(X;u,v) + uv + (uv)^2 + \cdots + (uv)^{d-1} \tag{1.4}$$

(c) E_{str}**-functions.** Allowing the existence of log-terminal singularities in order to pass to stringy invariants, one takes essentialy into account the “discrepancy coefficients”.

Definition 1.3 Let X be a normal complex variety with at most log-terminal singularities, $\varphi : \widetilde{X} \longrightarrow X$ an snc-desingularization of X as in (a), $D_1, D_2, \ldots, D_r$ the prime divisors of the exceptional locus, and $I := \{1,2,\ldots,r\}$. For any subset $J\subseteq I$ define

$$D_J := \begin{cases} \widetilde{X}, & \text{if } J=\varnothing \\ \bigcap_{j\in J} D_j, & \text{if } J\neq\varnothing \end{cases} \qquad \text{and} \qquad D_J^\circ := D_J \smallsetminus \bigcup_{j\in I\smallsetminus J} D_j .$$

The algebraic function

$$E_{\mathrm{str}}(X;u,v) := \sum_{J\subseteq I} E(D_J^\circ;u,v) \prod_{j\in J} \frac{uv-1}{(uv)^{a_j+1}-1} \tag{1.5}$$

(under the convention for $\prod_{j\in J}$ to be 1, if $J=\varnothing$, and $E(\varnothing;u,v):=0$) is called the *string-theoretic E-function* of X.

The main result of [4] says that:

Theorem 1.4 *The string-theoretic E-function* $E_{\mathrm{str}}(X;u,v)$ *is independent of the choice of the snc-desingularization* $\varphi : \widetilde{X} \longrightarrow X$.

Remark 1.5 (i) The proof of 1.4 relies on ideas of Kontsevich [30], Denef and Loeser by making use of the interpretation of the defining formula (1.5) as some kind of "motivic non-Archimedean integral" over the space of arcs of $\widetilde{X}$. (For an introduction to *motivic integration* and *measures*, we refer to Craw [11] and Looijenga [31]).

(ii) To define (1.5) it is sufficient for $\varphi : \widetilde{X} \longrightarrow X$ to fulfil the snc-condition only for those D_i's for which $a_i \neq 0$.

(iii) If X admits a *crepant* desingularization $\pi : \widehat{X} \longrightarrow X$, i.e., $K_{\widehat{X}} = \pi^* K_X$ with $\widehat{X}$ smooth, then $E_{\mathrm{str}}(X;u,v) = E(\widehat{X};u,v)$.

(iv) In general $E_{\mathrm{str}}(X;u,v)$ may be not a rational function in the two variables u,v. Nevertheless, if X has at most Gorenstein singularities, then the discrepancy coefficients $a_1,\ldots,a_r$ are non-negative integers and

$$E_{\mathrm{str}}(X;u,v) \in \mathbb{Z}[\![u,v]\!] \cap \mathbb{Q}(u,v).$$

(Of course, for X projective, stringy Hodge numbers $h^{p,q}_{\mathrm{str}}(X)$ can be defined only if $E_{\mathrm{str}}(X;u,v) \in \mathbb{Z}[u,v]$).

(v) The *existence* of snc-desingularizations of any X is guaranteed by Hironaka's main theorems [24]. But since definition 1.3 is intrinsic in its nature, it is practically fairly difficult to compute $E_{\mathrm{str}}(X;u,v)$ precisely without having *at least one* snc-desingularization of X at hand, accompanied firstly with the intersection graph of $D_1,\ldots,D_r$ and secondly with the knowledge of their analytic structure.

Definition 1.6 One defines the rational number

$$e_{\mathrm{str}}(X) := \lim_{u,v\to 1} E_{\mathrm{str}}(X;u,v) = \sum_{J\subseteq I} e(D_J^{\circ}) \prod_{j\in J} \frac{1}{a_j+1} \tag{1.6}$$

as the *string-theoretic Euler number* of X. Moreover, the *string-theoretic index* $\mathrm{ind}_{\mathrm{str}}(X)$ of X is defined to be the positive integer

$$\mathrm{ind}_{\mathrm{str}}(X) := \min \left\{ l \in \mathbb{Z}_{\geq 1} \;\middle|\; e_{\mathrm{str}}(X) \in \frac{1}{l}\mathbb{Z} \right\}.$$

Examples 1.7 (i) For $\mathbb{Q}$-Gorenstein toric varieties X, $\mathrm{ind}_{\mathrm{str}}(X) = 1$, and $e_{\mathrm{str}}(X)$ is equal to the normalized volume of the defining fan. Moreover, for Gorenstein toric varieties X, $E_{\mathrm{str}}(X;u,v)$ is a polynomial.

(ii) Normal algebraic surfaces X with at most log-terminal singularities have $\mathrm{ind}_{\mathrm{str}}(X) = 1$. There exist, however, normal complex varieties X of dimension $d \geq 3$ with at most Gorenstein canonical singularities having $\mathrm{ind}_{\mathrm{str}}(X) > 1$.

Batyrev formulated in [4, 5.9] the following conjecture:

Conjecture 1.8 (On the range of the string-theoretic index) *Let X be a d-dimensional normal complex variety having at most Gorenstein canonical singularities. Then* $\mathrm{ind}_{\mathrm{str}}(X)$ *is bounded by a constant* $C(d)$ *depending only on* d.

Remark 1.9 As it will be clear by Theorem 1.11, Conjecture 1.8 is *not* true in general. Nevertheless, there exist several classes of examples of such X's with string-theoretic index bounded by a constant which depends exclusively on the dimension. (See e.g. [4, 5.1, 5.10] for the case in which X is the cone over a $(d-1)$-dimensional smooth projective Fano variety being equipped with a projective embedding defined by a suitable very ample line bundle). The problem of chacterizing those X's having bounded $\mathrm{ind}_{\mathrm{str}}(X)$ is still open.

(d) **The A-D-E's.** The d-dimensional analogues of the classical hypersurface **A-D-E** singularities [16] have underlying spaces of the form

$$\boxed{\begin{array}{l} X_f := X_f^{(d)} := \mathrm{Spec}\left(\mathbb{C}[x_1,\ldots,x_{d+1}] \ / \ (f)\right), \ d \geq 2, \\ \text{with} \quad f(x_1,\ldots,x_{d+1}) := g(x_1,x_2) + g'(x_3,\ldots,x_{d+1}) \end{array}} \tag{1.7}$$

where $g(x_1,x_2)$ is the defining polynomial of a simple curve singularity

$$X_g := \mathrm{Spec}\left(\mathbb{C}[x_1,x_2] \ / \ (g)\right)$$

in the affine plane with

Types	$g(x_1,x_2)$
$\mathbf{A}_n$	$x_1^{n+1} + x_2^2, \ n \geq 1$
$\mathbf{D}_n$	$x_1^{n-1} + x_1 x_2^2, \ n \geq 4$
$\mathbf{E}_6$	$x_1^3 + x_2^4$
$\mathbf{E}_7$	$x_1^3 + x_1 x_2^3$
$\mathbf{E}_8$	$x_1^3 + x_2^5$

and $g'(x_3,\ldots,x_{d+1}) := \sum_{j=3}^{d+1} x_j^2$ is nothing but the defining quadratic polynomial of the affine $(d-2)$-dimensional quadric

$$X_{g'} := X_{g'}^{(d-2)} := \mathrm{Spec}\left(\mathbb{C}[x_3,\ldots,x_{d+1}] \ / \ (g')\right).$$

Remark 1.10 The d-dimensional **A-D-E** singularities have lots of interesting properties:

(i) Herszberg [23] and Treger [46, Thm. 1] proved that they are *absolutely isolated*, i.e., that they can be resolved by blowing up successively a finite number of closed points; in fact, up to analytic isomorphism, they are the only absolutely isolated singularities of multiplicity 2.

(ii) Generalizing the classical result of Artin [2], Burns [9, 3.3-3.4] showed that they are *rational*, i.e., that for any desingularization $\pi : Y \longrightarrow X_f^{(d)}$ in dimension $d \geq 2$, we have $\left(R^i \pi_* \mathcal{O}_Y\right)_{\mathbf{0}} = 0$ for all $i \geq 1$. In particular, this means that they have to be canonical (resp. terminal) of index 1 for $d \geq 2$ (resp. for $d \geq 3$); cf. Reid [33].

(iii) Finally, Arnold's results [1] (see also [14, 8.26-8.27]) imply that they are the only simple (i.e., "0-modular") hypersurface singularities.

These properties lead us to the conclusion that $X_f^{(d)}$'s might belong to the class of the best possible candidates for performing concrete computations for the string-theoretic invariants. On the other hand, we should stress that none of the above general techniques mentioned in 1.10 (i)-(ii) are "constructive" enough in the sense of 1.5 (v). That's why we restrict ourselves in this paper to the three-dimensional case, and based on a canonical snc-resolution being constructed by Giblin [18] and independently by the second-named author in [34], [35], we work out the needed details to prove the following:

Theorem 1.11 *The rational, string-theoretic E-functions of the underlying spaces $X = X_f^{(3)}$ of the* 3*-dimensional* **A**-**D**-**E***-singularities are functions in $w = uv$ given by the following formulae*:

(i) **Type $\mathbf{A}_n$, n even.**

$$\boxed{\begin{aligned} E_{\mathrm{str}}(X; u, v) &= w^3 + w - 1 + \sum_{i=2}^{\frac{n}{2}} \frac{(w-1)\left(w^2-1\right)}{w^{i+1}-1} + \frac{(w-1)w^2}{w^{n+3}-1} \\ &+ (w-1)\left(w^2-1\right)\left[\sum_{i=1}^{\frac{n}{2}-1} \frac{1}{(w^{i+1}-1)(w^{i+2}-1)} + \frac{1}{(w^{\frac{n}{2}+1}-1)(w^{n+3}-1)}\right] \end{aligned}}$$

(ii) **Type $\mathbf{A}_n$, n odd.**

$$\boxed{\begin{aligned} &E_{\mathrm{str}}(X; u, v) = (w-1)\,(w+1)^2 + w + \left\lfloor \tfrac{1}{n} \right\rfloor \\ &+ \left(w^2-1\right)\left[\sum_{i=2}^{\frac{n-1}{2}} \frac{(w-1)}{w^{i+1}-1} + \frac{w}{w^{\frac{n+3}{2}}-1} + \sum_{i=1}^{\frac{n-1}{2}} \frac{(w-1)}{(w^{i+1}-1)(w^{i+2}-1)}\right] \cdot \left\lceil \tfrac{n-1}{n} \right\rceil \end{aligned}}$$

(iii) **Type $\mathbf{D}_n$, n even.**

$$
\boxed{
\begin{aligned}
&E_{\mathrm{str}}\left(X;u,v\right)=\left(w-1\right)\,\left(w^{2}+3w+1\right)\\
&+\left(w-1\right)\,\left(w+1\right)^{2}\left[\frac{2}{w^{n}-1}+\sum_{i=3}^{\frac{n}{2}+1}\frac{1}{w^{2(n+4-2i)}-1}\right]\\
&+2\left(w-1\right)\,\left(1+4w+w^{2}\right)\left[\sum_{i=1}^{\frac{n}{2}-1}\frac{1}{w^{\left(\frac{n}{2}-i+1\right)}-1}\right]\\
&+\left(1+w\right)\left[4\left(\frac{w-w^{n}}{w^{n}-1}\right)\left(\frac{w-w^{\frac{n}{2}}}{w^{\frac{n}{2}}-1}\right)+\sum_{i=1}^{\frac{n}{2}-1}\left(\frac{w-w^{\left(\frac{n}{2}-i+1\right)}}{w^{\left(\frac{n}{2}-i+1\right)}-1}\right)^{2}\right]\\
&+\left(1+w\right)\left[2\sum_{(\kappa,\lambda)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)-7\left(\frac{n}{2}-1\right)\right]\\
&+\sum_{(\kappa,\lambda,\mu)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)\left(\frac{w-w^{\mu+1}}{w^{\mu+1}-1}\right)\\
&+2\sum_{(\kappa',\lambda',\mu')}\left(\frac{w-w^{\kappa'+1}}{w^{\kappa'+1}-1}\right)\left(\frac{w-w^{\lambda'+1}}{w^{\lambda'+1}-1}\right)\left(\frac{w-w^{\mu'+1}}{w^{\mu'+1}-1}\right)+2n-5
\end{aligned}
}
$$

where the pairs (κ,λ) of the fourth sum are taken from the set

$$
\begin{aligned}
&\left\{\left(\tfrac{n}{2}-i,\tfrac{n}{2}-(i+1)\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-2\right\}\\
&\cup\left\{\left(\tfrac{n}{2}-i,2\left(n-2i\right)-1\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-1\right\}\\
&\cup\left\{\left(\tfrac{n}{2}-\left(i+1\right),2\left(n-2i\right)-1\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-2\right\},
\end{aligned}
$$

the triples (κ,λ,μ) of the fifth sum from the set

$$
\begin{aligned}
&\left\{\left(\tfrac{n}{2}-i,\tfrac{n}{2}-i,2\left(n-2i\right)-1\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-1\right\}\\
&\cup\left\{\left(\tfrac{n}{2}-\left(i+1\right),\tfrac{n}{2}-\left(i+1\right),2\left(n-2i\right)-1\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-2\right\},
\end{aligned}
$$

and the triples (κ',λ',μ') of the sixth sum from the set

$$
\begin{aligned}
&\left\{\left(\tfrac{n}{2}-i,\tfrac{n}{2}-\left(i+1\right),2\left(n-2i\right)-1\right)\;\middle|\;1\leq i\leq\tfrac{n}{2}-2\right\}\\
&\cup\left\{\left(n-1,\tfrac{n}{2}-1,\tfrac{n}{2}-1\right)\right\}\;.
\end{aligned}
$$

(iv) **Type $\mathbf{D}_n$, n odd.**

$$
\boxed{\begin{aligned}
&E_{\mathrm{str}}(X;u,v) = (w-1)\,(w+1)^2 \\
&+(w-1)\,(w+1)^2\left[\tfrac{1}{w^{n-1}-1}+\tfrac{1}{w^{n}-1}+\sum_{i=3}^{\frac{n+1}{2}}\tfrac{1}{w^{2(n+3-2i)}-1}\right] \\
&+2\,(w-1)\,(1+4w+w^2)\left[\sum_{i=1}^{\frac{n-3}{2}}\tfrac{1}{w^{\left(\frac{n-1}{2}-i+1\right)}-1}\right] \\
&+2\,(1+w)\left(\tfrac{w-w^{\frac{n-1}{2}}}{w^{\frac{n-1}{2}}-1}\right)\left[\tfrac{w-w^{n}}{w^{n}-1}+\tfrac{w-w^{n-1}}{w^{n-1}-1}\right] \\
&+(1+w)\left[\left(\tfrac{w-w^{n}}{w^{n}-1}\right)\left(\tfrac{w-w^{n-1}}{w^{n-1}-1}\right)+\sum_{i=1}^{\frac{n-3}{2}}\left(\tfrac{w-w^{\left(\frac{n-1}{2}-i+1\right)}}{w^{\left(\frac{n-1}{2}-i+1\right)}-1}\right)^2\right] \\
&+(1+w)\left[2\sum_{(\kappa,\lambda)}\left(\tfrac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\tfrac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)-\tfrac{7}{2}\,(n-1)+6\right] \\
&+\sum_{(\kappa,\lambda,\mu)}\left(\tfrac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\tfrac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)\left(\tfrac{w-w^{\mu+1}}{w^{\mu+1}-1}\right) \\
&+2\sum_{(\kappa',\lambda',\mu')}\left(\tfrac{w-w^{\kappa'+1}}{w^{\kappa'+1}-1}\right)\left(\tfrac{w-w^{\lambda'+1}}{w^{\lambda'+1}-1}\right)\left(\tfrac{w-w^{\mu'+1}}{w^{\mu'+1}-1}\right)+2\,(n-1)-4
\end{aligned}}
$$

where the pairs (κ,λ) *are taken from the set*

$$
\begin{aligned}
&\left\{\left(\tfrac{n-1}{2}-i,\tfrac{n-1}{2}-(i+1)\right)\;\middle|\;1\le i\le\tfrac{n-5}{2}\right\}\\
&\cup\left\{\left(\tfrac{n-1}{2}-i,2\,(n-2i)-3\right)\;\middle|\;1\le i\le\tfrac{n-3}{2}\right\}\\
&\cup\left\{\left(\tfrac{n-1}{2}-(i+1)\,,2\,(n-2i)-3\right)\;\middle|\;1\le i\le\tfrac{n-5}{2}\right\},
\end{aligned}
$$

the triples (κ,λ,μ) *from the set*

$$
\begin{aligned}
&\{(n-1,\tfrac{n-3}{2},\tfrac{n-3}{2})\}\cup\left\{\left(\tfrac{n-1}{2}-i,\tfrac{n-1}{2}-i,2\,(n-2i)-3\right)\;\middle|\;1\le i\le\tfrac{n-3}{2}\right\}\\
&\cup\left\{\left(\tfrac{n-1}{2}-(i+1)\,,\tfrac{n-1}{2}-(i+1)\,,2\,(n-2i)-3\right)\;\middle|\;1\le i\le\tfrac{n-5}{2}\right\},
\end{aligned}
$$

and the triples (κ',λ',μ') *from the set*

$$
\begin{aligned}
&\left\{\left(\tfrac{n-1}{2}-i,\tfrac{n-1}{2}-(i+1)\,,2\,(n-2i)-3\right)\;\middle|\;1\le i\le\tfrac{n-5}{2}\right\}\\
&\cup\{(n-1,n-2,\tfrac{n-3}{2})\}\;.
\end{aligned}
$$

(v) **Type $\mathbf{E}_6$.**

$$
\boxed{\begin{aligned}
&E_{\mathrm{str}}(X;u,v) = w^3-1+\tfrac{w+1}{w^2+1}+\tfrac{(w+1)^2(w-1)}{w^7-1}+\tfrac{(w+1)^2(w-1)}{w^{10}-1}+\tfrac{2(1+4w+w^2)}{w+1} \\
&+(1+w)\left[\sum_{(\kappa,\lambda)}\left(\tfrac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\tfrac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)-9\right] \\
&+\sum_{(\kappa,\lambda,\mu)}\left(\tfrac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\tfrac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)\left(\tfrac{w-w^{\mu+1}}{w^{\mu+1}-1}\right)+5
\end{aligned}}
$$

where the pairs (κ, λ) *of the first sum are taken from the set*

$$\{(1,1),(1,3),(3,1),(1,6),(6,1),(1,9),(9,1),(3,6),(6,9)\}$$

and the triples (κ, λ, μ) *of the second sum from the set*

$$\{(1,1,9),(1,6,9),(1,9,6),(1,3,6),(1,6,3)\}\ .$$

(vi) **Type** $\mathbf{E}_7$.

$$\boxed{\begin{array}{l} E_{\mathrm{str}}\left(X;u,v\right) = (w-1)\ (w+1)^2\left[1+\frac{1}{w^6-1}+\frac{1}{w^{10}-1}+\frac{1}{w^{12}-1}+\frac{1}{w^{14}-1}\right] \\ +2\,(w-1)\ (1+4w+w^2)\left[\frac{1}{w^2-1}+\frac{1}{w^3-1}+\frac{1}{w^5-1}\right] \\ +(1+w)\left[\sum\limits_{(\kappa,\lambda)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)-21\right] \\ +\sum\limits_{(\kappa,\lambda,\mu)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)\left(\frac{w-w^{\mu+1}}{w^{\mu+1}-1}\right)+12 \end{array}}$$

where the pairs (κ, λ) *are taken from the set*

$$\begin{array}{l}\{(4,9),(9,4),(4,11),(11,4),(1,11),(11,1),(4,4)\\ (1,4),(4,1),(4,13),(13,4),(2,13),(13,2),(2,2)\\ (2,5),(5,2),(1,2),(2,1),(4,2),(2,4),(1,1)\}\,.\end{array}$$

and the triples (κ, λ, μ) *from the set*

$$\begin{array}{l}\{(1,1,11),(1,2,4),(1,4,2),(1,4,11),(1,11,4),(2,2,5),\\ (2,2,13),(2,4,13),(2,13,4),(4,4,9),(4,4,11),(4,4,13)\}\ .\end{array}$$

(vi) **Type** $\mathbf{E}_8$.

$$\boxed{\begin{array}{l} E_{\mathrm{str}}\left(X;u,v\right) = w^3-1+(w-1)\ (w+1)^2\left[\frac{1}{w^{12}-1}+\frac{1}{w^{16}-1}+\frac{1}{w^{20}-1}+\frac{1}{w^{24}-1}\right] \\ +2\,(w-1)\ (1+4w+w^2)\left[\frac{1}{w^2-1}+\frac{1}{w^3-1}+\frac{1}{w^5-1}+\frac{1}{w^8-1}\right] \\ +(1+w)\left[\sum\limits_{(\kappa,\lambda)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)-28\right] \\ +\sum\limits_{(\kappa,\lambda,\mu)}\left(\frac{w-w^{\kappa+1}}{w^{\kappa+1}-1}\right)\left(\frac{w-w^{\lambda+1}}{w^{\lambda+1}-1}\right)\left(\frac{w-w^{\mu+1}}{w^{\mu+1}-1}\right)+17 \end{array}}$$

where the pairs (κ, λ) *are taken from the set*

$$\begin{array}{l}\{(1,1),(1,2),(2,1),(1,4),(4,1),(1,11),(11,1),\\ (2,2),(2,4),(4,2),(2,7),(7,2),(2,19),(19,2),\\ (4,4),(4,7),(7,4),(4,11),(11,4),(4,23),(23,4),\\ (7,7),(7,15),(15,7),(7,19),(19,7),(7,23),(23,7)\}\end{array}$$

and the triples (κ, λ, μ) *from the set*

$$\begin{array}{l}\{(1,1,11),(1,2,4),(1,4,2),(1,4,11),(1,11,4),(2,2,19),\\ (2,4,7),(2,7,4),(2,7,19),(2,19,7),(4,4,11),(4,4,23),\\ (4,7,23),(4,23,7),(7,7,15),(7,7,19),(7,7,23)\}\ .\end{array}$$

In particular, the values of the corresponding string-theoretic Euler numbers (1.6) *are equal to*

Types	$e_{\mathrm{str}}(X)$
$\mathbf{A}_n$, n even	$2-\frac{3}{n+3}$
$\mathbf{A}_n$, n odd	2
$\mathbf{D}_n$, n even	$-\frac{80n^4-381n^3+96n^2-128}{16n^3} + \sum_{i=1}^{\frac{n}{2}-2} \frac{2\left(372-492n^2-32i-184n^3+20n^4+688in-160in^3+304in^2+208n+5n^5-50in^4\right)}{(n-2i)^3(n-2i+2)^2}$
$\mathbf{D}_n$, n odd	$\frac{-96n^3+765n^2-1562n+1085}{16(n-1)^2} + \sum_{i=1}^{\frac{n-5}{2}} \frac{2\left(585n-129+130n^2-306i-214n^3-5n^4-200in+40in^3+484in^2+5n^5-50in^4\right)}{(n+1-2i)^2(n-1-2i)^3}$
$\mathbf{E}_6$	$\frac{67}{40}=1.675$
$\mathbf{E}_7$	$\frac{609\,851}{189\,000}\approx 3.226\,7$
$\mathbf{E}_8$	$\frac{315\,467}{230\,400}\approx 1.369\,2$

and the string-theoretic indices take the following values:

Types	$\mathrm{ind}_{\mathrm{str}}(X)$
$\mathbf{A}_n$	$\begin{cases} 1, & \text{if } n \equiv 1 \,(\mathrm{mod}\ 2) \\ n+3, & \text{if } n \equiv 2 \text{ or } 4 \,(\mathrm{mod}\ 6) \\ \frac{n}{3}+1, & \text{if } n \equiv 0 \,(\mathrm{mod}\ 6) \end{cases}$
$\mathbf{D}_n$	It belongs to the intervall $\begin{cases} (n,\ n^3 \prod_{i=1}^{\frac{n}{2}-2} (n-2i)^3\,(n-2i+2)^2] \cap \mathbb{Z}, & \text{if } n \text{ even} \\ (n, 16\,(n-1)^2 \prod_{i=1}^{\frac{n-5}{2}} (n+1-2i)^2\,(n-1-2i)^3] \cap \mathbb{Z}, & \text{if } n \text{ odd} \end{cases}$
$\mathbf{E}_6$	$2^3 5$
$\mathbf{E}_7$	$2^3 3^3 5^3 7$
$\mathbf{E}_8$	$2^{10} 3^2 5^2$

2 The canonical desingularization procedure

Throughout this section we shall omit the superscript $d(=3)$, use the notation (1.7), and write the defining equation as:

$$X_f = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \;\middle|\; f(x_1, x_2, x_3, x_4) = g(x_1, x_2) + x_3^2 + x_4^2 = 0\right\}.$$

Let $\pi : \mathbf{Bl_0}(\mathbb{C}^4) \longrightarrow \mathbb{C}^4$ be the blow up of $\mathbb{C}^4$ at the origin, with

$$\mathbf{Bl_0}(\mathbb{C}^4) = \left\{((x_1, x_2, x_3, x_4), (t_1 : t_2 : t_3 : t_4)) \in \mathbb{C}^4 \times \mathbb{P}^3_{\mathbb{C}} \;\middle|\; \begin{array}{l} x_i\, t_j = x_j\, t_i, \\ \forall i, j,\ 1 \leq i, j \leq 4 \end{array} \right\},$$

$\mathcal{E} = \pi^{-1}(\mathbf{0}) = \{\mathbf{0}\} \times \mathbb{P}^3_{\mathbb{C}}$, and let $U_i \subset \mathbf{Bl_0}(\mathbb{C}^4)$ denote the open set given by $(t_i \neq 0)$. In terms of analytic coordinates we may write for $i \in \{1, 2, 3, 4\}$,

$$U_i = \left\{((x_1, x_2, x_3, x_4), (\xi_1, .., \widehat{\xi_i}, .., \xi_4)) \in \mathbb{C}^4 \times \mathbb{C}^3 \;\middle|\; \begin{array}{l} x_j = x_i\, \xi_j, \\ \forall j,\ j \in \{1, 2, 3, 4\} \smallsetminus \{i\} \end{array} \right\},$$

where $\xi_j = \frac{t_j}{t_i}$, and $\widehat{\xi_i}$ means that we omit ξ_i. Moreover, we may identify U_i with a $\mathbb{C}^4$ with respect to the coordinates $x_i, \xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_4$. The restriction $\pi|_{U_i}$ is therefore given by mapping

$$\begin{array}{c} \mathbb{C}^4 \ni (x_i, \xi_1, .., \widehat{\xi_i}, .., \xi_4) \\ \downarrow \cong \\ ((x_i\, \xi_1, .., x_i\, \xi_{i-1}, x_i, x_i\, \xi_{i+1}, .., x_i\, \xi_4), (\xi_1 : .. : \underbrace{1}_{i\text{-th pos.}} : .. : \xi_4)) \in U_i \\ \downarrow\ \pi|_{U_i} \\ (x_i\, \xi_1, \ldots, x_i\, \xi_{i-1}, x_i, x_i\, \xi_{i+1}, \ldots, x_i\, \xi_4) \end{array}$$

Note that $\mathcal{E}_i := \mathcal{E} \cap U_i$ is described as the coordinate hyperplane $(x_i = 0)$; i.e., the open cover $\{U_i\}_{1 \leq i \leq 4}$ of $\mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4)$ restricts to $\mathcal{E}$ to provide the standard open cover of $\mathbb{P}^3_{\mathbb{C}}$ by affine spaces $\mathbb{C}^3$, with $\{\xi_j\}_{j \in \{1,2,3,4\} \smallsetminus \{i\}}$ being the analytic coordinates of $\mathcal{E}_i$.

Notation. To work with a more convenient notation we define

$$\mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4) = \bigcup_{i=1}^{4} U_i, \qquad U_i = \mathrm{Spec}\left(\mathbb{C}[y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}]\right),$$

by setting as coordinates for U_i's:

$$y_{i,k} := \begin{cases} x_k, & \text{for } i = k \\ \xi_k, & \text{for } i \neq k \end{cases}$$

• **Step 1: The first blow-up.** Blowing up X_f at the origin, we take the diagram

$$\begin{array}{ccccc} \mathcal{E} & \subset & \mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4) & \stackrel{\pi}{\longrightarrow} & \mathbb{C}^4 \\ \cup & & \cup & & \cup \\ \mathcal{E} \cap \mathbf{Bl}_{\mathbf{0}}(X_f) & \subset & \mathbf{Bl}_{\mathbf{0}}(X_f) & \stackrel{\pi|_{\mathrm{restr.}}}{\longrightarrow} & X_f \end{array}$$

and consider the strict transform

$$\mathbf{Bl}_{\mathbf{0}}(X_f) = \overline{\pi^{-1}(X_f \cap (\mathbb{C}^4 \smallsetminus \{\mathbf{0}\}))} = \overline{\pi^{-1}(X_f) \cap (\mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4) \smallsetminus \mathcal{E})}$$

of X_f in $\mathbb{C}^4$ under π, and the corresponding exceptional (not necessarily prime) divisor $\mathcal{E}_f := \mathcal{E} \cap \mathbf{Bl}_{\mathbf{0}}(X_f)$ with respect to $\pi\,|_{\mathrm{restr.}}$

▶ **Local description of $\mathbf{Bl}_{\mathbf{0}}(X_f)$ and $\mathcal{E}_f$.** After pulling back f by π and restricting ourselves onto U_i, we get

$$\pi^*(f)\,|_{U_i} = x_i^2\ \widetilde{f}_i = y_{i,i}^2\ \widetilde{f}_i,$$

with $\widetilde{f}_i \in \mathbb{C}[y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}]$. More precisely, we obtain

Types	$\widetilde{f}_1$	$\widetilde{f}_2$
$\mathbf{A}_n$	$y_{1,1}^{n-1} + y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2$	$y_{2,1}^{n+1} y_{2,2}^{n-1} + 1 + y_{2,3}^2 + y_{2,4}^2$
$\mathbf{D}_n$	$y_{1,1}^{n-3} + y_{1,1} y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2$	$y_{2,1}^{n-1} y_{2,2}^{n-3} + y_{2,1} y_{2,2} + y_{2,3}^2 + y_{2,4}^2$
$\mathbf{E}_6$	$y_{1,1} + y_{1,1}^2 y_{1,2}^4 + y_{1,3}^2 + y_{1,4}^2$	$y_{2,1}^3 y_{2,2} + y_{2,2}^2 + y_{2,3}^2 + y_{2,4}^2$
$\mathbf{E}_7$	$y_{1,1} + y_{1,1}^2 y_{1,2}^3 + y_{1,3}^2 + y_{1,4}^2$	$y_{2,1}^3 y_{2,2} + y_{2,1} y_{2,2}^2 + y_{2,3}^2 + y_{2,4}^2$
$\mathbf{E}_8$	$y_{1,1} + y_{1,1}^3 y_{1,2}^5 + y_{1,3}^2 + y_{1,4}^2$	$y_{2,1}^3 y_{2,2} + y_{2,2}^3 + y_{2,3}^2 + y_{2,4}^2$

and

Types	$\widetilde{f}_3$	$\widetilde{f}_4$
$\mathbf{A}_n$	$y_{3,1}^{n+1} y_{3,3}^{n-1} + y_{3,2}^2 + 1 + y_{3,4}^2$	$y_{4,1}^{n+1} y_{4,4}^{n-1} + y_{4,2}^2 + y_{4,3}^2 + 1$
$\mathbf{D}_n$	$y_{3,1}^{n-1} y_{3,3}^{n-3} + y_{3,1} y_{3,2}^2 y_{3,3} + 1 + y_{3,4}^2$	$y_{4,1}^{n-1} y_{4,4}^{n-3} + y_{4,1} y_{4,2}^2 y_{4,4} + y_{4,3}^2 + 1$
$\mathbf{E}_6$	$y_{3,1}^3 y_{3,3} + y_{3,2}^4 y_{3,3}^2 + 1 + y_{3,4}^2$	$y_{4,1}^3 y_{4,4} + y_{4,4}^4 y_{4,4}^2 + y_{4,3}^2 + 1$
$\mathbf{E}_7$	$y_{3,1}^3 y_{3,3} + y_{3,1} y_{3,2}^2 y_{3,3}^2 + 1 + y_{3,4}^2$	$y_{4,1}^3 y_{4,4} + y_{4,1} y_{4,2}^2 y_{4,4}^2 + y_{4,3}^2 + 1$
$\mathbf{E}_8$	$y_{3,1}^3 y_{3,3} + y_{3,2}^5 y_{3,3}^3 + 1 + y_{3,4}^2$	$y_{4,1}^3 y_{4,4} + y_{4,2}^5 y_{4,4}^3 + y_{4,3}^2 + 1$

Locally,

$$\mathbf{Bl_0}(X_f)\,|_{U_i} \overset{\cong}{\rightarrow} \left\{ (y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}) \in \mathbb{C}^4 \;\middle|\; \widetilde{f}_i\,(y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}) = 0 \right\},$$

and using the restrictions of $\widetilde{f}_i$'s on the $\mathcal{E}_i$'s, $i = 1, 2, 3, 4$, we get the equations for $\mathcal{E}_f\,|_{U_i}$:

$$\mathbf{Bl_0}(X_f) \cap \mathcal{E}_i = \mathcal{E}_f\,|_{U_i} \overset{\cong}{\rightarrow} \left\{ (y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}) \in \mathbb{C}^4 \;\middle|\; y_{i,i} = \widetilde{f}_i\,(y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}) = 0 \right\}.$$

Lemma 2.1 (Local Reduction) *The types of the singularities of* $\mathbf{Bl_0}(X_f)$ *are given by the following table*:

Initial types of singularities of X_f	**New singularities (and their types) on $\mathbf{Bl_0}(X_f)$**	**located in the affine pieces**
$\mathbf{A}_1, \mathbf{A}_2$	$-$	$-$
$\mathbf{A}_n$, $n \geq 3$	$\mathbf{A}_{n-2}$	U_1
$\mathbf{D}_4$	$\mathbf{A}_1,\ \mathbf{A}_1,\ \mathbf{A}_1$	U_2, $U_1 \cap U_2$, $U_1 \cap U_2$
$\mathbf{D}_5$	$\mathbf{A}_3,\ \mathbf{A}_1$	U_1, U_2
$\mathbf{D}_n$, $n \geq 6$	$\mathbf{D}_{n-2},\ \mathbf{A}_1$	U_1, U_2
$\mathbf{E}_6$	$\mathbf{A}_5$	U_2
$\mathbf{E}_7$	$\mathbf{D}_6$	U_2
$\mathbf{E}_8$	$\mathbf{E}_7$	U_2

Proof. The affine pieces in which the singularities of $\mathbf{Bl_0}(X_f)$ are located are obviously those of the above table (simply by partial derivative checking). Let us now examine the types of the appearing singularities in each case separately.

$\triangleright$ Blowing up singularity $\mathbf{A}_n$, $n \geq 3$, we obtain an $\mathbf{A}_{n-2}$-singularity in its normal form $\widetilde{f}_1$.

$\triangleright$ Blowing up $\mathbf{D}_n$'s, and working first with the patch U_1, we get a $\mathbf{D}_{n-2}$-singularity in its normal form $\widetilde{f}_1$ whenever $n \geq 6$, no singularity for $n = 4$, and an $\mathbf{A}_3$-singularity for $n = 5$, just by utilizing the analytic coordinate change

$$y_{1,i} = \begin{cases} y'_{1,i}, & i \in \{2,3,4\} \\ y'_{1,1} - \frac{1}{2}(y'_{1,2})^2, & i = 1 \end{cases}$$

and writing the corresponding defining polynomial as:

$$y_{1,1}^2 + y_{1,1}y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2 = -\frac{1}{4}(y'_{1,2})^4 + (y'_{1,1})^2 + (y'_{1,3})^2 + (y'_{1,4})^2 \ .$$

Passing to U_2, we have $\mathbf{Bl_0}(X_f)\,|_{U_2} =$

$$= \left\{ (y_{2,1}, ..., y_{2,4}) \in \mathbb{C}^4 \;\middle|\; \theta(y_{2,1}, ..., y_{2,4}) := y_{2,1}^{n-1} y_{2,2}^{n-3} + y_{2,1}y_{2,2} + y_{2,3}^2 + y_{2,4}^2 = 0 \right\}$$

with partial derivatives w.r.t. $\theta = \theta(y_{2,1}, ..., y_{2,4})$:

$$\begin{cases} \dfrac{\partial \theta}{\partial y_{2,1}} = (n-1)\; y_{2,1}^{n-2} y_{2,2}^{n-3} + y_{2,2} = y_{2,2}\, \left((n-1)\; y_{2,1}^{n-2} y_{2,2}^{n-4} + 1\right) \\ \dfrac{\partial \theta}{\partial y_{2,2}} = (n-3)\; y_{2,1}^{n-1} y_{2,2}^{n-4} + y_{2,1} = y_{2,1}\, \left((n-3)\; y_{2,1}^{n-2} y_{2,2}^{n-4} + 1\right) \\ \dfrac{\partial \theta}{\partial y_{2,3}} = 2\, y_{2,3} \quad \text{and} \quad \dfrac{\partial \theta}{\partial y_{2,4}} = 2\, y_{2,4} \ . \end{cases}$$

Clearly, for $n = 4$, the singular locus of $\mathbf{Bl_0}(X_f)\,|_{U_2}$ consists of the points

$$(0,0,0,0)\,, \quad (\sqrt{-1},0,0,0) \quad \text{and} \quad (-\sqrt{-1},0,0,0)$$

which can be expressed as the singularities *at the origin* $\mathbf{0}$ of $\mathbb{C}^4$ for

$$\begin{cases} y_{2,1}^3 y_{2,2} + y_{2,1}y_{2,2} + y_{2,3}^2 + y_{2,4}^2 = 0 \\ y'_{2,2}\,(y'_{2,1})^3 \pm 3\sqrt{-1}\, y'_{2,2}\,(y'_{2,1})^2 - 2\, y'_{2,2}\, y'_{2,1} + (y'_{2,3})^2 + (y'_{2,4})^2 = 0 \end{cases} \tag{2.1}$$

(just by setting $y_{2,1} = y'_{2,1} \pm \sqrt{-1}$ and $y_{2,i} = y'_{2,i}$, for $i \in \{2,3,4\}$). Next, applying a result of Bădescu (in a very special case of it, [3, Thm. 1, p. 209]), we see that *all* normal isolated singularities which can be fully resolved after a single blow-up and have exceptional divisor $E \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with conormal bundle $\mathcal{N}_E^{\vee}$ isomorphic to $\mathcal{O}_E\,(1,1)$ are analytically isomorphic to each other. It is easy to verify that this is valid for all singularities (2.1). Hence, they are all analytically isomorphic to an $\mathbf{A}_1$-singularity (which has the same property). Alternatively, one can show that these are analytically isomorphic to $\mathbf{A}_1$-singularities by exploiting the fact that they are semiquasihomogeneous of weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and by using [36, Corollary 3.3]. (The completions are isomorphic to the singularities defined by that polynomial part consisting of all terms of weight 1, which is obviously equal to $y_{2,1}y_{2,2} + y_{2,3}^2 + y_{2,4}^2$ and $-2\, y'_{2,2}\, y'_{2,1} + (y'_{2,3})^2 + (y'_{2,4})^2$, respectively). On the other hand, for $n \geq 5$, the only singular point of $\mathbf{Bl_0}(X_f)\,|_{U_2}$ is $(0,0,0,0)$, which again turns out to be an $\mathbf{A}_1$-singularity (by the same reasoning).

$\triangleright$ Now the singularity $\mathbf{E}_6$ passes after blowing up to an $\mathbf{A}_5$-singularity, because using the analytic coordinate change

$$y_{2,i} = \begin{cases} y'_{2,i}, & i \in \{1,3,4\} \\ y'_{2,2} - \frac{1}{2}(y'_{2,1})^3, & i = 2 \end{cases}$$

we get

$$y_{2,1}^3 y_{2,2} + y_{2,2}^2 + y_{2,3}^2 + y_{2,4}^2 = -\frac{1}{4}(y'_{2,1})^6 + (y'_{2,2})^2 + (y'_{2,3})^2 + (y'_{2,4})^2 \ .$$

▷ Starting with $\mathbf{E}_7$ we obtain a $\mathbf{D}_6$-singularity, because the analytic coordinate change

$$y_{2,i} = \begin{cases} y'_{2,i}, & i \in \{1,3,4\} \\ y'_{2,2} - \frac{1}{2}(y'_{2,1})^2, & i = 2 \end{cases}$$

implies

$$y_{2,1}^3 y_{2,2} + y_{2,1} y_{2,2}^2 + y_{2,3}^2 + y_{2,4}^2 = -\frac{1}{4}(y'_{2,1})^5 + y'_{2,1}\,(y'_{2,2})^2 + (y'_{2,3})^2 + (y'_{2,4})^2 \ .$$

▷ Finally, blowing up singularity $\mathbf{E}_8$, we acquire an $\mathbf{E}_7$-singularity in its normal form $\widetilde{f}_2$. □

▶ **Global description of $\mathbf{Bl}_\mathbf{0}(X_f)$ and $\mathcal{E}_f$.** This can be realized after coming back to our global coordinates:

Types	$\mathbf{Bl}_\mathbf{0}(X_f) =$ all $((x_1,..,x_4),(t_1:t_2:t_3:t_4)) \in \mathbf{Bl}_\mathbf{0}(\mathbb{C}^4)$ with:
$\mathbf{A}_n$	$x_1^{n-1}\,t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{D}_n$	$x_1^{n-3}\,t_1^2 + x_1\,t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{E}_6$	$x_1\,t_1^2 + x_2^2\,t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{E}_7$	$x_1\,t_1^2 + x_1\,x_2\,t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{E}_8$	$x_1\,t_1^2 + x_2^3\,t_2^2 + t_3^2 + t_4^2 = 0$

In particular, this means that the exceptional locus $\mathcal{E}_f$ is given globally by

Types of X_f's	$\mathcal{E}_f =$ all $(\mathbf{0},(t_1:t_2:t_3:t_4)) \in \{\mathbf{0}\} \times \mathbb{P}^3_\mathbb{C}$ with:
$\mathbf{A}_1$	$t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{A}_n, n \geq 2$	$t_2^2 + t_3^2 + t_4^2 = 0$
$\mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$	$t_3^2 + t_4^2 = \left(t_3 + \sqrt{-1}\,t_4\right)\left(t_3 - \sqrt{-1}\,t_4\right) = 0$

In the latter four cases $\mathcal{E}_f$ consists of two exceptional prime divisors, say $\mathcal{E}'_f$ and $\mathcal{E}''_f$ (which are $\cong \mathbb{P}^2_\mathbb{C}$). Moreover, taking into account the above local description of singularities of $\mathbf{Bl_0}(X_f)$, we may rewrite them in homogeneous coordinates on $\{\mathbf{0}\} \times \mathbb{P}^3_{\mathbb{C}}$ as follows:

Types of X_f's	**Singular points of $\mathbf{Bl_0}(X_f)$**
$\mathbf{A}_1,\ \mathbf{A}_2$	$-$
$\mathbf{A}_n,\ n \geq 3$	$(\mathbf{0}, (1:0:0:0)) \in \mathcal{E}_f$
$\mathbf{D}_4$	$(\mathbf{0}, (0:1:0:0))\,,\ (\mathbf{0}, (\pm\sqrt{-1}:1:0:0)) \in \mathcal{E}_f' \cap \mathcal{E}_f''$
$\mathbf{D}_n,\ n \geq 5$	$(\mathbf{0}, (1:0:0:0))\,,\ (\mathbf{0}, (0:1:0:0)) \in \mathcal{E}_f' \cap \mathcal{E}_f''$
$\mathbf{E}_6,\ \mathbf{E}_7,\ \mathbf{E}_8$	$(\mathbf{0}, (0:1:0:0)) \in \mathcal{E}_f' \cap \mathcal{E}_f''$

• Step 2: The next blow-ups. The desired snc-desingularizations of X_f's, say $\varphi : \widetilde{X} \to X_f$, will be constructed by blowing up the possibly new singular points again and again until we reach to a smooth threefold $\widetilde{X}$ with exceptional locus $\mathfrak{Ex}(\varphi)$ consisting of smooth prime divisors with normal crossings. We give a complete characterization of φ's by the following data:

▹ the *local resolution diagrams* (abbreviated *LR-diagrams*) which are constructed after repeated applications of Lemma 2.1 (with each arrow indicating a *local* blow-up at a single closed point),

▹ the *intersection* (*plane*) *graphs* whose vertices represent the exceptional prime divisors w.r.t. the φ's and their edges insinuate that the corresponding vertices are divisors which have non-empty intersection,

▹ the *structure* of the exceptional prime divisors up to biregular isomorphism (which turn out to be certain compact rational surfaces of Picard number either 2 or 4), and finally

▹ the *intersection cycles* of all intersecting pairs of exceptional prime divisors $(D_i \cdot D_j)\,|_{D_k}$, $k \in \{i, j\}$, as *divisors* on D_k (cf. [34], [35]), though we are primarily interested in their underlying topological spaces (see below lemma 2.3).

The interplay of local and global data (simultaneous blow-ups, strict transfoms after each step etc.) will be explained explicitly only for types $\mathbf{A}_n, \mathbf{D}_4, \mathbf{E}_6$. (For reasons of economy, further details -in this connection- about the other types will be omitted. The not so difficult verification of the way one builds the corresponding intersection graphs step by step is left to the reader).

(i) Type $\mathbf{A}_1$. Blowing up the origin once, we achieve immediately the required desingularization. The exceptional prime divisor

$$\mathcal{E}_f \cong \left\{ (t_1 : t_2 : t_3 : t_4) \in\ \mathbb{P}^3_{\mathbb{C}}\ \mid\ t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0 \right\}$$

is biregularly isomorphic to $\left\{ (t_1' : t_2' : t_3' : t_4') \in \mathbb{P}^3_{\mathbb{C}}\ \mid\ t_1'\, t_2' - t_3'\, t_4' = 0 \right\}\ =\ \mathrm{Im}(\gamma)\,,$ where γ denotes the Segre embedding

$$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \ni \left((\varpi_1 : \varpi_2)\,, (\varpi_1' : \varpi_2') \right) \overset{\gamma}{\longmapsto} (z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3_{\mathbb{C}}$$

with

$$\begin{cases} z_1 = \varpi_1 \varpi_1', \ z_2 = \varpi_1 \varpi_2', \ z_3 = \varpi_2 \varpi_1', \ z_4 = \varpi_2 \varpi_2', \\ t_1' = z_1, t_2' = z_4, \ t_3' = z_2, \ t_4' = z_3. \end{cases}$$

Indeed, defining δ to be the biregular isomorphism

$$(t_1' : t_2' : t_3' : t_4') \overset{\delta}{\longmapsto} (t_1 - \sqrt{-1}\,t_2 : t_1 + \sqrt{-1}\,t_2 : t_3 - \sqrt{-1}\,t_4 : -(t_3 + \sqrt{-1}\,t_4)),$$

we obtain $\delta(\mathrm{Im}(\gamma)) = \mathcal{E}_f$. Consequently, $\mathcal{E}_f \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and has conormal bundle $\mathcal{O}_{\mathcal{E}_f}(1,1)$.

(ii) Type $\mathbf{A}_2$. Blowing up the origin once, $\mathbf{Bl}_{\mathbf{0}}(X_f)$ is smooth (as threefold), though

$$\mathcal{E}_f = \left\{ (\mathbf{0}, (t_1 : t_2 : t_3 : t_4)) \in \ \{\mathbf{0}\} \times \mathbb{P}^3_{\mathbb{C}} \ \mid \ t_2^2 + t_3^2 + t_4^2 = 0 \right\} \subset \mathbf{Bl}_{\mathbf{0}}(X_f)$$

(as surface on the threefold $\mathbf{Bl}_{\mathbf{0}}(X_f)$) has a singular, ordinary double point at $q = (\mathbf{0}, (1:0:0:0))$ in $\mathcal{E}_f\,|_{U_1}$. For this reason, in order to form an snc-resolution of the original singularity, we have to blow-up once more our threefold at q and consider

$$\varphi : \widetilde{X} = \mathbf{Bl}_q(\mathbf{Bl}_{\mathbf{0}}(X_f)) \longrightarrow X_f \ .$$

The new exceptional prime divisor is obviously a $\mathbb{P}^2_{\mathbb{C}}$, while the strict transform of the old one is nothing but the (2-dimensional) blow-up of $\mathcal{E}_f$ at q. Since $\mathcal{E}_f$ can be viewed as the projective cone $\subset \mathbb{P}^3_{\mathbb{C}}$ over the smooth quadratic hypersurface $V = \left\{ (t_2 : t_3 : t_4) \in \mathbb{P}^2_{\mathbb{C}} \ \mid \ t_2^2 + t_3^2 + t_4^2 = 0 \right\}$ with $(1:0:0:0)$ as its vertex, blowing up $(1:0:0:0)$, we obtain a ruled (compact) surface over $V \cong \mathbb{P}^1_{\mathbb{C}}$ having the inverse image of $(1:0:0:0)$ as a section C_0 with self-intersection $\mathsf{C}_0^2 = -2$ (see Hartshorne [21, V.2.11.4, pp. 374-375]). Hence, the strict transform of $\mathcal{E}_f$ under φ has to be the rational ruled surface $\mathbb{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2))$ (because $\mathbb{F}_2$ is the unique $\mathbb{P}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$ having an irreducible curve of self-intersection -2, cf. [19, p. 519]).

Remark 2.2 Among the three-dimensional **A**-**D**-**E**'s, type $\mathbf{A}_2$, and, in general, type $\mathbf{A}_n$, n even, constitutes the only exception in which one has to blow up a *smooth* threefold point at the last step to ensure an snc-resolution. In all the other cases the snc-condition will be present immediately after the last blow-ups of singular points (becoming clear from the LR-diagrams which have only $\mathbf{A}_1$'s at their last but one ends).

(iii) Types $\mathbf{A}_n$, $n \geq 3$. The LR-diagram for these types depends on the (mod 2)-behaviour of n, and the number of the required blow-ups equals $m := \left\lfloor \frac{n+2}{2} \right\rfloor$.

$$\boxed{\begin{array}{l} \mathbf{A}_n \to \mathbf{A}_{n-2} \to \mathbf{A}_{n-4} \to \cdots \to \mathbf{A}_3 \to \mathbf{A}_1 \to \mathbf{A}_0 \quad (\text{if } \ n \equiv 1 (\mathrm{mod}\ 2)) \\[2ex] \mathbf{A}_n \to \mathbf{A}_{n-2} \to \mathbf{A}_{n-4} \to \cdots \to \mathbf{A}_2 \to \mathbf{A}_0 \to \mathbf{A}_0 \quad (\text{if } \ n \equiv 0 (\mathrm{mod}\ 2)) \end{array}}$$

($\mathbf{A}_0$ stands for a "smooth chart" on the threefold). But $\varphi : \widetilde{X} \to X_f$ is decomposed also globally into m blow-ups

$$\begin{array}{r} \widetilde{X} = \mathbf{Bl}_{q_m}(\mathbf{Bl}_{q_{m-1}}(\cdots(\mathbf{Bl}_{q_1}(X_f)))) \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_3} \mathbf{Bl}_{q_2}(\mathbf{Bl}_{q_1}(X_f)) \xrightarrow{\pi_2} \mathbf{Bl}_{q_1}(X_f) \\ {\scriptstyle \pi_1 = \pi} \downarrow \\ X_f \end{array}$$

of m points $q_1 = \mathbf{0}$, $q_2 = (\mathbf{0}, (1:0:0:0)), \ldots, q_m$, and is endowed with the "separation property". By this we mean that, if $E_1 = \mathcal{E}_f, E_2, \ldots, E_m$ are the exceptional loci of $\pi_1, \pi_2, \ldots, \pi_m$, respectively, then for $i \geq 2$ a singular point q_i is resolved by π_i and the (possibly existing) new singular point q_{i+1} is *not* contained in the strict transforms of $E_1, E_2, \ldots, E_{i-1}$ under π_i. Thus, defining D_i to be the strict transform of E_i under $\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_{m-1} \circ \pi_m$ on $\widetilde{X}$, we obtain an intersection graph of the form:

Case $\mathbf{A}_n$.

It is clear by **(i)** and **(ii)** that $D_m \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, for $n \equiv 1 (\mathrm{mod}\ 2)$, and $D_m \cong \mathbb{P}^2_{\mathbb{C}}$, for $n \equiv 0 (\mathrm{mod}\ 2)$, while $D_j \cong \mathbb{F}_2$ for all j, $1 \leq j \leq m-1$. The Picard group $\mathrm{Pic}(\mathbb{F}_2) \cong \mathbb{Z}^2$ of each $\mathbb{F}_2$ is generated by two projective lines: a fiber f and a section C_0 with $\mathsf{C}_0^2 = -2$. The intersection cycles read as follows:

$$(D_j \cdot D_{j+1})\,\big|_{D_j} = \mathsf{C}_0,\ (D_j \cdot D_{j+1})\,\big|_{D_{j+1}} \sim \mathsf{C}_0 + 2\,\mathsf{f},\ \ \forall j,\ 1 \leq j \leq m-2\ ,$$

and

$$(D_{m-1} \cdot D_m)\,\big|_{D_{m-1}} = \mathsf{C}_0,\ (D_{m-1} \cdot D_m)\,|_{D_m} \sim \begin{cases} \mathsf{H}_1 + \mathsf{H}_2, & \text{if } n \equiv 1 (\mathrm{mod}\ 2) \\ 2\,\mathsf{H}, & \text{if } n \equiv 0 (\mathrm{mod}\ 2) \end{cases}$$

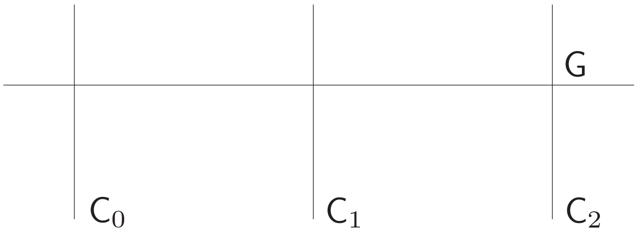
where $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(\mathsf{H}) = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ in $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})$, and

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}}(\mathsf{H}_1) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}}(1,0), \qquad \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}}(\mathsf{H}_2) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}}(0,1)$$

in $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$. (We shall keep the notation below whenever the arising exceptional prime divisors are biregularly isomorphic to $\mathbb{F}_2$ or to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$). Obviously, $(\mathsf{H} \cdot \mathsf{H})\,\Big|_{\mathbb{P}^2_{\mathbb{C}}} = (\mathsf{H}_1 \cdot \mathsf{H}_2)\,\Big|_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}} = 1$ and $(\mathsf{H}_1 \cdot \mathsf{H}_1)\,\Big|_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}} = (\mathsf{H}_2 \cdot \mathsf{H}_2)\,\Big|_{\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}} = 0$.

▶ **Three characteristic rational surfaces.** The remaining types **D**-**E** of singularities $(X_f, \mathbf{0})$ are more complicated as the φ's under construction will not fulfil the above "separation property". Furthermore, since the exceptional locus after the first blow-up consists of two irreducible components $\mathcal{E}'_f$ and $\mathcal{E}''_f$, and the appearing new singular points (3 in case $\mathbf{D}_4$, 2 in case $\mathbf{D}_n$, $n \geq 5$, and 1 in cases $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$) lie on the line $\mathcal{G} = \mathcal{E}'_f \cap \mathcal{E}''_f$, the strict transforms of $\mathcal{G}$ together with their intersections with other components (due to the next desingularization steps) will accompany us until we arrive at $\widetilde{X}$. In addition, to ensure a uniform resolution procedure from the "global" point of view, one has to blow up the new singularities *simultaneously* (in each step) and take into account the related intrinsic geometry. That's why, before proceeding to the examination of the remaining cases, we define three rational compact complex surfaces which will appear in a natural way as exceptional prime divisors of our φ's. (In fact, they will be inherited from the strict transforms of the original $\mathcal{E}'_f$ and $\mathcal{E}''_f$ as well as from the other intermediate components which arise on one's way on the "surface level".)

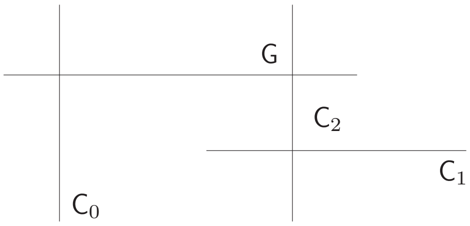
▷ Let $\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]$ be the surface resulting after the blow-up $\mathbf{Bl}_{\{q_0,q_1,q_2\}}(\mathbb{P}^2_{\mathbb{C}})$ of $\mathbb{P}^2_{\mathbb{C}}$ simultaneously at three different points q_0, q_1, q_2 of a line $\mathcal{G} \subset \mathbb{P}^2_{\mathbb{C}}$. (This surface is unique up to biregular isomorphism, because for any other triple q_0', q_1', q_2' of different points of a line $\mathcal{G}' \subset \mathbb{P}^2_{\mathbb{C}}$ the linear isomorphism $\mathcal{G} \overset{\cong}{\to} \mathcal{G}'$ mapping q_i to $q_i', i = 1, 2, 3$, can be extended to an isomorphism $\mathbb{P}^2_{\mathbb{C}} \overset{\cong}{\to} \mathbb{P}^2_{\mathbb{C}}$). If we denote by C_i the inverse image of q_i in $\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]$, then $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]) \cong \mathbb{Z}^4$ with $\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}\}$ as generating system, where G is the strict transform of the original line $\mathcal{G}$. Topologically $\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}\}$ looks like:



The intersection numbers of these generators on $\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]$ are the following:

$$\begin{cases} \mathsf{C}_0^2 = \mathsf{C}_1^2 = \mathsf{C}_2^2 = -1, \mathsf{G}^2 = -2, \\ (\mathsf{G} \cdot \mathsf{C}_0) = (\mathsf{G} \cdot \mathsf{C}_1) = (\mathsf{G} \cdot \mathsf{C}_2) = 1 \\ (\text{and } = 0 \text{ otherwise}) \end{cases}$$

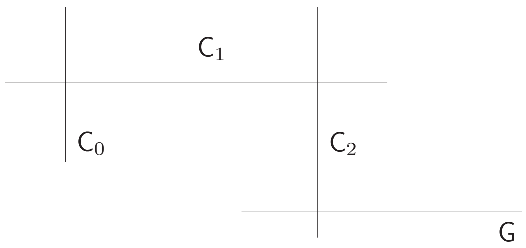
▹ Let now $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]$ be the surface $\mathbf{Bl}_{\{q_2\}}(\mathbf{Bl}_{\{q_0,q_1\}}(\mathbb{P}^2_{\mathbb{C}}))$ being constructed by simultaneously blowing-up of $\mathbb{P}^2_{\mathbb{C}}$ at two different points q_0, q_1, followed by the blow-up at the intersection point q_2 of the strict transform of $\overline{q_0\, q_1}$ and the blow-up of q_1 on $\mathbf{Bl}_{\{q_0,q_1\}}(\mathbb{P}^2_{\mathbb{C}})$. (The isomorphism type of $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]$ is unique, and one can use arbitrary points $q_0 \neq q_1$ for the construction). If we denote by G the strict transform of $\overline{q_0\, q_1}$, by C_i the strict transform of $q_i, i \in \{0, 1\}$, and by C_2 the blow-up of q_2 within $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]$, then $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]) \cong \mathbb{Z}^4$ with $\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}\}$ as generating system:



and intersection numbers:

$$\begin{cases} \mathsf{C}_0^2 = \mathsf{C}_2^2 = -1, \mathsf{C}_1^2 = \mathsf{G}^2 = -2, \\ (\mathsf{G} \cdot \mathsf{C}_0) = (\mathsf{G} \cdot \mathsf{C}_2) = (\mathsf{C}_1 \cdot \mathsf{C}_2) = 1 \\ (\text{and } = 0 \text{ otherwise}) \end{cases}$$

▹ Finally, let $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]$ denote the surface $\mathbf{Bl}_{\{q_2\}}(\mathbf{Bl}_{\{q_1\}}(\mathbf{Bl}_{\{q_0\}}(\mathbb{P}^2_{\mathbb{C}})))$ determined by blowing up a point q_0 of $\mathbb{P}^2_{\mathbb{C}}$, taking a line $\mathcal{G} \subset \mathbb{P}^2_{\mathbb{C}}$, with $q_0 \in \mathcal{G}$, such that (strict transform of $\mathcal{G}) \cap \mathbf{Bl}_{\{q_0\}}(\mathbb{P}^2_{\mathbb{C}}) = \{q_1\}$, blowing up in turn q_1, and blowing up (at the last step) q_2, where (strict transform of $\mathcal{G}) \cap \mathbf{Bl}_{\{q_1\}}(\mathbf{Bl}_{\{q_0\}}(\mathbb{P}^2_{\mathbb{C}})) = \{q_2\}$. The isomorphism type of $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]$ is again unique, $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]) \cong \mathbb{Z}^4$ is generated by $\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}\}$, where G is the (final) strict transform of $\mathcal{G}$, C_i the strict transform of $q_i, i \in \{0, 1\}$, and C_2 the blow-up of q_2 within $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]$. Topologically $\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}\}$ looks like:



and the corresponding intersection numbers equal:

$$\begin{cases} \mathsf{C}_2^2 = -1, \mathsf{C}_0^2 = \mathsf{C}_1^2 = \mathsf{G}^2 = -2, \\ (\mathsf{G}\cdot\mathsf{C}_2) = (\mathsf{C}_0\cdot\mathsf{C}_1) = (\mathsf{C}_1\cdot\mathsf{C}_2) = 1 \\ (\text{and } = 0 \text{ otherwise}) \end{cases}$$

(iv) Types $\mathbf{D}_n$ for $n = 2k,\ k \geq 2$. Let us first explain what happens in the $\mathbf{D}_4$-case. Blowing up the origin $\mathbf{0} \in X_f$ we get

$$\mathbf{Bl}_{\mathbf{0}}(X_f) = \left\{ \left((x_1, .., x_4), (t_1 : .. : t_4) \right) \in \ \mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4) \,\middle|\, x_1\, t_1^2 + x_1\, t_2^2 + t_3^2 + t_4^2 = 0 \right\}$$

with $\mathcal{E}_f = \mathcal{E}'_f \cup \mathcal{E}''_f$ as exceptional locus. As we have already mentioned above, $\mathbf{Bl}_{\mathbf{0}}(X_f)$ possess the three $\mathbf{A}_1$-singularities

$$q_0 = (\mathbf{0}, (0:1:0:0))\,,\ \ q_1 = (\mathbf{0}, (\sqrt{-1}:1:0:0)),\ \ q_2 = (\mathbf{0}, (-\sqrt{-1}:1:0:0)),$$

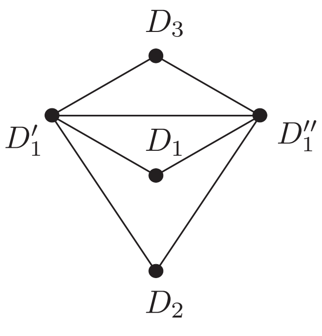
which belong to the line $\mathcal{G} = \mathcal{E}'_f \cap \mathcal{E}''_f$. To obtain our global desingularization $\varphi : \widetilde{X} \longrightarrow X_f$ it is enough to blow up once more all three points q_0, q_1, q_2 simultaneously:

$$\widetilde{X} = \mathbf{Bl}_{\{q_0,q_1,q_2\}}(\mathbf{Bl}_{\mathbf{0}}(X_f)) \xrightarrow{\pi_2} \mathbf{Bl}_{\mathbf{0}}(X_f) \xrightarrow{\pi_1 = \pi} X_f\ .$$

Let us denote by D'_1 (resp. D''_2) the strict transform of $\mathcal{E}'_f$ (resp. $\mathcal{E}''_f$) under π_2, $D_3 = \pi_2^{-1}(q_0)\,,\, D_j = \pi_2^{-1}(q_j)\,$, for $j \in \{1,2\}$, and define

$$\mathsf{C}_i := \pi_2^{-1}\left|_{D'_1 \text{ (resp. } D''_1)}\right. (q_i),\quad i \in \{0,1,2\}\ .$$

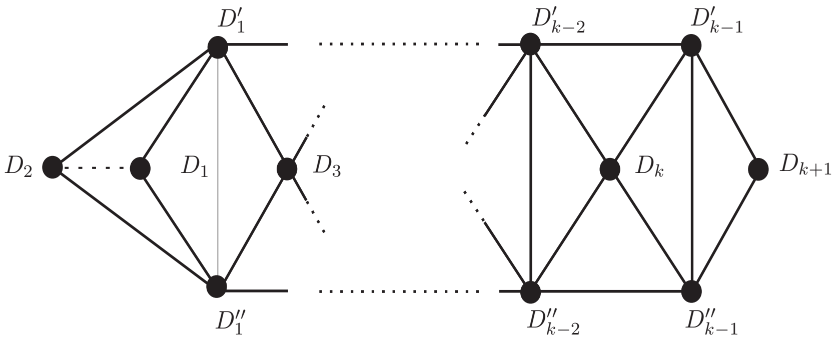
Then obviously $D_1 \cong D_2 \cong D_3 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $D'_1 \cong D''_1 \cong \mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]$ with Picard group generated by $\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2$ and G, where G is the strict transform of $\mathcal{G}$ under π_2. The intersection graph of these five exceptional divisors is illustrated as follows:



Generalizing to $\mathbf{D}_{2k}$, the LR-diagram has the form:

$$\boxed{\begin{array}{ccccccccccccc}
 & & & & & & & & \mathbf{A}_0 & & & & \\
 & & & & & & & & \uparrow & & & & \\
 & & & & & & & & \mathbf{A}_1 & & & & \\
 & & & & & & & & \uparrow & & & & \\
\mathbf{D}_{2k} & \rightarrow & \mathbf{D}_{2(k-1)} & \rightarrow & \cdots & \rightarrow & \mathbf{D}_6 & \rightarrow & \mathbf{D}_4 & \rightarrow & \mathbf{A}_1 & \rightarrow & \mathbf{A}_0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
\mathbf{A}_1 & & \mathbf{A}_1 & & & & \mathbf{A}_1 & & \mathbf{A}_1 & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
\mathbf{A}_0 & & \mathbf{A}_0 & & & & \mathbf{A}_0 & & \mathbf{A}_0 & & & &
\end{array}}$$

with a D_4 at its right-hand side and the intersection graph looks like:



Case $\mathbf{D}_n$.

(The dotted line from D_2 to D_1 will be used only for the case of odd n's and it should be ignored for the time being). The ordering of the subscripts of the divisors of the top and the bottom row is $1, 2, ..., k-2, k-1$, whereas that of the divisors of the middle row is $2, 1, 3, 4, ..., k, k+1$. In this general case one needs altogether $k+1$ global (= simultaneous) blow-ups to construct $\varphi : \widetilde{X} \longrightarrow X_f$. The exceptional prime divisors which occur are $D_j \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $\forall j,\ 1 \leq j \leq k+1$, and

$$D_1' \cong D_1'' \cong \mathbb{P}^2_{\mathbb{C}}[\mathbf{3}], \qquad D_j' \cong D_j'' \cong \mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,], \quad \forall j, \ \ 2 \leq j \leq k-1 \ ,$$

with the $k+1$ $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$'s coming from the $\mathbf{A}_1$'s of the LR-diagram, and the $k-2$ pairs of $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]$'s inherited from the strict transforms of the $\mathcal{E}_f'$ and $\mathcal{E}_f''$ with respect to the first $k-2$ global blow-ups (where in each step the singularities appear pairwise). The corresponding intersection cycles are:

$$
\begin{array}{ll}
(D_1 \cdot D_1')\,\big|_{D_1} = \mathsf{H}_2, & (D_1 \cdot D_1')\,\big|_{D_1'} = \mathsf{C}_1, \\
(D_1 \cdot D_1'')\,\big|_{D_1} = \mathsf{H}_1, & (D_1 \cdot D_1'')\,\big|_{D_1''} = \mathsf{C}_1, \\
(D_2 \cdot D_1')\,\big|_{D_2} = \mathsf{H}_2, & (D_2 \cdot D_1')\,\big|_{D_1'} = \mathsf{C}_2, \\
(D_2 \cdot D_1'')\,\big|_{D_2} = \mathsf{H}_1, & (D_2 \cdot D_1'')\,\big|_{D_1''} = \mathsf{C}_2, \\
(D_{k+1} \cdot D_{k-1}')\,\big|_{D_{k+1}} = \mathsf{H}_1, & (D_{k+1} \cdot D_{k-1}')\,\big|_{D_{k-1}'} = \mathsf{C}_0, \\
(D_{k+1} \cdot D_{k-1}'')\,\big|_{D_{k+1}} = \mathsf{H}_2, & (D_{k+1} \cdot D_{k-1}'')\,\big|_{D_{k-1}''} = \mathsf{C}_0,
\end{array}
$$

while for $k \geq 3$, and all j, $3 \leq j \leq k$,

$$
\begin{array}{ll}
(D_j \cdot D_{j-1}')\,\big|_{D_j} = \mathsf{H}_2, & (D_j \cdot D_{j-1}')\,\big|_{D_{j-1}'} = \mathsf{C}_2, \\
(D_j \cdot D_{j-2}')\,\big|_{D_j} = \mathsf{H}_1, & (D_j \cdot D_{j-2}')\,\big|_{D_{j-2}'} = \mathsf{C}_0, \\
(D_j \cdot D_{j-1}'')\,\big|_{D_j} = \mathsf{H}_1, & (D_j \cdot D_{j-1}'')\,\big|_{D_{j-1}''} = \mathsf{C}_2, \\
(D_j \cdot D_{j-2}'')\,\big|_{D_j} = \mathsf{H}_2, & (D_j \cdot D_{j-2}'')\,\big|_{D_{j-2}''} = \mathsf{C}_0, \\
(D_1' \cdot D_2')\,\big|_{D_1'} \sim \mathsf{G} + \mathsf{C}_1 + \mathsf{C}_2, & (D_1' \cdot D_2')\,\big|_{D_2'} = \mathsf{C}_1, \\
(D_1'' \cdot D_2'')\,\big|_{D_1''} \sim \mathsf{G} + \mathsf{C}_1 + \mathsf{C}_2, & (D_1'' \cdot D_2'')\,\big|_{D_2''} = \mathsf{C}_1,
\end{array}
$$

and for all j, $2 \leq j \leq k-2$,

$$
(D_j'^{\,(\prime\prime)} \cdot D_{j+1}'^{\,(\prime\prime)})\,\Big|_{D_j'^{\,(\prime\prime)}} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_j'^{\,(\prime\prime)} \cdot D_{j+1}'^{\,(\prime\prime)})\,\Big|_{D_{j+1}'^{\,(\prime\prime)}} = \mathsf{C}_1.
$$

and finally, for all j, $1 \leq j \leq k-1$,

$$
(D_j' \cdot D_j'')\,\big|_{D_j'} = \mathsf{G}, \qquad (D_j' \cdot D_j'')\,\big|_{D_j''} = \mathsf{G}.
$$

(v) Types $\mathbf{D}_n$ for $n = 2k+1$. The LR-diagram in this case reads as follows:

$$
\boxed{
\begin{array}{ccccccccccccc}
\mathbf{D}_{2k+1} & \rightarrow & \mathbf{D}_{2(k-1)+1} & \rightarrow & \cdots & \rightarrow & \mathbf{D}_5 & \rightarrow & \mathbf{A}_3 & \rightarrow & \mathbf{A}_1 & \rightarrow & \mathbf{A}_0 \\
\downarrow & & \downarrow & & & & \downarrow & & & & & & \\
\mathbf{A}_1 & & \mathbf{A}_1 & & & & \mathbf{A}_1 & & & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & & & \\
\mathbf{A}_0 & & \mathbf{A}_0 & & & & \mathbf{A}_0 & & & & & &
\end{array}
}
$$

Up to the introduction of the extra dotted edge into the game, the intersection diagram remains the same, and the exceptional prime divisors are

$$
D_1 \cong \mathbb{F}_2\,, \quad D_j \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \quad \forall j, \quad 2 \leq j \leq k+1,
$$

and

$$
D_j' \cong D_j'' \cong \mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,], \quad \forall j, \quad 1 \leq j \leq k-1\ .
$$

Moreover, the intersection cycles are identical with those we have encountered before in **(iv)**, up to the following ones:

$$(D_1 \cdot D_2)\,|_{D_1} = \mathsf{C}_0, \quad (D_1 \cdot D_2)\,|_{D_2} \sim \mathsf{H}_1 + \mathsf{H}_2, \quad (D_1 \cdot D_1')\,|_{D_1} = \mathsf{f}, \quad (D_1 \cdot D_1')\Big|_{D_1'} = \mathsf{C}_1,$$

$$(D_1 \cdot D_1'')\,|_{D_1} = \mathsf{f}', \qquad (D_1 \cdot D_1'')\Big|_{D_1''} = \mathsf{C}_1, \qquad (\mathsf{f} \underset{\text{set th.}}{\neq} \mathsf{f}' \text{ fibers of } \mathbb{F}_2)$$

$$(D_1'^{(\prime\prime)} \cdot D_2'^{(\prime\prime)})\Big|_{D_1'^{(\prime\prime)}} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \qquad (D_1'^{(\prime\prime)} \cdot D_2'^{(\prime\prime)})\Big|_{D_2'^{(\prime\prime)}} = \mathsf{C}_1,$$

(vi) Type $\mathbf{E}_6$. The LR-diagram in this case reads as:

$$\boxed{\mathbf{E}_6 \rightarrow \mathbf{A}_5 \rightarrow \mathbf{A}_3 \rightarrow \mathbf{A}_1 \rightarrow \mathbf{A}_0}$$

Globally, the desingularization procedure is described as follows. To obtain the morphism $\varphi : \widetilde{X} \longrightarrow X_f$, we need 3 additional blow-ups at three points q_0, q_1, q_2 *after* $\mathbf{Bl}_{\mathbf{0}}(X_f) \stackrel{\pi}{\longrightarrow} X_f$, i.e.,

$$\begin{array}{l} \mathbf{Bl}_{q_1}(\mathbf{Bl}_{q_0}(\mathbf{Bl}_{\mathbf{0}}(X_f))) \stackrel{\pi_2}{\longrightarrow} \mathbf{Bl}_{q_0}(\mathbf{Bl}_{\mathbf{0}}(X_f)) \stackrel{\pi_1}{\longrightarrow} \mathbf{Bl}_{\mathbf{0}}(X_f) \stackrel{\pi_0=\pi}{\longrightarrow} X_f \\ \uparrow \pi_3 \\ \widetilde{X} = \mathbf{Bl}_{q_2}(\mathbf{Bl}_{q_1}(\mathbf{Bl}_{q_0}(\mathbf{Bl}_{\mathbf{0}}(X_f)))) \end{array}$$

where $q_0 = (\mathbf{0}, (0:1:0:0)) \in U_2$ on

$$\mathbf{Bl}_{\mathbf{0}}(X_f) = \left\{ ((x_1, .., x_4), (t_1 : t_2 : t_3 : t_4)) \in \ \mathbf{Bl}_{\mathbf{0}}(\mathbb{C}^4) \ \mid \ x_1\, t_1^2 + x_2^2\, t_2^2 + t_3^2 + t_4^2 = 0 \right\}.$$

Analogously, one gets $q_1 = (\mathbf{0}, (0:1:0:0))$ on $\mathbf{Bl}_{q_0}(\mathbf{Bl}_{\mathbf{0}}(X_f)\,|_{U_2})$, which equals

$$\left\{ ((y_{2,1}, .., y_{2,4}), (\lambda_1 : .. : \lambda_4)) \in \ U_2 \times \mathbb{P}^3_{\mathbb{C}} \ \mid \ (y_{2,1})^2\ \lambda_1\lambda_2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 0 \right\}$$

(and similarly for $q_2 \in \mathbf{Bl}_{q_1}(\mathbf{Bl}_{q_0}(\mathbf{Bl}_{\mathbf{0}}(X_f)\,|_{U_2}))$ in the last step). The point q_0 belongs to the line $\mathcal{G} = \mathcal{E}_f' \cap \mathcal{E}_f''$ (where, as usual, $\pi^{-1}(\mathbf{0}) = \mathcal{E}_f' \cup \mathcal{E}_f''$) and $(\mathbf{Bl}_{\mathbf{0}}(X_f), q_0)$ is an $\mathbf{A}_5$-singularity. According to **(iii)**, this will be resolved by $\pi_1 \circ \pi_2 \circ \pi_3$ to give two $\mathbb{F}_2$'s and one $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ as exceptional divisors. More precisely,

$$q_1 \in (\text{strict transform of } \mathcal{G} \text{ under } \pi_1) \cap (\text{exceptional locus of } \pi_1)$$

is the new, $\mathbf{A}_3$-singularity, while

$$q_2 \in (\text{strict transform of } \mathcal{G} \text{ under } \pi_1 \circ \pi_2) \smallsetminus \begin{pmatrix} \text{strict transform} \\ \text{of the exceptional} \\ \text{locus of } \pi_1 \text{ under } \pi_2 \end{pmatrix}$$

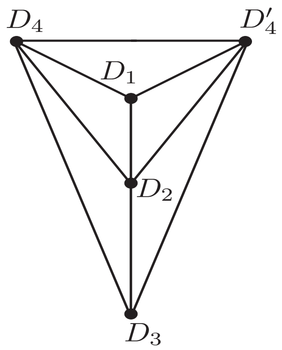
is the final $\mathbf{A}_1$-singularity. Let us denote by D_1 the strict transform of the exceptional locus of π_1 under $\pi_2 \circ \pi_3$, by D_2 the strict transform of the exceptional locus of π_2 under π_3, by D_3 the exceptional locus of π_3, and finally by D_4 (resp. D_4', G) the strict transform of the original $\mathcal{E}_f'$ (resp. $\mathcal{E}_f'', \mathcal{G}$) under $\pi_1 \circ \pi_2 \circ \pi_3$, and define

$$\begin{cases} \mathsf{C}_0 := (\text{strict transform of } q_0 \text{ under } \pi_1 \circ \pi_2 \circ \pi_3 \text{ on } D_4 \ \ (\text{resp. } D_4')) \\ \mathsf{C}_1 := (\text{strict transform of } q_1 \text{ under } \pi_2 \circ \pi_3 \text{ on } D_4 \ \ (\text{resp. } D_4')) \\ \mathsf{C}_2 := (\text{the blow-up of } q_2 \text{ by } \pi_3 \text{ on } D_4 \ \ (\text{resp. } D_4')) \ . \end{cases}$$

Then

$$D_1 \cong D_2 \cong \mathbb{F}_2, \quad D_3 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \quad D_4 \cong D_4' \cong \mathbb{P}^2_{\mathbb{C}}[\overline{\overline{\mathbf{3}}}],$$

with $\mathrm{Pic}(D_4)$ (resp. $\mathrm{Pic}(D_4')$) generated by $\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{G}$, intersection graph



Case $\mathbf{E}_6$.

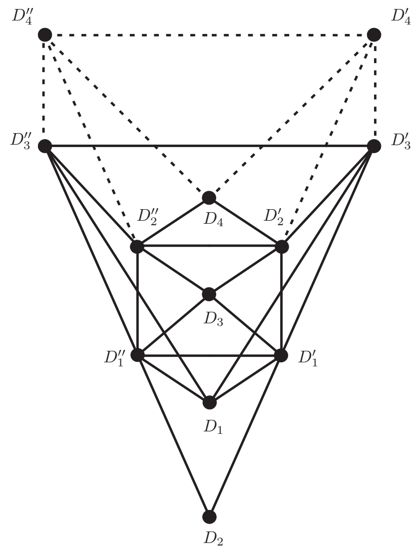
and intersections cycles:

$$(D_1 \cdot D_2)\,|_{D_1} = \mathsf{C}_0, \quad (D_1 \cdot D_2)\,|_{D_2} \sim \mathsf{C}_0 + 2\mathsf{f},$$
$$(D_1 \cdot D_4)\,|_{D_1} = \mathsf{f}, \qquad (D_1 \cdot D_4)\,|_{D_4} = \mathsf{C}_0,$$
$$(D_1 \cdot D_4')\,|_{D_1} = \mathsf{f}', \qquad (D_1 \cdot D_4')\,\Big|_{D_4'} = \mathsf{C}_0,$$
$$(D_2 \cdot D_3)\,|_{D_2} = \mathsf{C}_0, \quad (D_2 \cdot D_3)\,|_{D_3} \sim \mathsf{H}_1 + \mathsf{H}_2,$$
$$(D_2 \cdot D_4)\,|_{D_2} = \mathsf{f}, \quad (D_2 \cdot D_4)\,|_{D_4} = \mathsf{C}_1,$$
$$(D_2 \cdot D_4')\,|_{D_2} = \mathsf{f}', \quad (D_2 \cdot D_4')\,\Big|_{D_4'} = \mathsf{C}_1,$$
$$(D_3 \cdot D_4)\,|_{D_3} = \mathsf{H}_1, \quad (D_3 \cdot D_4)\,|_{D_4} = \mathsf{C}_2,$$
$$(D_3 \cdot D_4')\,|_{D_3} = \mathsf{H}_2, \quad (D_3 \cdot D_4')\,\Big|_{D_4'} = \mathsf{C}_2,$$
$$(D_4 \cdot D_4')\,|_{D_4} = \mathsf{G}, \quad (D_4 \cdot D_4')\,\Big|_{D_4'} = \mathsf{G}.$$

(where $\mathsf{f} \underset{\text{set th.}}{\neq} \mathsf{f}'$ fibers of $\mathbb{F}_2$).

(vii) The cases $\mathbf{E}_7$ and $\mathbf{E}_8$. Since $\mathbf{E}_8$ passes to an $\mathbf{E}_7$ after the first blow-up, the LR-diagram looks like:

$$\boxed{\begin{tikzcd}
 & & & \mathbf{A}_0 & & \\
 & & & \mathbf{A}_1 \arrow[u] & & \\
\mathbf{E}_8 \arrow[r, dashed] & \mathbf{E}_7 \arrow[r] & \mathbf{D}_6 \arrow[r] \arrow[d] & \mathbf{D}_4 \arrow[r] \arrow[u] \arrow[d] & \mathbf{A}_1 \arrow[r] & \mathbf{A}_0 \\
 & & \mathbf{A}_1 \arrow[d] & \mathbf{A}_1 \arrow[d] & & \\
 & & \mathbf{A}_0 & \mathbf{A}_0 & &
\end{tikzcd}}$$



Cases $\mathbf{E}_7$ and $\mathbf{E}_8$.

Globally, for the resolution of $\mathbf{E}_7$- (resp. $\mathbf{E}_8$-) singularity, we need 4 (resp. 5) blow-ups. The intersection graph contains 10 (resp. 12) vertices (with the dotted edges only in the $\mathbf{E}_8$-case)
corresponding to the 12 exceptional prime divisors

$$D_1 \cong D_2 \cong D_3 \cong D_4 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}},$$

$$D_1' \cong D_1'' \cong \mathbb{P}^2_{\mathbb{C}}[\mathbf{3}], \;\; D_2' \cong D_2'' \cong \mathbb{P}^2_{\mathbb{C}}[\overline{\mathbf{3}}],$$

$$D_3' \cong D_3'' \cong D_4' \cong D_4'' \cong \mathbb{P}^2_{\mathbb{C}}[\overline{\overline{\mathbf{3}}}].$$

The "central" four $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$'s come from the four lastly appearing $\mathbf{A}_1$'s, and the four top $\mathbb{P}^2_{\mathbb{C}}[\overline{\overline{\mathbf{3}}}]$'s are due to the last three sucessive blow-ups of $\mathcal{E}_f'$ and $\mathcal{E}_f''$. The two $\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]$'s (resp. the two $\mathbb{P}^2_{\mathbb{C}}[\overline{\mathbf{3}}]$'s) are in turn inherited from the strict transforms of $\mathcal{E}_f'$ and $\mathcal{E}_f''$ after passing from $\mathbf{D}_4$ to the three $\mathbf{A}_1$'s (resp. from $\mathbf{D}_6$ to $\mathbf{D}_4$). Making use of the previously introduced notation, the intersection cycles read as follows:

$$(D_1 \cdot D_1')\,|_{D_1} = \mathsf{H}_1, \quad (D_1 \cdot D_1')\Big|_{D_1'} = \mathsf{C}_1,$$

$$(D_1 \cdot D_1'')\,|_{D_1} = \mathsf{H}_2, \quad (D_1 \cdot D_1'')\Big|_{D_1''} = \mathsf{C}_2,$$

$$(D_1 \cdot D_3')\,|_{D_1} = \mathsf{H}_2, \quad (D_1 \cdot D_3')\Big|_{D_3'} = \mathsf{C}_2,$$

$$(D_1 \cdot D_3'')\,|_{D_1} = \mathsf{H}_1, \quad (D_1 \cdot D_3'')\Big|_{D_3''} = \mathsf{C}_2,$$

$$(D_1' \cdot D_1'')\Big|_{D_1'} = \mathsf{G}, \quad (D_1' \cdot D_1'')\Big|_{D_1''} = \mathsf{G},$$

$$(D_1' \cdot D_2)\Big|_{D_1'} = \mathsf{C}_2, \quad (D_1' \cdot D_2)\,|_{D_2} = \mathsf{H}_1,$$

$$(D_1' \cdot D_2')\Big|_{D_1'} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_1' \cdot D_2')\Big|_{D_2'} = \mathsf{C}_1,$$

$$(D_1' \cdot D_3)\Big|_{D_1'} = \mathsf{C}_0, \qquad (D_1' \cdot D_3)\,|_{D_3} = \mathsf{H}_2,$$

$$(D_1' \cdot D_3')\big|_{D_1'} \sim \mathsf{G} + \mathsf{C}_0 + \mathsf{C}_2, \quad (D_1' \cdot D_3')\big|_{D_3'} = \mathsf{C}_1,$$

$$(D_1'' \cdot D_2)\big|_{D_1''} = \mathsf{C}_2, \qquad (D_1'' \cdot D_2)\,|_{D_2} = \mathsf{H}_2,$$

$$(D_1'' \cdot D_2'')\big|_{D_1''} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_1'' \cdot D_2'')\big|_{D_2''} = \mathsf{C}_1,$$

$$(D_1'' \cdot D_3)\big|_{D_1''} = \mathsf{C}_0, \qquad (D_1'' \cdot D_3)\,|_{D_3} = \mathsf{H}_1,$$

$$(D_1'' \cdot D_3'')\big|_{D_1''} \sim \mathsf{G} + \mathsf{C}_0 + \mathsf{C}_2, \quad (D_1'' \cdot D_3'')\big|_{D_3''} = \mathsf{C}_1,$$

$$(D_2' \cdot D_2'')\big|_{D_2'} = \mathsf{G}, \qquad (D_2' \cdot D_2'')\big|_{D_2''} = \mathsf{G},$$

$$(D_2' \cdot D_3')\big|_{D_2'} \sim \mathsf{G} + \mathsf{C}_0 + \mathsf{C}_2, \quad (D_2' \cdot D_3')\big|_{D_3'} = \mathsf{C}_0,$$

$$(D_2' \cdot D_3)\big|_{D_2'} = \mathsf{C}_2, \qquad (D_2' \cdot D_3)\,|_{D_3} = \mathsf{H}_1,$$

$$(D_2'' \cdot D_3'')\big|_{D_2''} \sim \mathsf{G} + \mathsf{C}_0 + \mathsf{C}_2, \quad (D_2'' \cdot D_3'')\big|_{D_3''} = \mathsf{C}_0,$$

$$(D_2'' \cdot D_3)\big|_{D_2''} = \mathsf{C}_2, \qquad (D_2'' \cdot D_3)\,|_{D_3} = \mathsf{H}_1,$$

$$(D_2' \cdot D_4)\big|_{D_2'} = \mathsf{H}_1, \quad (D_2' \cdot D_4)\,|_{D_4} = \mathsf{C}_0,$$

$$(D_2'' \cdot D_4)\big|_{D_2''} = \mathsf{C}_0, \quad (D_2'' \cdot D_4)\,|_{D_4} = \mathsf{H}_1,$$

with $(D_3' \cdot D_3'')\big|_{D_3'} = \mathsf{G}$, $(D_3' \cdot D_3'')\big|_{D_3''} = \mathsf{G}$, and

$$(D_2' \cdot D_4')\big|_{D_2'} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_2' \cdot D_4')\big|_{D_4'} = \mathsf{C}_1,$$

$$(D_4 \cdot D_4'')\,|_{D_4} = \mathsf{H}_1, \qquad (D_4 \cdot D_4'')\big|_{D_4''} = \mathsf{C}_2,$$

$$(D_2'' \cdot D_4'')\big|_{D_2''} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_2'' \cdot D_4'')\big|_{D_4''} = \mathsf{C}_1,$$

$$(D_4 \cdot D_4')\,|_{D_4} = \mathsf{H}_2, \qquad (D_4 \cdot D_4')\big|_{D_4'} = \mathsf{C}_2,$$

$$(D_3' \cdot D_4')\big|_{D_3'} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_3' \cdot D_4')\big|_{D_4'} = \mathsf{C}_0,$$

$$(D_4' \cdot D_4'')\big|_{D_4'} = \mathsf{G}, \qquad (D_4' \cdot D_4'')\big|_{D_4''} = \mathsf{G},$$

$$(D_3'' \cdot D_4'')\big|_{D_3''} \sim \mathsf{G} + \mathsf{C}_1 + 2\mathsf{C}_2, \quad (D_3'' \cdot D_4'')\big|_{D_4''} = \mathsf{C}_0,$$

where these last $2 \cdot 7$ intersections concern only the snc-resolution of the $\mathbf{E}_8$-type singularity.

Lemma 2.3 (i) *All the edges of the intersection graphs represent smooth, irreducible, rational compact complex curves.*

(ii) *Let* $\mathbf{b}(X)$ *denote the total number of the edges of the intersection graph associated to the desingularization* $\varphi : \widetilde{X} \to X_f = X$, *and let* $\mathbf{t}(X)$ *be the number of those triangles of the graph for which the corresponding three exceptional prime divisors have non-empty intersection in common. Then each of the* $\mathbf{t}(X)$ *triple* *non-empty intersections consists topologically of exactly one point. In addition,* $\mathbf{b}(X)$ *and* $\mathbf{t}(X)$ *take the following values*:

Types	$\mathbf{b}(X)$	$\mathbf{t}(X)$
$\mathbf{A}_n$ (n odd)	$m-1\ (=\frac{n-1}{2})$	0
$\mathbf{A}_n$ (n even)	$m-1\ (=\frac{n}{2})$	0
$\mathbf{D}_{2k}$	$7(k-1)$	$3+4(k-2)$
$\mathbf{D}_{2k+1}$	$7k-6$	$4+4(k-2)$
$\mathbf{E}_6$	9	5
$\mathbf{E}_7$	21	12
$\mathbf{E}_8$	28	17

(iii) *In all the cases, there are no four exceptional prime divisors having non-empty intersection in common.*

Proof. (i) The underlying topological spaces of all divisors H, H_1, H_2, f, f', C_0, C_1, C_2, G are in all the cases homeomorphic to $\mathbb{P}^1_{\mathbb{C}}$. But also all the other divisors $(D_i \cdot D_j)\,|_{D_k}$, $k \in \{i,j\}$, for which we gave (just for geometric reasons and completeness' sake) certain expressions in terms of the generators of $\mathrm{Pic}(D_k)$ up to linear equivalence '$\sim$', are actually lines (living on D_k and being strict transforms of other lines which are intersections of the exceptional divisors with affine patches in the previous steps). Therefore they have underlying topological spaces homeomorphic to $\mathbb{P}^1_{\mathbb{C}}$. (It is better to compare with the corresponding intersections $(D_i \cdot D_j)\,|_{D_{\{i,j\}\smallsetminus\{k\}}}$ for a quick check!)

(ii) We find $\mathbf{b}(X)$ by simply counting all the edges of each of our graphs. The graph for type $\mathbf{A}_n$ contains no triangles. For the remaining types $\mathbf{D}_{2k}$, $\mathbf{D}_{2k+1}$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, the intersection graphs contain $3+4(k-2)$, $5+4(k-2)$, 7, 12 and 17 triangles, respectively, whose vertices are the only graph-vertices lying on their boundaries. Using the explicitly just described behaviour of the intersections between the corresponding exceptional prime divisors, one verifies easily that the number $\mathbf{t}(X)$ equals $3+4(k-2)$, $4+4(k-2)$, 5, 12 and 17, respectively. The only triangles which have to be excluded are those associated to $D_1 \cap D_1' \cap D_1'' = \varnothing$ (for type $\mathbf{D}_{2k+1}$) and to $D_1 \cap D_4 \cap D_4' = D_2 \cap D_4 \cap D_4' = \varnothing$ (for type $\mathbf{E}_6$), and each triple non-empty intersection consists obviously of exactly one point.

(iii) Examining each (not necessarily convex or non-degenerate) quadrilateral of the intersection graphs (with no interior points in its edges), we obtain by the above given data: $D_i \cap D_j \cap D_k \cap D_l = \varnothing$, for all possible pairwise distinct indices i,j,k,l. □

Lemma 2.4 (i) *The E-polynomials of* $\mathbb{F}_2$ *and* $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ *are equal*:

$$E(\mathbb{F}_2; u,v) = E(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}; u,v) = 1 + 2\,uv + (uv)^2 = (1+uv)^2 \tag{2.2}$$

(ii) $\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}], \mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]$ *and* $\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]$ *have identical E-polynomials, with*

$$E(\mathbb{P}^2_{\mathbb{C}}[\mathbf{3}]; u,v) = E(\mathbb{P}^2_{\mathbb{C}}[\,\overline{\mathbf{3}}\,]; u,v) = E(\mathbb{P}^2_{\mathbb{C}}[\,\overline{\overline{\mathbf{3}}}\,]; u,v) = 1 + 4\,uv + (uv)^2 \tag{2.3}$$

Proof. (i) is obvious. (For the fibration $\mathbb{F}_2 \to \mathbb{P}^1_{\mathbb{C}}$ one may use directly (1.3)). (ii) follows easily from the fact that the E-polynomial of a non-singular surface increases by uv after a blow-up (cf. (1.4)). $\square$

3 Computing the discrepancy coefficients

This section is devoted to the exact computation of the discrepancy coefficients w.r.t. the above snc-desingularizations $\varphi : \widetilde{X} \longrightarrow X = X_f^{(3)}$ of 3-dimensional **A**-**D**-**E**'s and to a subsequent simplification of applying formula (1.5).

Proposition 3.1 *The discrepancies of the snc-desingularizations*

$$\varphi : \widetilde{X} \longrightarrow X$$

of the underlying spaces $X = X_f^{(3)}$ of the three-dimensional **A**-**D**-**E** *singularities* (*discussed in* §2) *are given by the following table*:

Types	**Discrepancy** $K_{\widetilde{X}} - \varphi^*(K_X)$
$\mathbf{A}_n$, n even	$\sum_{i=1}^{\frac{n}{2}} i\, D_i + (n+2)\, D_{\frac{n}{2}+1}$
$\mathbf{A}_n$, n odd	$\sum_{i=1}^{\frac{n+1}{2}} i\, D_i$
$\mathbf{D}_n$, n even	$(n-1)\, D_1 + (n-1)\, D_2 + \sum_{i=3}^{\frac{n}{2}+1} (2(n-2i)+7)\, D_i$ $+ \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{n}{2} - i\right) (D_i' + D_i'')$
$\mathbf{D}_n$, n odd	$(n-2)\, D_1 + (n-1)\, D_2 + \sum_{i=3}^{\frac{n+1}{2}} (2(n-2i-1)+7)\, D_i$ $+ \sum_{i=1}^{\frac{n-3}{2}} \left(\frac{n-1}{2} - i\right) (D_i' + D_i'')$
$\mathbf{E}_6$	$3D_1 + 6D_2 + 9D_3 + D_4 + D_4'$
$\mathbf{E}_7$	$11D_1 + 9D_2 + 13D_3 + 5D_4 + 4D_1' + 4D_1''$ $+2D_2' + 2D_2'' + D_3' + D_3''$
$\mathbf{E}_8$	$19D_1 + 15D_2 + 23D_3 + 11D_4 + 7D_1' + 7D_1''$ $+4D_2' + 4D_2'' + 2D_3' + 2D_3'' + D_4' + D_4''$

Proof. By construction, $\varphi : \widetilde{X} \longrightarrow X$ is composed of "partial" resolution morphisms. To use a uniform notation (from a *global* point of view) in what follows, we shall write $\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_\nu$ and

$$\widetilde{X} = X_\nu \xrightarrow{\varphi_\nu} X_{\nu-1} \xrightarrow{\varphi_{\nu-1}} \cdots \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X \tag{3.1}$$

for these partial resolutions (where $\nu = \left\lfloor \frac{n+2}{2} \right\rfloor, \left\lfloor \frac{n+1}{2} \right\rfloor, 4, 4, 5$ for types $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$, respectively, as one deduces from §2). The discrepancy w.r.t. φ

equals:

$$
\begin{aligned}
&K_{\widetilde{X}} - \varphi^{*}\left(K_X\right) = \\
&\sum_{i=1}^{\nu-1} \left(\varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_{\nu}\right)^{*} \left(K_{X_i} - \varphi_i^{*}\left(K_{X_{i-1}}\right)\right) + K_{X_\nu} - \varphi_\nu^{*}\left(K_{X_{\nu-1}}\right)
\end{aligned}
\tag{3.2}
$$

Therefore, for its computation, it suffices to determine the discrepancies w.r.t. each of the φ_i's, and then to specify the pull-backs which are involved in (3.2).

I) Computation of the intermediate discrepancies. Since the arising singularities are isolated, we may investigate the zeros of canonical differentials locally around them.

(i) Type $\mathbf{A}_n$. The defining polynomial of the singularity is

$$f(x_1, \ldots, x_4) = x_1^{n+1} + x_2^2 + x_3^2 + x_4^2 \,. \tag{3.3}$$

Let $n \geq 2$, and consider the rational canonical differential

$$\mathfrak{s} := \mathrm{Res}_X \left(\frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{f} \right) = \frac{dx_2 \wedge dx_3 \wedge dx_4}{(\partial f \,/\, \partial x_1)} \in \Omega^3_{\mathbb{C}(X)/\mathbb{C}} \,.$$

$\mathfrak{s}$ is a basis of the dualizing sheaf $\omega_X = \mathcal{O}_X(K_X) = (\Omega_X^3)^{\vee\vee}$ whose sections are defined by

$$\left\{ \begin{array}{c} \text{open sets} \\ \text{of } X \end{array} \right\} \ni V \longmapsto \Gamma\left(V, \omega_X\right) := \left\{ \mathfrak{y} \in \Omega^3_{\mathbb{C}(X)/\mathbb{C}} \;\middle|\; \begin{array}{l} \mathfrak{y} \text{ is a} \\ \text{regular canonical} \\ \text{differential} \\ \text{on } V \cap (X \smallsetminus \{\mathbf{0}\}) \end{array} \right\}.$$

Blow up X at $\mathbf{0}$ and consider the affine piece $U_1 \cap \mathbf{Bl_0}(X)$, with

$$U_1 = \mathrm{Spec}\left(\mathbb{C}[y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}]\right).$$

The restriction of the exceptional locus $\mathcal{E}_f$ on U_1 is nothing but

$$\mathbf{Bl_0}(X) \cap \mathcal{E}_1 = \mathcal{E}_f\,|_{U_1} = \left\{ (y_{1,1}, .., y_{1,4}) \in \mathbb{C}^4 \;\middle|\; y_{1,1} = \widetilde{f}_1\left(y_{1,1}, .., y_{1,4}\right) = 0 \right\}$$

where

$$\widetilde{f}_1\left(y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}\right) = y_{1,1}^{n-1} + y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2 \,.$$

(As we explained before, the possibly existing new ($\mathbf{A}_{n-2}$-) singularity on $\mathbf{Bl_0}(X)$ lies in $\mathcal{E}_f\,|_{U_1}$). To find the discrepancy coefficient w.r.t. $\mathbf{Bl_0}(X) \longrightarrow X$, it suffices to compare $\mathfrak{s}$ with the rational canonical differential

$$\overline{\mathfrak{s}} := \frac{dy_{1,2} \wedge dy_{1,3} \wedge dy_{1,4}}{(\partial \widetilde{f}_1 \,/\, \partial y_{1,1})} \in \Omega^3_{\mathbb{C}(U_1)/\mathbb{C}} \,.$$

(U_1 is non-singular with local coordinates $y_{1,2}, y_{1,3}, y_{1,4}$ at any point q for which $\partial \widetilde{f}_1(q) \,/\, \partial y_{1,1} \neq 0$). In U_1 we have $x_1 = y_{1,1}$ and $x_j = x_1\, \xi_j = y_{1,1}\, y_{1,j}$, for all $j \in \{2, 3, 4\}$. Hence,

$$
\begin{aligned}
& dx_2 \wedge dx_3 \wedge dx_4 \\
& = (y_{1,1} dy_{1,2} + y_{1,2}\, dy_{1,1}) \wedge (y_{1,1} dy_{1,3} + y_{1,3}\, dy_{1,1}) \wedge (y_{1,1} dy_{1,4} + y_{1,4}\, dy_{1,1}) \\
& = y_{1,1}^2\, (y_{1,2}\, dy_{1,1} \wedge dy_{1,3} \wedge dy_{1,4} - y_{1,3}\, dy_{1,1} \wedge dy_{1,2} \wedge dy_{1,4} \\
& + y_{1,4}\, dy_{1,1} \wedge dy_{1,2} \wedge dy_{1,3} + y_{1,1}\, dy_{1,2} \wedge dy_{1,3} \wedge dy_{1,4})
\end{aligned}
\tag{3.4}
$$

and

$$\partial f \,/\, \partial x_1 = (n+1)\ x_1^n = (n+1)\ y_{1,1}^n = \left(\frac{n+1}{n-1}\right)\ y_{1,1}^2\ (\partial \widetilde{f}_1 \,/\, \partial y_{1,1}) \tag{3.5}$$

On the other hand, note that

$$d\widetilde{f}_1 = (n-1)\ y_{1,1}^{n-2}\, dy_{1,1} + 2\ (y_{1,2}\, dy_{1,2} + y_{1,3}\, dy_{1,3} + y_{1,4}\, dy_{1,4}) = 0$$

if and only if

$$dy_{1,1} = -\frac{2}{n-1}\ y_{1,1}^{2-n}\ \ (y_{1,2}\, dy_{1,2} + y_{1,3}\, dy_{1,3} + y_{1,4}\, dy_{1,4}) \tag{3.6}$$

Substituting the expression (3.6) for $dy_{1,1}$ into the right-hand side of (3.4), we obtain easily

$$\begin{aligned} & dx_2 \wedge dx_3 \wedge dx_4 \\ & = \left(-\tfrac{2}{n-1}\ y_{1,1}^{4-n}\ (y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2) + y_{1,1}^3\right)\ dy_{1,2} \wedge dy_{1,3} \wedge dy_{1,4} \end{aligned} \tag{3.7}$$

Combining now (3.7) with $y_{1,2}^2 + y_{1,3}^2 + y_{1,4}^2 = -y_{1,1}^{n-1}$ and (3.5), we get

$$\mathfrak{s} = \frac{\left(\frac{n+1}{n-1}\ y_{1,1}^3\right)\ dy_{1,2} \wedge dy_{1,3} \wedge dy_{1,4}}{\left(\frac{n+1}{n-1}\right)\ y_{1,1}^2\ (\partial \widetilde{f}_1 \,/\, \partial y_{1,1})} = y_{1,1}\ \overline{\mathfrak{s}} \tag{3.8}$$

The equality (3.8) shows that the discrepancy coefficient of the exceptional prime divisor $\mathcal{E}_f$ w.r.t. $\mathbf{Bl}_{\mathbf{0}}(X) \longrightarrow X$ equals 1.

If $n = 1$, then we compare

$$\mathfrak{s} = \frac{dx_1 \wedge dx_2 \wedge dx_3}{(\partial f \,/\, \partial x_4)} \qquad \text{with} \qquad \overline{\mathfrak{s}} = \frac{dy_{1,1} \wedge dy_{1,2} \wedge dy_{1,3}}{(\partial \widetilde{f}_1 \,/\, \partial y_{1,4})}.$$

Since $\frac{\partial f}{\partial x_4} = 2x_4 = 2y_{1,1}y_{1,4}$, $\frac{\partial \widetilde{f}_1}{\partial y_{1,4}} = 2y_{1,4}$, and

$$dx_1 \wedge dx_2 \wedge dx_3 = y_{1,1}^2 dy_{1,1} \wedge dy_{1,2} \wedge dy_{1,3},$$

we conclude again $\mathfrak{s} = y_{1,1}\overline{\mathfrak{s}}$. In fact, this kind of argumentation covers all but one steps of the resolution procedure for $\mathbf{A}_n$'s. The indicated "special" case occurs only in the last step and only for n even, where we blow-up once more to get rid of the singularity of the exceptional locus for the purpose of ensuring the snc-condition for $\varphi : \widetilde{X} \longrightarrow X$ ("$n = 0$"-case). But since we blow-up a point which is *smooth* on the *3-fold*, the discrepancy coefficient of the lastly created exceptional prime divisor $D_{\frac{n}{2}+1}$ equals 2 (see remark 2.2 and Griffiths & Harris [19, Lemma of p. 187]).

(ii) Type $\mathbf{D}_n$. For this type we proceed analogously by making use of the affine piece U_1. The only difference here is that the exceptional divisor $\mathcal{E}_f$ under the first blow-up has two irreducible components $\mathcal{E}_f'$ and $\mathcal{E}_f''$. Nevertheless, the corresponding local computation with rational canonical differentials gives again

$$\frac{dx_2 \wedge dx_3 \wedge dx_4}{(\partial f \,/\, \partial x_1)} = y_{1,1}\ \frac{dy_{1,2} \wedge dy_{1,3} \wedge dy_{1,4}}{(\partial \widetilde{f}_1 \,/\, \partial y_{1,1})}$$

and the discrepancy coefficient for both of them equals 1. As it is clear from Lemma 2.1 and **(i)**, the discrepancy coefficients in all resolution steps will be again 1.

(iii) Types $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$. For these types one may work along the same lines with respect to the affine piece $U_2 = \mathrm{Spec}(\mathbb{C}[y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}])$. The exceptional divisor $\mathcal{E}_f$ w.r.t. $\mathbf{Bl}_0(X) \longrightarrow X$ consists again of two prime ones. Each of them has discrepancy coefficient equal to 1. This property remains also valid for all other composites (3.1) of φ, exactly as in the case of type $\mathbf{D}_n$. Further details will be omitted.

Recapitulating, we should stress that in **(i)**, **(ii)**, **(iii)**, the discrepancy coefficient for *each* of the prime divisors of the exceptional locus of the φ_i's in (3.1) equals 1, up to the last resolution morphism for type $\mathbf{A}_n$, n even, which has discrepancy 2. This fact will be used below in an essential way.

II) Computation of the pull-backs. To determine the required pullbacks of our discrepancies (see (3.1), (3.2)), we shall denote by E_j (resp., $E_j^{\prime(\prime\prime)}$) those exceptional prime divisors which are created (for the first time) after the application of a φ_i (i.e., actually the members of $\mathfrak{Ex}(\varphi_i)$), so that their strict transforms (on $\widetilde{X}$) are exactly the exceptional prime divisors (w.r.t. φ) which are denoted by D_j (resp., $D_j^{\prime(\prime\prime)}$) in §2.

(i) Type $\mathbf{A}_n$. Defining $m = \left\lfloor \frac{n+2}{2} \right\rfloor$, as in §2, φ is decomposed into m birational morphisms:

$$\widetilde{X} = X_m \xrightarrow{\varphi_m} X_{m-1} \xrightarrow{\varphi_{m-1}} \cdots \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X.$$

Each $\varphi_i (= \pi_i$ of §2) gives rise to an exceptional prime divisor E_i. By **I)** we get

$$K_{X_i} - \varphi_i^*(K_{X_{i-1}}) = E_i, \quad \forall i, \ 1 \leq i \leq m-1, \tag{3.9}$$

and

$$K_{X_m} - \varphi_m^*(K_{X_{m-1}}) = \begin{cases} D_m, & \text{if } n \text{ is odd,} \\ \\ 2D_m, & \text{if } n \text{ is even.} \end{cases} \tag{3.10}$$

We claim that for all i, $1 \leq i \leq m-1$,

$$(\varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_m)^* (E_i) = \begin{cases} \sum\limits_{j=i}^{m} D_j, & \text{if } n \text{ is odd,} \\ \\ \sum\limits_{j=i}^{m} D_j + 2D_m, & \text{if } n \text{ is even.} \end{cases} \tag{3.11}$$

To prove (3.11) we shall work with local equations for the corresponding divisors. Consider two successive blow-ups

$$X_{j+1} \xrightarrow{\varphi_{j+1}} X_j \xrightarrow{\varphi_j} X_{j-1}$$

and assume that X_j has a singularity of type $\mathbf{A}_n$, $n \geq 1$, (with equation (3.3)), where φ_j denotes the blow-up of the $\mathbf{A}_{n+2}$-singularity of X_{j-1}. The local equation $(\widetilde{f}_2 = 0)$ is the equation of X_{j+1} on the affine chart $U_2 = \mathrm{Spec}(\mathbb{C}[y_{2,1}, .., y_{2,4}])$, where

$$\widetilde{f}_2 (y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}) = y_{2,1}^{n+1} y_{2,2}^{n-1} + 1 + y_{2,3}^2 + y_{2,4}^2$$

(cf. §2). The new exceptional locus E_{j+1} of φ_{j+1} on $U_2 \cap X_{j+1}$ is given by the local equation $(y_{2,1} = 0)$. On the other hand, $(x_1 = 0)$ and $(y_{2,2} = 0)$ express the local equations for E_j on X_j and for its strict transform $E_{j,\,\text{ST}}$ on $U_2 \cap X_{j+1}$, respectively. Since the preimage of $(x_1 = 0)$ under φ_{j+1} equals $(y_{2,1} \cdot y_{2,2} = 0)$, we have:

$$\varphi_{j+1}^*(E_j) = E_{j+1} + E_{j,\,\text{ST}}. \tag{3.12}$$

It remains to see what happens in the case in which φ_{j+1} is the blow up of a (regular) $\mathbf{A}_0$-point, i.e., whenever $j = m-1 = k$ and X_{k+1} is the last step of the resolution process for a singularity of type $\mathbf{A}_{2k}$. For $n = 0$, we get equations

$$x_1 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \text{and} \quad z_{2,1} + z_{2,2}(1 + z_{2,3}^2 + z_{2,4}^2) = 0,$$

on X_k and $U_2 \cap X_{k+1}$, respectively. The divisors D_{k+1}, E_k, $E_{k,\,\text{ST}}$ have local equations $(z_{2,2} = 0)$, $(x_1 = 0)$ and $(z_{2,1} = 0)$, respectively. Since

$$x_1 = z_{2,1}\, z_{2,2} = z_{2,2}^2(1 + z_{2,3}^2 + z_{2,4}^2),$$

we deduce

$$\varphi_{k+1}^*(E_k) = 2D_{k+1} + E_{k,\,\text{ST}} = 2D_m + E_{m-1,\,\text{ST}}. \tag{3.13}$$

(3.11) follows after repeated application of equations like (3.12) and (3.13). Now inserting the data of (3.9), (3.10), (3.11) into (3.2) we obtain:

$$K_{\widetilde{X}} - \varphi^*(K_X) = \begin{cases} \sum_{i=1}^{\frac{n+1}{2}} i\, D_i, & \text{if } n \text{ is odd}, \\ \sum_{i=1}^{\frac{n}{2}} i\, D_i + (n+2)\, D_{\frac{n}{2}+1}, & \text{if } n \text{ is even}. \end{cases}$$

(ii) Type $\mathbf{D}_n$, $n = 2k$. In this case φ is decomposed into k birational morphisms:

$$\widetilde{X} = X_k \xrightarrow{\varphi_k} X_{k-1} \xrightarrow{\varphi_{k-1}} \cdots \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X.$$

By construction, $\mathfrak{Ex}(\varphi_1) = \left\{E'_{k-1}, E''_{k-1}\right\}$,

$$\mathfrak{Ex}(\varphi_{i+1}) = \left\{E'_{k-i-1}, E''_{k-i-1}, E_{k-i+2}\right\},\ \forall i,\ 1 \le i \le k-2,$$

and $\mathfrak{Ex}(\varphi_k) = \{D_1, D_2, D_3\}$. By **I)** we have

$$K_{X_1} - \varphi_1^*(K_{X_0}) = E'_{k-1} + E''_{k-1},$$

$$K_{X_{i+1}} - \varphi_{i+1}^*(K_{X_i}) = E'_{k-i-1} + E''_{k-i-1} + E_{k-i+2},\ \forall i,\ 1 \le i \le k-2,$$

$$K_{X_k} - \varphi_k^*(K_{X_{k-1}}) = D_1 + D_2 + D_3.$$

We shall prove that

$$K_{\widetilde{X}} - \varphi^*(K_X) = (2k-1)(D_1 + D_2) + \sum_{i=1}^{k-1} i(D'_{k-i} + D''_{k-i}) + \sum_{j=1}^{k-1}(4j-1)D_{k-j+2}. \tag{3.14}$$

For $k=2$ this can be shown easily. Suppose that $k\geq 3$. Then

$$\varphi_{i+1}^{*}(E_{k-i}^{\prime(\prime\prime)}) = E_{k-i-1}^{\prime(\prime\prime)} + E_{k-i+2} + E_{k-i,\mathrm{ST}}^{\prime(\prime\prime)},\ \forall i,\ 1\leq i\leq k-2,$$

$$\varphi_{k}^{*}(E_{1}^{\prime(\prime\prime)}) = D_1 + D_2 + D_3 + D_1^{\prime(\prime\prime)},$$

and for all i, $1\leq i\leq k-2$,

$$(\varphi_{i+1}\circ\varphi_{i+2})^{*}\,(E_{k-i}^{\prime(\prime\prime)}) = E_{k-i+1} + E_{k-i+2} + E_{k-i-1,\mathrm{ST}}^{\prime(\prime\prime)} + E_{k-i,\mathrm{ST},\ \mathrm{ST}}^{\prime(\prime\prime)}.$$

This means that

$$(\varphi_2\circ\varphi_3\circ\cdots\circ\varphi_k)^{*}\left(E_{k-1}^{\prime}+E_{k-1}^{\prime\prime}\right)$$

$$=\sum_{j=1}^{k-1}\left(D_{k-j}^{\prime}+D_{k-j}^{\prime\prime}\right)+2(D_1+D_2+D_3)+2\left(D_3+2\sum_{j=2}^{k-2}D_{k-j+2}+D_{k+1}\right),$$

and that for all i, $2\leq i\leq k-2$,

$$(\varphi_{i+1}\circ\varphi_{i+2}\circ\cdots\circ\varphi_k)^{*}\left(E_{k-i}^{\prime}+E_{k-i}^{\prime\prime}+E_{k-i+3}\right)$$

$$=\sum_{j=i}^{k-1}\left(D_{k-j}^{\prime}+D_{k-j}^{\prime\prime}\right)+2(D_1+D_2+D_3)$$

$$+2\left(D_3+2\sum_{j=i+1}^{k-2}D_{k-j+2}+D_{k-i+2}\right)+D_{k-i+3}$$

and

$$\varphi_k^{*}(E_1^{\prime}+E_1^{\prime\prime}+E_4) = (D_1^{\prime}+D_1^{\prime\prime})+2(D_1+D_2+D_3)+D_4.$$

Thus, (3.2) implies (3.14).

(iii) Type $\mathbf{D}_n$, $n=2k+1$. Here φ is decomposed into $k+1$ birational morphisms:

$$\widetilde{X}=X_{k+1}\xrightarrow{\varphi_{k+1}}X_k\xrightarrow{\varphi_k}\cdots\xrightarrow{\varphi_3}X_2\xrightarrow{\varphi_2}X_1\xrightarrow{\varphi_1}X_0=X.$$

Computing the total discrepancy, we find analogously:

$$K_{\widetilde{X}}-\varphi^{*}(K_X)=(2k-1)\,D_1+2kD_2+\sum_{i=1}^{k-1}i(D_{k-i}^{\prime}+D_{k-i}^{\prime\prime})+\sum_{j=1}^{k-1}(4j-1)D_{k-j+2}.$$

(iv) Type $\mathbf{E}_6$. In this case φ is decomposed into 4 birational morphisms:

$$\widetilde{X}=X_4\xrightarrow{\varphi_4}X_3\xrightarrow{\varphi_3}X_2\xrightarrow{\varphi_2}X_1=\mathbf{Bl}_{\mathbf{0}}(X)\xrightarrow{\varphi_1}X_0=X$$

By construction,

$$\mathfrak{Ex}(\varphi_1)=\{E_4,E_4^{\prime}\},\quad \mathfrak{Ex}(\varphi_2)=\{E_1\},\quad \mathfrak{Ex}(\varphi_3)=\{E_2\},$$

and $\mathfrak{Ex}(\varphi_4)=\{D_3\}$ (where $\varphi_i=\pi_{i-1}$ of §2). By **I)** we have

$$K_{X_1}-\varphi_1^{*}(K_{X_0})=E_4+E_4^{\prime},\quad K_{X_2}-\varphi_2^{*}(K_{X_1})=E_1,$$

$$K_{X_3}-\varphi_3^{*}(K_{X_2})=E_2,\qquad K_{X_4}-\varphi_4^{*}(K_{X_3})=D_3.$$

The intersection diagrams imply

$$(\varphi_2 \circ \varphi_3 \circ \varphi_4)^* (E_4 + E'_4) = 2D_1 + 4D_2 + 6D_3 + D_4 + D'_4,$$

$$(\varphi_3 \circ \varphi_4)^* (E_1) = D_1 + D_2 + D_3,$$

$$\varphi_4^* (E_2) = D_2 + D_3.$$

Hence, by (3.2), the discrepancy w.r.t. φ equals $3D_1 + 6D_2 + 9D_3 + D_4 + D'_4$.

(v) Type $\mathbf{E}_7$. Here φ is decomposed into 4 birational morphisms:

$$\widetilde{X} = X_4 \xrightarrow{\varphi_4} X_3 \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1 = \mathbf{Bl}_{\mathbf{0}}(X) \xrightarrow{\varphi_1} X_0 = X$$

By construction,

$$\mathfrak{Ex}(\varphi_1) = \{E'_3, E''_3\}, \quad \mathfrak{Ex}(\varphi_2) = \{E'_2, E''_2\}, \quad \mathfrak{Ex}(\varphi_3) = \{E'_1, E''_1, E_4\},$$

and $\mathfrak{Ex}(\varphi_4) = \{D_1, D_2, D_3\}$. By **I)** we obtain

$$K_{X_1} - \varphi_1^*(K_{X_0}) = E'_3 + E''_3, \qquad K_{X_2} - \varphi_2^*(K_{X_1}) = E'_2 + E''_2,$$

$$K_{X_3} - \varphi_3^*(K_{X_2}) = E'_1 + E''_1 + E_4, \quad K_{X_4} - \varphi_4^*(K_{X_3}) = D_1 + D_2 + D_3.$$

The computation of the pull-backs gives

$$(\varphi_2 \circ \varphi_3 \circ \varphi_4)^* (E'_3 + E''_3) =$$
$$6D_1 + 4D_2 + 6D_3 + 2D_4 + 2(D'_1 + D''_1) + D'_2 + D''_2 + D'_3 + D''_3,$$

$$(\varphi_3 \circ \varphi_4)^* (E'_2 + E''_2) = 2D_1 + 2D_2 + 4D_3 + 2D_4 + D'_1 + D''_1 + D'_2 + D''_2,$$

$$\varphi_4^* (E'_1 + E''_1 + E_4) = 2D_1 + 2D_2 + 2D_3 + D_4 + D'_1 + D''_1.$$

Now apply (3.2).

(vi) Type $\mathbf{E}_8$. In this case φ is decomposed into 5 birational morphisms:

$$\widetilde{X} = X_5 \xrightarrow{\varphi_5} X_4 \xrightarrow{\varphi_4} X_3 \xrightarrow{\varphi_3} X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X$$

By construction, $\mathfrak{Ex}(\varphi_1) = \{E'_4, E''_4\}$,

$$\mathfrak{Ex}(\varphi_2) = \{E'_3, E''_3\}, \quad \mathfrak{Ex}(\varphi_3) = \{E'_2, E''_2\}, \quad \mathfrak{Ex}(\varphi_4) = \{E'_1, E''_1, E_4\},$$

and $\mathfrak{Ex}(\varphi_5) = \{D_1, D_2, D_3\}$. By **I)** we have

$$K_{X_1} - \varphi_1^*(K_{X_0}) = E'_4 + E''_4, \quad K_{X_2} - \varphi_2^*(K_{X_1}) = E'_3 + E''_3,$$

$$K_{X_3} - \varphi_3^*(K_{X_2}) = E'_2 + E''_2, \quad K_{X_4} - \varphi_4^*(K_{X_3}) = E'_1 + E''_1 + E_4,$$

and $K_{X_5} - \varphi_5^*(K_{X_4}) = D_1 + D_2 + D_3$. We obtain

$$(\varphi_2 \circ \varphi_3 \circ \varphi_4 \circ \varphi_5)^* (E'_4 + E''_4) =$$
$$8D_1 + 6D_2 + 10D_3 + 6D_4 + 3(D'_1 + D''_1) + 2(D'_2 + D''_2) + D'_3 + D''_3 + D'_4 + D''_4.$$

The remaining inverse images $(\varphi_3 \circ \varphi_4 \circ \varphi_5)^* (E'_3 + E''_3)$, $(\varphi_4 \circ \varphi_5)^* (E'_2 + E''_2)$ and $\varphi_5^* (E'_1 + E''_1 + E_4)$ coincide with (v), where in each case $\varphi_i \circ \cdots \circ \varphi_4$ has to be replaced by $\varphi_{i+1} \circ \cdots \circ \varphi_5$. Finally, apply again (3.2). $\square$

Proposition 3.2 *Suppose that* $X = X_f^{(3)}$ *is the underlying space of an* **A-D-E**-*singularity,* $\varphi : \widetilde{X} \longrightarrow X$ *its snc-desingularization,* $\mathfrak{Ex}(\varphi) = \{D_1, .., D_r\}$ *the corresponding exceptional set with discrepacy coefficients* $a_1, \ldots, a_r$, $I := \{1, 2, ..., r\}$, *and*

$$\mathfrak{R}_\varphi := \{(i,j) \in I^2 \big| D_{\{i,j\}} \neq \varnothing\}, \quad \mathfrak{Q}_\varphi := \{(i,j,k) \in I^3 \big| D_{\{i,j,k\}} \neq \varnothing\}.$$

Then the string-theoretic E*-function of* X *satisfies the following equality*:

$$\begin{aligned} E_{\mathrm{str}}(X; u, v) &= E\left(D_\varnothing^\circ; u, v\right) + \sum_{i=1}^{r} \tfrac{E(D_i; u,v)(uv-1)}{(uv)^{a_i+1}-1} \\ &+ (1+uv)\left[\sum_{(i,j)\in\mathfrak{R}_\varphi} \left(\tfrac{uv-(uv)^{a_i+1}}{(uv)^{a_i+1}-1}\right)\left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right) - \mathbf{b}(X)\right] \\ &+ \sum_{(i,j,k)\in\mathfrak{Q}_\varphi} \left(\tfrac{uv-(uv)^{a_i+1}}{(uv)^{a_i+1}-1}\right)\left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right)\left(\tfrac{uv-(uv)^{a_k+1}}{(uv)^{a_k+1}-1}\right) + \mathbf{t}(X) \end{aligned} \tag{3.15}$$

with $\mathbf{b}(X)$, $\mathbf{t}(X)$ *as defined in* 2.3 (ii). *In particular,*

$$\begin{aligned} e_{\mathrm{str}}(X) - e\left(D_\varnothing^\circ\right) &= \sum_{i=1}^{r} \tfrac{e(D_i)}{a_i+1} + 2\left[\sum_{(i,j)\in\mathfrak{R}_\varphi} \left(\tfrac{a_i}{a_i+1}\right)\left(\tfrac{a_j}{a_j+1}\right) - \mathbf{b}(X)\right] \\ &- \sum_{(i,j,k)\in\mathfrak{Q}_\varphi} \left(\tfrac{a_i}{a_i+1}\right)\left(\tfrac{a_j}{a_j+1}\right)\left(\tfrac{a_k}{a_k+1}\right) + \mathbf{t}(X) \end{aligned} \tag{3.16}$$

(As we shall see below in 4.3, $e\left(D_\varnothing^\circ\right) = 0$).

Proof. Using inclusion-exclusion principle (1.2) for the E-polynomial of D_J°, we obtain

$$E\left(D_J^\circ; u, v\right) = E\left(D_J; u, v\right) - \sum_{\varnothing \neq J' \subseteq I \smallsetminus J} (-1)^{|J'|-1} E\left(D_{J'}; u, v\right) \tag{3.17}$$

Formula (1.5) can be rewritten via (3.17) as follows:

$$\begin{aligned} &E_{\mathrm{str}}(X; u, v) \\ &= \sum_{J\subseteq I} \left(E\left(D_J; u, v\right) - \sum_{\varnothing \neq J' \subseteq I \smallsetminus J} (-1)^{|J'|-1} E\left(D_{J'\cup J}; u, v\right)\right) \prod_{j\in J} \left(\tfrac{uv-1}{(uv)^{a_j+1}-1}\right) \\ &= \sum_{J\subseteq I} E\left(D_J; u, v\right) \prod_{j\in J} \left(\tfrac{uv-1}{(uv)^{a_j+1}-1} - 1\right) \\ &= \sum_{J\subseteq I} E\left(D_J; u, v\right) \prod_{j\in J} \left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right). \end{aligned}$$

Hence,

$$\begin{aligned}
&E_{\text{str}}(X;u,v) - E\left(D^{\circ}_{\varnothing};u,v\right) \\
&= E(\bigcup_{i\in I} D_i;u,v) + \sum_{\varnothing\neq J\subseteq I} E\left(D_J;u,v\right) \prod_{j\in J}\left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right) \\
&= \sum_{i=1}^{r} E(D_i;u,v) - \sum_{(i,j)\in\mathfrak{R}_\varphi} E(D_{\{i,j\}};u,v) + \sum_{(i,j,k)\in\mathfrak{Q}_\varphi} E(D_{\{i,j,k\}};u,v) \\
&+ \sum_{i=1}^{r} E\left(D_j;u,v\right)\left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right) + \sum_{\substack{J\subseteq I \\ |J|\in\{2,3\}}} E\left(D_J;u,v\right)\textstyle\prod_{j\in J}\left(\tfrac{uv-(uv)^{a_j+1}}{(uv)^{a_j+1}-1}\right)
\end{aligned} \tag{3.18}$$

Since $|\mathfrak{R}_\varphi| = \mathbf{b}(X)$, $|\mathfrak{Q}_\varphi| = \mathbf{t}(X)$, and

$$E(D_{\{i,j\}};u,v) = 1+uv,\ \forall(i,j)\in\mathfrak{R}_\varphi, \quad E(D_{\{i,j,k\}};u,v) = 1,\ \forall(i,j,k)\in\mathfrak{Q}_\varphi,$$

Formula (3.15) follows from (3.18), and (3.16) from (3.15) by passing to the limit $u,v \longrightarrow 1$. $\square$

4 Proof of the Theorem

Theorem 1.11 will be proved by direct evaluation of formula (3.15). For this it is obviously enough to determine the coefficients of the E-polynomials of all exceptional prime divisors, on the one hand, and those of $E\left(D^{\circ}_{\varnothing};u,v\right)$, on the other. Hence, in view of lemma 2.4 and of our explicit description of a canonical desingularization, what remains to be done is the study of the coefficients of this "first summand" $E\left(D^{\circ}_{\varnothing};u,v\right)$ which depend exclusively on the intrinsic geometry around the singularities. We begin with a general proposition being valid in all dimensions.

Proposition 4.1 *Let (X,x) be an isolated complete intersection singularity of pure dimension $d\geq 2$ and $(\widetilde{X},\mathfrak{Ex}(\varphi)) \xrightarrow{\varphi} (X,x)$ a resolution with exceptional locus $\mathfrak{Ex}(\varphi) = \cup_{i=1}^{r} D_i$. Then the coefficients of the E-polynomial*

$$\begin{aligned}
E(\widetilde{X}\smallsetminus\mathfrak{Ex}(\varphi);u,v) &= E\left(D^{\circ}_{\varnothing};u,v\right) = \\
&= E\left(X\smallsetminus\{x\};u,v\right) = (uv)^d\, E\left(L;u^{-1},v^{-1}\right)
\end{aligned} \tag{4.1}$$

of $\widetilde{X}\smallsetminus\mathfrak{Ex}(\varphi)$ depend on those of the E-polynomial of its link L, and, in fact, only on the Hodge numbers of the $(d-1)$-cohomology group of L.

If (X,x) is, in addition, a rational singularity, then

$$\begin{cases} E(\widetilde{X}\smallsetminus\mathfrak{Ex}(\varphi)\,;u,v) = E\,(X\smallsetminus\{x\}\,;u,v) = (uv)^d - 1 + \\ \\ +\,(-1)^d \left[\sum\limits_{\substack{1\leq p,q\leq d-1 \\ 2\leq p+q\leq d-1}} h^{p,q}(H^{d-1}(L,\mathbb{C}))\; u^p\, v^q \right] + \\ \\ +\,(-1)^{d-1} \left[\sum\limits_{\substack{1\leq p,q\leq d-1 \\ d+1\leq p+q\leq 2d-2}} h^{d-p,d-q}(H^{d-1}(L,\mathbb{C}))\; u^p\, v^q \right] \end{cases} \tag{4.2}$$

Proof. Let $L = L\,(X,x)$ denote the link of the singularity (X,x), i.e., the intersection of a closed neighbourhood of x containing it with a small sphere. L is a differentiable, compact, oriented manifold of dimension $2d-1$, and there are isomorphisms:

$$H^{i+1}\,(X, X\smallsetminus\{x\}\,,\mathbb{Q}) \cong H^i\,(X\smallsetminus\{x\}\,,\mathbb{Q}) \cong H^i\,(L,\mathbb{Q}) \;\; .$$

For this reason it is sufficient to consider the natural MHS on the cohomologies of L. Note that

$$h^{p,q}(H^i\,(L,\mathbb{C})) = h^{q,p}(H^i\,(L,\mathbb{C})) \tag{4.3}$$

while Poincaré duality implies (4.1) because

$$h^{p,q}(H^i\,(L,\mathbb{C})) = h^{d-p,d-q}(H^{2d-i-1}\,(L,\mathbb{C}))$$

equals

$$h^{p,q}(H^i\,(L,\mathbb{C})) = h^{p,q}(H^i\,(X\smallsetminus\{x\}\,,\mathbb{C})) = h^{d-p,d-q}(H_c^{2d-i}\,(X\smallsetminus\{x\}\,,\mathbb{C})) \tag{4.4}$$

For the computation of these dimensions it is therefore enough to assume, from now on, that $i \leq d$. According to [44, Cor. (15.9)], the restriction map

$$H^i(\widetilde{X},\mathbb{Q}) \longrightarrow H^i(\widetilde{X}\smallsetminus\mathfrak{Ex}\,(\varphi)\,,\mathbb{Q}) \cong H^i\,(L,\mathbb{Q})$$

is surjective for $i < d$ and equals the zero-map for $i = d$. From the induced exact MHS-sequences

$$0 \longrightarrow H^i_{\mathfrak{Ex}(\varphi)}(\widetilde{X},\mathbb{Q}) \longrightarrow H^i\,(\mathfrak{Ex}\,(\varphi)\,,\mathbb{Q}) \longrightarrow H^i\,(L,\mathbb{Q}) \longrightarrow 0 \qquad (i < d)$$

$$0 \longrightarrow H^d_{\mathfrak{Ex}(\varphi)}(\widetilde{X},\mathbb{Q}) \longrightarrow H^d\,(\mathfrak{Ex}\,(\varphi)\,,\mathbb{Q}) \longrightarrow 0 \qquad\qquad\qquad (i = d)$$

one gets the vanishing of $Gr_j^{\mathcal{W}_\bullet}(H^i_{\mathfrak{Ex}(\varphi)}(\widetilde{X},\mathbb{Q}))$, $j \neq i$, and of $Gr_j^{\mathcal{W}_\bullet}(H^i\,(L,\mathbb{Q}))$, for $j \geq i-1$ (cf. [42, Cor. 1.12]), and consequently, for $i < d$, $h^{p,q}\left(H^i\,(L,\mathbb{C})\right)$ equals

$$\begin{cases} h^{p,q}(H^i\,(\mathfrak{Ex}\,(\varphi)\,,\mathbb{C})), & \text{if} \quad p+q < i \\ h^{p,q}(H^i(\mathfrak{Ex}\,(\varphi)\,,\mathbb{C})) - h^{d-p,d-q}(H^{2d-i}(\mathfrak{Ex}\,(\varphi)\,,\mathbb{C})), & \text{if} \quad p+q = i \\ 0, & \text{if} \quad p+q > i \end{cases} \tag{4.5}$$

(The right-hand side of (4.5) is therefore independent of the choice of the resolution). Since X is also a complete intersection, L is $(d-2)$-connected (cf. [20, Kor. 1.3]), and the local Lefschetz theorem gives:

$$\begin{aligned} &H^i\,(L,\mathbb{C}) \cong \mathbb{C}, \;\; \text{for } i \in \{0, 2d-1\}\,, \\ &H^i\,(L,\mathbb{C}) = 0\;, \;\; \text{for } i \notin \{0, d-1, d, 2d-1\}\,. \end{aligned} \tag{4.6}$$

Thus, for $i \in \{0, 2d-1\}$, the only non-zero Hodge numbers are

$$h^{0,0}\left(H^0\left(L,\mathbb{C}\right)\right) = h^{d,d}(H^{2d-1}\left(L,\mathbb{C}\right)) = 1\ . \tag{4.7}$$

By (4.5), (4.6) and (4.7) we deduce

$$\begin{aligned}
&E\left(L;u,v\right) = \textstyle\sum_{0\leq p,q\leq d} e^{p,q}(L)\ u^p\, v^q = \\
&= \sum_{0\leq p,q\leq d} \left[\left(h^{p,q}(H^0(L,\mathbb{C})) - h^{p,q}(H^{2d-1}(L,\mathbb{C}))\right)\right]\ u^p\, v^q + \\
&+ \sum_{0\leq p,q\leq d} \left[(-1)^{d-1}\ \left(h^{p,q}(H^{d-1}(L,\mathbb{C})) - h^{p,q}(H^{d}(L,\mathbb{C}))\right)\right]\ u^p\, v^q = \\
&= \sum_{0\leq p,q\leq d} \left[(h^{p,q}(H^0(L,\mathbb{C})) - h^{p,q}(H^{2d-1}(L,\mathbb{C})))\right]\ u^p\, v^q + \\
&+ \sum_{0\leq p,q\leq d} \left[(-1)^{d-1}\, (h^{p,q}(H^{d-1}(L,\mathbb{C})) - h^{d-p,d-q}(H^{d-1}(L,\mathbb{C})))\right]\ u^p\, v^q = \\
&= 1-(uv)^d + (-1)^{d-1}\ [\sum_{0\leq p,q\leq d} h^{p,q}(H^{d-1}(L,\mathbb{C}))\ u^p\, v^q] + \\
&+ (-1)^d\ [-\sum_{0\leq p,q\leq d} h^{d-p,d-q}(H^{d-1}(L,\mathbb{C}))\ u^p\, v^q] = \\
&= 1-(uv)^d + (-1)^{d-1}\left[\sum_{\substack{0\leq p,q\leq d-1\\ 0\leq p+q\leq d-1}} h^{p,q}(H^{d-1}(L,\mathbb{C}))\ u^p\, v^q\right] + \\
&+ (-1)^d\left[\sum_{\substack{1\leq p,q\leq d\\ d+1\leq p+q\leq 2d-1}} h^{d-p,d-q}(H^{d-1}(L,\mathbb{C}))\ u^p\, v^q\right]
\end{aligned}$$

which proves the first assertion. Now setting

$$\ell^{p,q}(L) := \ \dim_{\mathbb{C}} Gr^p_{\mathcal{F}^\bullet}\left(H^{p+q}\left(L,\mathbb{C}\right)\right),$$

one has

$$\ell^{p,q}(L) = \dim_{\mathbb{C}} H^q\left(\mathfrak{Er}\left(\varphi\right), \Omega^p_{\widetilde{X}}\left(\log\ \mathfrak{Er}\left(\varphi\right)\right)\otimes\mathcal{O}_{\mathfrak{Er}(\varphi)}\right)$$

(cf. [42, §1] and [45, §3]). Obviously,

$$\ell^{p,i-p}(L) = \sum_{q=0}^{d} h^{p,q}\left(H^i\left(L,\mathbb{C}\right)\right)$$

for $i \geq p$. If (X,x) is, in addition, a *rational* singularity, then for all $i \geq 1$ we have

$$\ell^{0,i}(L) = \dim_{\mathbb{C}} H^i\left(\mathfrak{Er}\left(\varphi\right), \mathcal{O}_{\mathfrak{Er}(\varphi)}\right) = 0 = \ell^{i,0}(L) \tag{4.8}$$

because $\ell^{i,0}(L) \leq \ell^{0,i}(L)$, $H^i(\widetilde{X},\mathcal{O}_{\widetilde{X}}) = 0$ and

$$H^i(\widetilde{X},\mathcal{O}_{\widetilde{X}}) \longrightarrow H^i\left(\mathfrak{Er}\left(\varphi\right), \mathcal{O}_{\mathfrak{Er}(\varphi)}\right)$$

is surjective by [42, Lemma 2.14]. Hence,

$$h^{j,0}\left(H^i\left(L,\mathbb{C}\right)\right) \overset{(4.3)}{=} h^{0,j}\left(H^i\left(L,\mathbb{C}\right)\right) \overset{(4.8)}{=} 0, \quad \text{for } 0\leq j\leq d \text{ and } i\geq 1. \tag{4.9}$$

This means that the E-polynomial of L can be written as

$$\begin{cases} E\,(L;u,v) = 1-(uv)^d + \\ \\ +\,(-1)^{d-1}\left[\sum_{\substack{1\leq p,q\leq d-1\\ 2\leq p+q\leq d-1}} h^{p,q}(H^{d-1}(L,\mathbb{C}))\ u^p\,v^q\right] + \\ \\ +\,(-1)^d\left[\sum_{\substack{1\leq p,q\leq d-1\\ d+1\leq p+q\leq 2d-2}} h^{d-p,d-q}(H^{d-1}(L,\mathbb{C}))\ u^p\,v^q\right] \end{cases} \tag{4.10}$$

and formula (4.2) follows from (4.10) and (4.1). □

Remark 4.2 (i) Let us now denote by F_f the *Milnor fiber* being associated to the **A**-**D**-**E** singularity $(X_f^{(d)},\mathbf{0})$. As it is known (cf. [32, Thm. 6.5]), F_f has the homotopy type of a bouquet of d-spheres, and its *Milnor number*

$$\mu\,(f) := \mu\,(F_f) := \ \#\,\{\text{of these spheres}\} = \ \dim_{\mathbb{C}}(\mathcal{O}_{d+1}\,/\,(\tfrac{\partial f}{\partial x_1},\ldots,\tfrac{\partial f}{\partial x_{d+1}}))$$

is in each case equal to the subscript of the type under consideration. According to the *Sebastiani-Thom theorem* [39] (see also [15, pp. 86-88]), the splitting $f = g+g'$ (as in (1.7)) gives rise to the construction of an homotopy equivalence between the Milnor fiber F_f and the join $F_g * F_{g'}$ of the corresponding Milnor fibers F_g and $F_{g'}$. In particular, this implies

$$\mu\,(f) = \mu\,(g)\cdot\mu\,(g') = \mu\,(g) \tag{4.11}$$

(ii) For any isolated complete intersection singularity (X,x) of pure dimension d, with link L, Milnor fiber F and Milnor number $\mu\,(F)$, *Steenbrink's invariant*

$$s_j(X,x),\quad 0\leq j\leq d,$$

is defined in [43] by regarding any 1-parameter smoothing $\psi : (\mathfrak{X},x)\to(\mathbb{C},0)$ of (X,x) (with $\mathfrak{X}_0 = \psi^{-1}\,(0)\cong X$) and setting

$$s_j(X,x) := \ \dim_{\mathbb{C}} Gr^j_{\mathcal{F}^\bullet}\,\mathbb{H}^d\left(\Phi^\bullet_\psi\,(\mathbb{C})\right),$$

where $\mathcal{F}^\bullet$ denotes here the Hodge-filtration of the highest hypercohomology group of the complex $\Phi^\bullet_\psi\,(\mathbb{C})$ of sheaves of vanishing cycles associated to ψ. (For all q, the direct image sheaves $\Phi^q_\psi\,(\mathbb{C}) = R^q\,(\vartheta_t)_*\,\underline{\mathbb{C}}_{\mathfrak{X}}$ are defined on $\mathfrak{X}_0$, with $\vartheta_t : \mathfrak{X}_t\to\mathfrak{X}_0$ denoting the restriction of the retraction $\vartheta : \mathfrak{X}\to\mathfrak{X}_0$ onto a fiber $\mathfrak{X}_t$. In fact, the definition of $\Phi^q_\psi\,(\mathbb{C})$ can be made independent of the choice of the fiber $\mathfrak{X}_t$ by passing to the "canonical" fiber $\underset{\sim}{\mathfrak{X}}_\infty$ of ψ. In this setting, the fiber of the sheaf $\Phi^q_\psi\,(\mathbb{C})$ over x is isomorphic to $\widetilde{H}^q\,(\mathfrak{X}_{t,x},\mathbb{C})$, where $\mathfrak{X}_{t,x}$ is diffeomorphic to the Milnor fiber F). $s_j(X,x)$ is an upper semicontinuous invariant under deformations of (X,x), *does not* depend on the particular choice of ψ (cf. [43, (1.8)-(1.10), and (2.6)]), and

$$\mu\,(F) = s_0(X,x)+s_1(X,x)+\cdots+s_{d-1}(X,x)+s_d(X,x) \tag{4.12}$$

On the other hand, taking into account the $\mathbb{Q}(-d)$-duality between $H^d(F,L,\mathbb{C})$ and $H^d(F,\mathbb{C})$, and the exact MHS-sequence

$$0\longrightarrow H^{d-1}(L,\mathbb{C})\longrightarrow H^d(F,L,\mathbb{C})\longrightarrow H^d(F,\mathbb{C})\longrightarrow H^d(L,\mathbb{C})\longrightarrow 0,$$

one deduces the equalities

$$\begin{aligned} & s_j(X,x) - s_{d-j}(X,x) = \\ & = \ell^{j,d-j}(L) - \ell^{j,d-j-1}(L) = \ell^{d-j,j-1}(L) - \ell^{j,d-j-1}(L) \end{aligned} \tag{4.13}$$

Corollary 4.3 *Let $X = X_f^{(3)}$ be the underlying spaces of the three-dimensional* **A**-**D**-**E** *singularities. Then we have*

$$E(X \smallsetminus \{\mathbf{0}\}; u, v) = (uv-1)\left[1 + \left(1 + h^{1,1}\left(H^2(L,\mathbb{C})\right)\right) uv + (uv)^2\right] \tag{4.14}$$

where

Types	$\mathbf{A}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
$h^{1,1}\left(H^2(L,\mathbb{C})\right)$	$\begin{cases} 1, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$	$\begin{cases} 1, & \text{for } n \text{ odd} \\ 2, & \text{for } n \text{ even} \end{cases}$	0	1	0

Proof. Formula (4.14) is nothing but (4.2) for $d = 3$. So it remains to compute $h^{1,1}\left(H^2(L,\mathbb{C})\right)$. Using the notation $\mu(f) := \mu(F_f)$ and $s_j(f) := s_j(X,\mathbf{0})$ for the singularity $(X,\mathbf{0})$, the equalities (4.8), (4.9) and (4.13) give

$$\ell^{1,1}(L) = h^{1,1}\left(H^2(L,\mathbb{C})\right) = s_2(f) - s_1(f) \tag{4.15}$$

and $s_0(f) = s_3(f)$. Furthermore, by (4.12),

$$\mu(f) = s_0(f) + s_1(f) + s_2(f) + s_3(f) = s_1(f) + s_2(f) + 2s_3(f)\,.$$

In fact, since $(X,\mathbf{0})$ is a *Du Bois singularity* (as it is a rational isolated singularity), or equivalently, since $s_3(f)$ equals the *geometric genus* of $(X,\mathbf{0})$ (see [45, §4], [42, (2.17) and (3.7)]), we have $s_0(f) = s_3(f) = 0$, i.e., $\mu(f) = s_0(f) + s_1(f)$. Now the splitting $f = g + g'$ (as in (1.7)) leads to a "Sebastiani-Thom formula" for Steenbink's invariant; namely,

$$s_j(f) = s_{j-1}(g) \tag{4.16}$$

Applying Milnor's formula [32, Thm. 10.5] for the curve singularity $(X_g,\mathbf{0})$, we obtain

$$\mu(g) = 2\,\delta(g) - r(g) + 1 \tag{4.17}$$

where

$$r(g) := \#\{\text{branches of the curve } X_g \text{ passing through the origin}\}$$

and

$$\delta(g) := \#\{\text{"virtual" double points w.r.t. } X_g\} = \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\widetilde{X_g}} \,/\, \mathcal{O}_{X_g})$$

with $\nu : \widetilde{X}_g \longrightarrow X_g$ the normalization of X_g. Note that this first number $r(g)$ is directly computable because the only types for which $g(x_1,x_2)$'s are reducible, are $\mathbf{A}_n$'s, for n odd, with

$$g(x_1,x_2) = (x_1^{\frac{n+1}{2}} + \sqrt{-1}\,x_2)\,(x_1^{\frac{n+1}{2}} - \sqrt{-1}\,x_2),$$

$\mathbf{D}_n$'s with

$$g(x_1, x_2) = \begin{cases} x_1 \, (x_1^{n-2} + x_2^2), & \text{if } n \text{ is odd} \\ \\ x_1 \, (x_1^{\frac{n}{2}-1} + \sqrt{-1}\, x_2) \, (x_1^{\frac{n}{2}-1} - \sqrt{-1}\, x_2), & \text{if } n \text{ is even} \end{cases}$$

and $\mathbf{E}_7$ with

$$g(x_1, x_2) = x_1(x_1^2 + x_2^3) \ ,$$

while $\delta(g)$ can be read off from (4.17) via the Milnor number. Finally, since

$$s_1(f) \overset{(4.16)}{=} s_0(g) = \delta(g) - r(g) + 1, \quad s_2(f) \overset{(4.16)}{=} s_1(g) = \delta(g), \tag{4.18}$$

(cf. [42, (2.17), p. 526]), we may form the following table:

Types	$\mu(f) = \mu(g)$	$r(g)$	$s_1(f) = s_0(g)$	$s_2(f) = s_1(g) = \delta(g)$
$\mathbf{A}_n$, n odd	n	2	$\frac{n-1}{2}$	$\frac{n+1}{2}$
$\mathbf{A}_n$, n even	n	1	$\frac{n}{2}$	$\frac{n}{2}$
$\mathbf{D}_n$, n odd	n	2	$\frac{n-1}{2}$	$\frac{n+1}{2}$
$\mathbf{D}_n$, n even	n	3	$\frac{n-2}{2}$	$\frac{n+2}{2}$
$\mathbf{E}_6$	6	1	3	3
$\mathbf{E}_7$	7	2	3	4
$\mathbf{E}_8$	8	1	4	4

This table allows us to evaluate $h^{1,1}\left(H^2(L,\mathbb{C})\right)$ for all possible types via (4.18) and (4.15). □

Proof of Theorem 1.11: It follows directly from the explicit arithmetical data for each of the canonical resolutions given in Lemma 2.3 and Proposition 3.1, and from formulae (3.15), (3.16), in combination with the formula (4.14) of Corollary 4.3. □

Final remarks and questions 4.4 (i) Is the resolution algorithm (or a slight modification of it) extendible to a wider class of three-dimensional Gorenstein terminal (or canonical) singularities?

(ii) The d-dimensional generalization of Theorem 1.11 seems to be feasible as the pattern of the local reduction of simple singularities remains invariant (after all, adding quadratic terms does not cause very crucial changes in the desingularization procedure), though the investigation of the structure of the corresponding exceptional prime divisors and of their intersections for the **D-E**'s might be rather complicated.

(iii) Since the string-theoretic "adjusting property" of $E_{\rm str}$-functions is of local nature and focuses solely on the *singular loci* of the varieties being under consideration, it is clear how to treat of $E_{\rm str}$ and $e_{\rm str}$ in global geometric constructions with prescribed **A-D-E**-singularities. We close the paper by giving some examples of this sort.

5 Global geometric applications

In view of Theorem 1.11, the E_{str}-function of a complex threefold Y having only **A**-**D**-**E**-singularities $q_1, q_2, .., q_k$ is computable provided that one knows how to determine the Hodge numbers $h^{p,q}(H_c^i(Y,\mathbb{C}))$ of Y, as we obtain:

$$E_{\mathrm{str}}(Y;u,v) = E(Y \smallsetminus \{q_1,q_2,\ldots,q_k\};u,v) \ + \sum_{i=1}^{k} E_{\mathrm{str}}((Y,q_i);u,v) =$$

$$= E(Y;u,v) \ + \sum_{i=1}^{k} (E_{\mathrm{str}}((Y,q_i);u,v) - 1) \tag{5.1}$$

(a) **Complete intersections in projective spaces.** A very simple closed formula for e_{str} can be built whenever Y is a (global) complete intersection in a projective space.

Proposition 5.1 *Let $Y = Y_{(d_1,d_2,\ldots,d_{r-3})}$ be a three-dimensional complete intersection of multidegree $(d_1,d_2,\ldots,d_{r-3})$ in $\mathbb{P}^r_{\mathbb{C}}$ having only k isolated singularities $q_1,q_2,\ldots,q_k$ of type* **A**-**D**-**E**. *Then its string-theoretic Euler number equals*

$$e_{\mathrm{str}}(Y) = \left[\binom{r+1}{3} + \sum_{\nu=1}^{3}(-1)^{\nu}\binom{r+1}{3-\nu}\left(\sum_{1\leq j_1\leq\ldots\leq j_\nu\leq r-3} d_{j_1}\cdots d_{j_\nu}\right)\right]\left(\prod_{j=1}^{r-3} d_j\right) +$$

$$+ \sum_{i=1}^{k}\left[e_{\mathrm{str}}(Y,q_i) + \mu(Y,q_i) - 1\right] \tag{5.2}$$

where $\mu(Y,q_i)$ is the Milnor number of the singularity (Y,q_i) and $e_{\mathrm{str}}(Y,q_i)$ can be read off from the Theorem 1.11.

Proof. Considering a small deformation of Y one can always obtain a non-singular complete intersection Y' in $\mathbb{P}^r_{\mathbb{C}}$ having multidegree $(d_1,d_2,\ldots,d_{r-3})$. If we take a ball B_i in $\mathbb{P}^r_{\mathbb{C}}$ centered at the point q_i, then, choosing B_i small enough, $\overline{B_i}\cap Y$ is contractible and $\overline{B_i}\cap Y'$ can be identified with the (closed) Milnor fiber of the singularity (Y,q_i). $\widehat{Y} := Y\smallsetminus(\bigcup_{i=1}^k B_i)$ and $\widehat{Y}' := Y'\smallsetminus(\bigcup_{i=1}^k B_i)$ are homeomorphic. Therefore $e(\widehat{Y}) = e(\widehat{Y}')$. Using Mayer-Vietoris sequence for the splitting $Y = \widehat{Y}\cup\bigcup_{i=1}^k(\overline{B_i}\cap Y)$, on the one hand, and for the splitting $Y' = \widehat{Y}'\cup\bigcup_{i=1}^k(\overline{B_i}\cap Y')$, on the other, we get $e(Y) = e(\widehat{Y}) + k$ and

$$e(Y') \ = \ e(\widehat{Y}') \ + \ k \ - \sum_{i=1}^{k}\mu(Y,q_i),$$

respectively (see [15, Ch. 5, Cor. 4.4 (ii), p. 162]). Hence,

$$e(Y) = e(Y') \ + \sum_{i=1}^{k}\mu(Y,q_i) \ .$$

The Euler number of Y' can be computed in terms of its multidegree data either by determining the χ_y-characteristic of Y' via Riemann-Roch Theorem (see Hirzebruch [25, §2]) or directly by Gauss-Bonnet theorem, i.e., by evaluating the highest Chern class of Y' at its fundamental cycle (cf. [19, p. 416] & Chen-Ogiue [10, Thm. 2.1]), and is expressible by the closed formula:

$$e(Y') = \left[\binom{r+1}{3} + \sum_{\nu=1}^{3} (-1)^{\nu} \binom{r+1}{3-\nu} \left(\sum_{1 \le j_1 \le \cdots \le j_\nu \le r-3} d_{j_1} \cdots d_{j_\nu} \right)\right] \left(\prod_{j=1}^{r-3} d_j \right).$$

Now (5.2) follows clearly from (5.1). □

Examples 5.2 (i) If Y possesses only $\mathbf{A}_1$-singularities (i.e., "ordinary double points" or "nodes"), then the second summand in (5.2) equals $2\,\#(\text{nodes of } Y)$. Let us apply (5.2) for some well-known hypersurfaces Y in $\mathbb{P}^4_{\mathbb{C}}$ with *many* nodes. [$e_{\rm str}(Y)$ is nothing but the Euler number of the overlying spaces of the so-called (simultaneous) "small resolutions" of the nodes of Y's.].

▶ **Schoen's quintic** [37]. This is the quintic

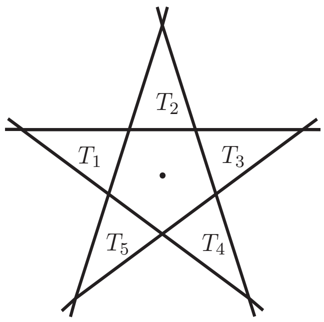
$$Y = \left\{ (z_1 : \ldots : z_5) \in \mathbb{P}^4_{\mathbb{C}} \;\middle|\; \sum_{i=1}^{5} z_i^5 - 5 \prod_{i=1}^{5} z_i = 0 \right\}$$

having 125 nodes, namely the members of the orbit of point $(1:1:1:1:1)$ under the action of the group which is generated by the coordinate transformations

$$(z_1 : \ldots : z_5) \longmapsto (z_1 : \zeta_5^{\alpha_1} z_2 : \ldots : \zeta_5^{\alpha_4} z_5),$$

where $\zeta_5 = e^{\frac{2\pi\sqrt{-1}}{5}}$, $\sum_{j=1}^{4} \alpha_j \equiv 0 (\text{mod } 5)$. Hence, $e_{\rm str}(Y) = -200 + 2 \cdot 125 = 50$.

▶ **Hirzebruch's quintic** [26]. Let $\{\mathbf{\Phi}(x,y) = \prod_{i=1}^{5} \Phi_i(x,y) = 0\}$ be the equation of the curve of degree 5 in the real (x,y)-plane constructed by the five lines $\Phi_i(x,y) = 0, 1 \le i \le 5$, of a regular pentagon:



This real picture shows that both partial derivatives of $\mathbf{\Phi}$ vanish at the 10 points of line intersections, as well as at one point t_i at every triangle T_i and at the center of the pentagon. Moreover, by symmetry, one has $\mathbf{\Phi}(t_i) = \mathbf{\Phi}(t_j)$ for all $1 \le i \le j \le 5$. The hypersurface $Y \subset \mathbb{P}^4_{\mathbb{C}}$ obtained after homogenization of the three-dimensional affine complex variety

$$\left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \;\middle|\; \mathbf{\Phi}(z_1, z_2) - \mathbf{\Phi}(z_3, z_4) = 0 \right\}$$

has $10^2 + 5^2 + 1^2 = 126$ nodes. This means that $e_{\rm str}(Y) = -200 + 2 \cdot 126 = 52$.

▶ **Symmetric Hypersurfaces.** In $\mathbb{P}^5_{\mathbb{C}}$ with $(z_1 : \ldots : z_6)$ as homogeneous coordinates we define the threefolds

$$\begin{cases} Y_1 := \{(z_1 : \ldots : z_6) \in \mathbb{P}^5_{\mathbb{C}} \mid \sigma_1(z_1, \ldots, z_6) = \sum_{i=1}^{6} z_i^3 = 0\}, \\ Y_2 := \left\{(z_1 : \ldots : z_6) \in \mathbb{P}^5_{\mathbb{C}} \mid \sigma_1(z_1, \ldots, z_6) = \sigma_4(z_1, \ldots, z_6) = 0\right\}, \\ Y_3 := \left\{(z_1 : \ldots : z_6) \in \mathbb{P}^5_{\mathbb{C}} \;\middle|\; \begin{array}{c} \sigma_1(z_1, \ldots, z_6) = \sigma_5(z_1, \ldots, z_6) + \\ +\sigma_2(z_1, \ldots, z_6)\, \sigma_3(z_1, \ldots, z_6) = 0 \end{array} \right\}, \end{cases}$$

where

$$\sigma_j(z_1, \ldots, z_6) = \sum_{1 \le \kappa_1 < \kappa_2 < \cdots < \kappa_j \le 6} z_{\kappa_1} \cdot z_{\kappa_2} \cdot \cdots \cdot z_{\kappa_j}, \quad 1 \le j \le 6,$$

denote the elementary symmetric polynomials with respect to the variables $z_1, .., z_6$. Obviously, Y_i's are invariant under the symmetry group $\mathfrak{S}_6$ acting on $\mathbb{P}^5_{\mathbb{C}}$ by permuting coordinates. Moreover, since the first equation

$$\sigma_1(z_1, \ldots, z_6) = z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0$$

is linear, Y_i's can be thought of as hypersurfaces in

$$\mathbb{P}^4_{\mathbb{C}} = \left\{(z_1 : \ldots : z_6) \in \mathbb{P}^5_{\mathbb{C}} \mid \sigma_1(z_1, \ldots, z_6) = 0\right\}.$$

The threefold Y_1 has 10 nodes, namely the points of $\mathbb{P}^5_{\mathbb{C}}$ for which three of their coordinates are 1 and the other three are -1 (i.e., just the members of the $\mathfrak{S}_6$-orbit of $(1 : 1 : 1 : -1 : -1 : -1)$). Correspondingly, Y_2 has 45 nodes, and Y_3 has 130 nodes, 10 constituting the $\mathfrak{S}_6$-orbit of $(1 : 1 : 1 : -1 : -1 : -1)$, 90 in the $\mathfrak{S}_6$-orbit of $(1 : 1 : -1 : -1 : \sqrt{-3} : -\sqrt{-3})$ and 30 more in the $\mathfrak{S}_6$-orbit of $(1 : 1 : 1 : 1 : \sqrt{-3} - 2 : -\sqrt{-3} - 2)$. The following table gives their special names, their string-theoretic Euler numbers, as well as the main references for further reading about their geometric properties. (Note that Y_1 and Y_2 attain exactly the upper bound for the cardinality of nodes for cubics and quartics in $\mathbb{P}^4_{\mathbb{C}}$, respectively. Y_3 is, to the best of our knowledge, the quintic in $\mathbb{P}^4_{\mathbb{C}}$ with the largest known number of nodes).

Threefolds	**Name**	Ref.	e_{str}
Y_1	**Segre's cubic**	[41]	$-6 + 2 \cdot 10 = 14$
Y_2	**Burkhart's quartic**	[8], [17]	$-56 + 2 \cdot 45 = 34$
Y_3	**van Straten's quintic**	[48]	$-200 + 2 \cdot 130 = 60$

(ii) Let now Y_1, Y_2 be the three-dimensional complete intersections of two quadrics

$$Y_i := \left\{\mathbf{z} = (z_1 : z_2 : .. : z_6) \in \mathbb{P}^5_{\mathbb{C}} \mid {}^t\mathbf{z}\, M_i\, \mathbf{z} = {}^t\mathbf{z}\, M_i'\, \mathbf{z} = 0\right\}, \; i = 1, 2,$$

where

$$M_1 = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \qquad M_1' = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}, \qquad M_2' = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.$$

Y_1 and Y_2 have $q = (1:0:0:0:0:0)$ as single isolated point and belong to a family of complete intersections which have been studied extensively by Segre [40] and Knörrer [28, pp. 38-51]. (Y_1, q) turns out to be an $\mathbf{A}_5$-singularity and (Y_2, q) a $\mathbf{D}_6$-singularity. For both Y_1 and Y_2 the first summand in (5.2) equals

$$\left[\binom{6}{3} - 2\cdot 2\cdot \binom{6}{2} + 3\cdot 2^2\cdot \binom{6}{1} - 4\cdot 2^3\right]\ (2^2) = 0\ .$$

Hence, $e_{\mathrm{str}}(Y_1) = 2 + 5 - 1 = 6 \in \mathbb{Z}$, whereas

$$e_{\mathrm{str}}(Y_2) = \tfrac{2633}{864} + 6 - 1 = 8 + \tfrac{41}{864} \in \mathbb{Q}\smallsetminus\mathbb{Z}\ .$$

(b) **Fiber products of elliptic surfaces over $\mathbb{P}^1_{\mathbb{C}}$.** Another kind of compact complex threefolds having both $\mathbf{A}_1$ and $\mathbf{A}_2$-singularities arises from a slight generalization of Schoen's construction [38]. Let $Z \to \mathbb{P}^1_{\mathbb{C}}$ and $Z' \to \mathbb{P}^1_{\mathbb{C}}$ denote two relatively minimal, rational elliptic surfaces with global sections, and let S (resp. S') be the images of the exceptional fibers of Y (resp. of Y') in $\mathbb{P}^1_{\mathbb{C}}$. The fiber product

$$Y := Z \times_{\mathbb{P}^1_{\mathbb{C}}} Z' \xrightarrow{\pi} \mathbb{P}^1_{\mathbb{C}}$$

is a complex threefold with sigularities located only in the fibers

$$Y_s = \pi^{-1}(s) = Z_s \times Z'_s$$

lying over points $s \in S'' := S \cap S'$. Since the Euler number of any smooth fiber is zero, we have obviously

$$e(Y) = \sum_{s \in S''} e(Z_s)\ e(Z'_s)\,. \tag{5.3}$$

We shall henceforth assume that $S'' = \{s_1, s_2, \ldots, s_\kappa\}$, where for $1 \le i \le \kappa$, Z_{s_i} is of Kodaira type I_{b_i} (i.e., a rational curve with an ordinary double point, if $b_i = 1$, and a cycle of b_i smooth rational curves, if $b_i \ge 2$), while Z'_{s_j} is of Kodaira type $I_{b'_j}$, for all j, $1 \le j \le \nu$, $(\nu < \kappa < 12)$, and of Kodaira type II (i.e., a rational curve with one cusp), for all j, $\nu + 1 \le j \le \kappa$. (See [29, Thm. 6.2] for the classification and Kodaira's notation of exceptional fibers). Under this assumption, Y is a 3-dimensional Calabi-Yau variety with $b_1 b_1' + \cdots + b_\nu b_\nu'$ $\mathbf{A}_1$-singularities (each of which contributing a 2 as string-theoretic Euler number) and $b_{\nu+1} + \cdots + b_\kappa$ $\mathbf{A}_2$-singularities (each of which contributing a $\frac{7}{5}$ as string-theoretic Euler number). Since $e(Z_{s_i}) = b_i$, for all i, $1 \le i \le \kappa$, $e(Z'_{s_j}) = b'_j$, for all j, $1 \le j \le \nu$, and $e(Z'_{s_j}) = 2$, for all j, $\nu + 1 \le j \le \kappa$, the string-theoretic Euler number of Y can be computed by (5.1) and (5.3), and can be written as follows:

$$e_{\mathrm{str}}(Y) = 2\left(\sum_{i=1}^{\nu} b_i b_i'\right) + \frac{12}{5}\left(\sum_{i=\nu+1}^{\kappa} b_i\right) \tag{5.4}$$

Example 5.3 Using Kodaira's homological and functional invariants (cf. [29, §8]), as well as the normal forms of the corresponding Weierstrass models (due to Kas [27]), Herfurtner has shown in detail in [22, cf. Table 3, pp. 336-337] the existence of relatively minimal, rational elliptic surfaces Z_1 (resp. Z_2, Z_3) with sections which possess exactly four exceptional fibers having types I_1, I_1, I_5, I_5 over the ordered 4-tuple of points

$$\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^2, \left(\tfrac{1-\sqrt{5}}{2}\right)^2, 0, \infty\right) \in (\mathbb{P}^1_{\mathbb{C}})^4$$

(resp. types I_1, I_1, I_2, I_8 over $(-1, 1, 0, \infty) \in (\mathbb{P}^1_{\mathbb{C}})^4$, resp. types I_1, I_2, II, I_7 over $(-\frac{9}{4}, -\frac{8}{9}, 0, \infty) \in (\mathbb{P}^1_{\mathbb{C}})^4$). Hence,

$$Y_1 := Z_1 \times_{\mathbb{P}^1_{\mathbb{C}}} Z_3, \qquad \text{(resp. } Y_2 := Z_2 \times_{\mathbb{P}^1_{\mathbb{C}}} Z_3\text{)},$$

has singularities only in the fibers over 0 and ∞; more precisely, it has five $\mathbf{A}_2$-singularities over 0 and 35 $\mathbf{A}_1$-singularities over ∞ (resp., two $\mathbf{A}_2$-singularities over 0 and 56 $\mathbf{A}_1$-singularities over ∞). Consequently, (5.4) gives:

$$e_{\mathrm{str}}(Y_1) = 2 \cdot 35 + \frac{12}{5} \cdot 5 = 82 \in \mathbb{Z}$$

whereas

$$e_{\mathrm{str}}(Y_2) = 2 \cdot 56 + \frac{12}{5} \cdot 2 = 116 + \frac{4}{5} \in \mathbb{Q} \smallsetminus \mathbb{Z}.$$

Acknowledgements. We are indebted to E. Brieskorn for pointing out the relation of [34], [35] with the work of P.J. Giblin, and to P.J. Giblin, who has sent to us a part of his thesis [18] containing results on the canonical resolutions of ADE-singularities.

We would like to express our warmest thanks to Nobuyuki Kakimi (University of Tokyo) who informed us about some mistakes concerning our computations for the discrepancy coefficients in a previous preprint-version of this article.

We are also indebted to J. H. M. Steenbrink for drawing our attention to the formula (4.16) which represents the analogue of Sebastiani-Thom formula for the spectrum of the join of isolated hypersurface singularities. Its use has simplified considerably our initial computations in §4.

References

[1] Arnold V.I.: *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k, and Lagrangian singularities*, Functional Anal. and Appl. **6**, (1972), 254-272.

[2] Artin M.: *On isolated rational singularities of surfaces*, Amer. Jour. of Math. **88**, (1966), 129-136.

[3] Badescu L.: *On certain isolated normal singularities*, Nagoya Math. Jour. **61**, (1976), 205-220.

[4] Batyrev V.V.: *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in "Integrable Systems and Algebraic Geometry", Proceedings of the Taniguchi Symposium 1997, Eds. M.-H. Saito, Y. Shimizu & K. Ueno; World Scientific, (1998), pp. 1-32.

[5] Batyrev V.V.: *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, Journal of the European Math. Soc. **1**, (1999), 5-33.

[6] Batyrev V.V., Borisov L.A.: *Mirror duality and string-theoretic Hodge numbers*, Inventiones Math. **126**, (1996), 183-203.

[7] Batyrev V.V., Dais D.I.: *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35**, (1996), pp. 901-929.

[8] Burkhardt H.: *Untersuchungen aus dem Gebiet der hyperelliptischen Modulfunctionen*, II-III, Math. Ann. **38**, (1890), 161-224, and **40**, (1892), 313-343.

[9] Burns D.: *On rational singularities in dimensions* > 2, Math. Ann. **211**, (1974), pp. 237-244.

[10] Chen B.-Y., Ogiue K.: *Some implications of the Euler-Poincaré characteristic for complete intersection manifolds*, Proc. of the A.M.S. **44**, (1974), 1-8.

[11] Craw A.: *An introduction to motivic integration*, preprint, math.AG/9911179.

[12] Deligne P.: *Théorie de Hodge* III, Publ. Math. I.H.E.S. **44**, (1975), 5-77.

[13] Denef J., Loeser F.: *Motivic integration, quotient singularities and the Mckay correspondence*, preprint, math.AG/9903187.

[14] Dimca A.: *Topics on Real and Complex Singularities*, Vieweg Advanced Lectures in Math., (1987).

[15] Dimca A.: *Singularities and Topology of Hypersurfaces*, Universitext, Springer-Verlag, (1992).

[16] du Val P.: *On isolated singularities of surfaces which do not affect the conditions of adjunction*, Proc. Cambridge Phil. Soc. **30**, (1934), 453-465.

[17] Finkelberg H.: *On the Geometry of Burkhardt Quartic*, PhD Thesis, Leiden, (1989).

[18] Giblin P. J.: *The topology of double points of rank zero on threefolds in* $\mathbb{C}^4$, PhD Thesis, London University, (1967), 188 pages; a brief version of it has been published in Jour. of London Math. Soc., Second Series, Vol. **44**, (1969), 523-530.

[19] GRIFFITHS P., HARRIS J.: *Principles of Algebraic Geometry*, J. Wiley & Sons, (1978).

[20] HAMM H.: *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191**, (1971), 235-252.

[21] HARTSHORNE R.: *Algebraic Geometry*, GTM **52**, Springer-Verlag, (1977).

[22] HERFURTNER S.: *Elliptic surfaces with four singular fibres*, Math. Ann. **291**, (1991), 319-342.

[23] HERSZBERG J.: *Classification of isolated double points of rank zero on primals in* S_n, Jour. of London Math. Soc. **32**, (1957), pp. 198-203.

[24] HIRONAKA H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. **79**, (1964), 109-326.

[25] HIRZEBRUCH F.: *Der Satz von Riemann-Roch in faisceau-theoretischer Formulierung.* In: Proc. International Congress of Math. 1954, Vol. **III**, pp. 457-473. (See also: "Gesammelte Abhandlungen", Bd. **I**, Springer-Verlag, 1987, pp. 128-144.)

[26] HIRZEBRUCH F.: *Some examples of threefolds with trivial canonical bundle*, "Gesammelte Abhandlungen", Bd. **II**, Springer-Verlag, (1987), pp. 757-770.

[27] KAS A.: *Weierstrass normal forms and invariants of elliptic surfaces*, Trans. A.M.S. **225**, (1977), 259-266.

[28] KNÖRRER H.: *Isolierte Singularitäten von Durchschnitten zweier Quadriken*, Bonner Mathematische Schriften, Bd. **117**, (1980).

[29] KODAIRA K.: *On compact complex analytic surfaces* II, Annals of Math. **77**, (1963), 563-626. (See also: "Collected Works", Vol. **III**, Princeton University Press, 1975, pp. 1269-1332).

[30] KONTSEVICH M.: *Motivic integration*, lecture at Orsay, Dec. 7, 1995.

[31] LOOIJENGA E.: *Motivic measures,* preprint, math.AG/0006220.

[32] MILNOR J.: *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies, Vol. **61**, Princeton University Press, (1968).

[33] REID M.: *Canonical threefolds*, Journée de Géométrie Algébrique d'Angers, A. Beauville ed., Sijthoff and Noordhoff, Alphen aan den Rijn, (1980), 273-310.

[34] ROCZEN M.: *Canonical resolutions of the 3-dimensional Arnold-singularities over a field of characteristic* $\neq 2$, Preprint Nr. **20**, Humboldt Universität zu Berlin, Sektion Mathematik, (1981).

[35] ROCZEN M.: *Some properties of the canonical resolutions of the 3-dimensional singularities* A_n, D_n, E_n *over a field of characteristic* $\neq 2$, in "Algebraic Geometry, Proc. of the Int. Conference in Bucharest, Aug. 1982", Lecture Notes in Math., Vol. **1056**, Springer-Verlag, (1984), pp. 297-365.

[36] ROCZEN M.: *Recognition of simple singularities in positive characteristic*, Math. Z. **210**, (1992), 641-654.

[37] SCHOEN C.: *On the geometry of a special determinantal hypersurface associated to the Mumford-Horrocks vector bundle*, Jour. für die reine und ang. Math. **346**, (1986), 85-111.

[38] SCHOEN C.: *On the fiber products of rational elliptic surfaces with section*, Math. Z. **197**, (1988), 177-199.

[39] SEBASTIANI M., THOM R.: *Un résultat sur la monodromie*, Inventiones Math. **13**, (1971), 90-96.

[40] SEGRE C.: *Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensione*, Memorie della Reale Acad. Sc. Torino, Ser. II, **36**, (1883), 3-86. (See also "Opere **III**", pp. 25-126.)

[41] SEGRE C.: *Sulla varieta cubica con dieci punti doppi dello spazio a quattro dimensioni*, Atti Acc. Torino **22**, (1887), 791-801.

[42] STEENBRINK J.H.M.: *Mixed Hodge structures associated with isolated singularities.* In: Proc. of Symposia in Pure Math., Vol. **40**, Part **II**, AMS, (1983), pp. 513-536.

[43] STEENBRINK J.H.M.: *Semicontinuity of the singularity spectrum*, Inventiones Math. **79**, (1985), 557-565.

[44] STEENBRINK J.H.M.: *Mixed Hodge structures and singularities*, manuscript, 1990.

[45] STEENBRINK J.H.M.: *Du Bois invariants of isolated complete intersection singularities*, Ann. Inst. Fourier **45**, No 5, (1997), 1367-1377.

[46] TREGER R.: *Rational singularities in dimension* ≥ 2, in "Algebraic Geometry, Summer Meeting, Copenhagen, Aug. 1978", Lecture Notes in Math., Vol. **732**, Springer-Verlag, (1979), pp. 592-604.

[47] VAFA C.: *String vacua and orbifoldized LG models*, Modern Phys. Lett. **A4**, (1989), 1169-1185.

[48] VAN STRATEN D.: *A quintic hypersurface in* $\mathbb{P}^4$ *with 130 nodes*, Topology **32**, (1993), 857-864.

[49] WITTEN E.: *Phases of* $N = 2$ *theories in two dimensions*, Nuclear Phys. **B**, Vol. **403**, (1993), 159-222.

[50] ZASLOW E.: *Topological orbifold models and quantum cohomology rings*, Commun. in Math. Phys. **156**, (1993), 301-331.