Marko Roczen

Examples of string-theoretic Euler numbers

dedicated to Mirela Stefanescu
on the occasion of her sixtieth birthday

Abstract: This note gives a short motivation to study stringy invariants and presents results on string theoretic Euler numbers of the 3-dimensional simple singularities, obtained by D. I. Dais and the author in [8] .

0. Introduction

Deformation an resolution of singularities is one of the central topics in algebraic geometry. The local question to study isolated singularities is closely related to the the investigation of properties of their resolutions, which is of global nature. The local part can be well formulated in terms of commutative algebra, for the other geometric techniques related to cohomology show their strength.

Let X be a Calabi-Yau variety, i.e. a d-dimensional, compact complex variety such that $\omega_X \cong \mathcal{O}_X$, $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, d-1$, which has at most canonical singularities. If $f : \tilde{X} \to X$ is a crepant, projective resolution, then an expected mirror partner $\tilde{Y} \to Y$ of f should have the property that Hodge numbers satisfy the condition

$$h^{p,q}(\tilde{X}) = h^{d-p,q}(\tilde{Y}), \quad 0 \le p, q \le d.$$

It is reasonable to ask whether $h^{p,q}(\tilde{X})$ are independent on the resolution $\tilde{X} \to X$. The affirmative answer results from the work of Batyrev and Kontsevich (see [2]). Unfortunately, there exist examples of Calabi-Yau varieties which do not admit a crepant resolution.

Working in the singular category, it is possible to define string-theoretic Hodge invariants which arise from appropriate correction terms in the neighbourhood of singularities.

1. The definitions

Let X be a complex variety, not necessarily compact nor smooth. The cohomology groups $H_c^k(X, \mathbb{C})$ with compact support have a natural mixed Hodge structure which gives rise to the E-polynomial

$$E(X; u, v) := \sum_{0 \le p,q \le d} e^{p,q}(X) u^p v^q, \quad e^{p,q}(X) := \sum_{0 \le k \le 2d} h^{p,q}(H_c^k(X, \mathbb{C}))$$

of X. This polynomial encodes invariants like $e(X) = E(X;1,1)$, the topological Euler characteristic of X. Now suppose X has at most log-terminal singularities and $f : \tilde{X} \to X$ is a resolution of singularities having exceptional divisors D_i $(i \in I = \{1,\dots,r\})$ which are smooth, irreducible with only normal crossings. Then the string-theoretic E-function of X is defined by

$$E_{str}(X;u,v) := \sum_{J\subseteq I} E(D_J^o;u,v) \prod_{j\in J} \frac{uv-1}{(uv)^{a_j+1}-1},$$

where a_j are the discrepancy coefficients,

$$K_{\tilde{X}} - f^*K_X = \sum_{i\in I} a_i D_i, \quad a_i > -1,$$

and $D_J := \cap_{j\in J} D_j$ for $J \neq \emptyset$, $D_\emptyset := \tilde{X}$ and $D_J^o := D_J - \cup_{i\notin J} D_i$. The rational number

$$e_{str}(X) := \lim_{u,v\to 1} E_{str}(X;u,v) = \sum_{J\subseteq I} e(D_J^o) \prod_{j\in J} \frac{1}{a_j+1}$$

is said to be the string-theoretic Euler number of X, and the natural number

$$\mathrm{ind}_{str}(X) := \min\{q \mid e_{str}(X) = \frac{p}{q},\ p,q \in \mathbb{Z},\ q > 0\}$$

is defined to be the stringtheoretic Index of X. In fact, by Batyrev's results, E_{str} is independent on the choice of the resolution, which gives a good sense to the definition of $e_{str}(X)$ and $\mathrm{ind}_{str}(X)$.

Motivated by the examples, in 1997 Batyrev asked the question whether (among all relevant X) the number $\mathrm{ind}_{str}(X)$ is bounded by a constant, depending only on the dimension of X.

2. Stringy invariants of 3-dimensional ADE's

Let (X,x) be a simple, 3-dimensional hypersurface singularity, i.e. one of the singularities of type ADE ([1],[14]). Their canonical resolutions $f : \tilde{X} \to X$ are known since a long time (cf. [16], [12]). Using detailed information on those resolutions, all discrepancy coefficients can be obtained. The E-polynomials are

$$E(\tilde{X} - \{x\};u,v) = E(D_\emptyset^o;u,v) = (uv)^d(E(L;u^{-1},v^{-1});$$

they only on the Link L of the singularity (X,x). For the case given this is nothing but

$$E\left(X - \{x\}\,;u,v\right) = (uv-1)\,\left[1 + \left(1 + h^{1,1}\left(H^2(L,\mathbb{C})\right)\right)\,uv + (uv)^2\right].$$

From the resolution data, remaining terms of E_{str} are obtained by calculations on the surfaces D_i. This gives rise to a complete list of the algebraic functions $E_{str}(X;u,v)$. Especially, we obtain that among A_n-singularities, as well as for D_n, the number ind_{str} has arbitrarily large values, thus answering Batyrev's question.

Here is an example how the results look like. Let $X = A_n$ be defined by the equation $x_1^{n+1} + x_2^2 + x_3^2 + x_4^2$ in 4-space, $n \geq 1$ and x the origin. Then

(i) n even implies

$$E_{str}\,(X;u,v) = w^3 + w - 1 + \sum_{i=2}^{\frac{n}{2}} \tfrac{(w-1)\left(w^2-1\right)}{w^{i+1}-1} + \tfrac{(w-1)w^2}{w^{n+3}-1}$$
$$+ (w-1)\left(w^2-1\right)\left[\sum_{i=1}^{\frac{n}{2}-1} \tfrac{1}{(w^{i+1}-1)(w^{i+2}-1)} + \tfrac{1}{(w^{\frac{n}{2}+1}-1)(w^{n+3}-1)}\right],$$

(ii) n odd implies

$$E_{str}\,(X;u,v) = (w-1)\,(w+1)^2 + w$$
$$+ \left(w^2-1\right)\left[\sum_{i=2}^{\frac{n-1}{2}} \tfrac{(w-1)}{w^{i+1}-1} + \tfrac{w}{w^{\frac{n+3}{2}}-1} + \sum_{i=1}^{\frac{n-1}{2}} \tfrac{(w-1)}{(w^{i+1}-1)(w^{i+2}-1)}\right].$$

The stringy Euler numbers are obtained as

(i) $e_{str}(X) = 2 - \dfrac{3}{n+3}$, and

(ii) $e_{str}(X) = 2$, respectively.

Therefore, the following values for string theoretic indices of A_n are obtained,

$$\mathrm{ind}_{str}(X) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 2 \\ n+3, & \text{if } n \equiv 2 \text{ or } 4 \pmod 6 \\ \frac{n}{3}+1, & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

3. Application

Globally, from the previous results string-theoretic Euler numbers can be obtained for complex 3-folds with known Hodge numbers, if they have only ADE-singularities. Among them, there are symmetric hypersurfaces like Segre's cubic ($e_{str} = 14$), Burkhart's quartic [6], where $e_{str} = 34$, van Straten's quintic [20], where $e_{str} = 60$, or Knörrer's quadric's [15] with $e_{str} = 6$ and $e_{str} = 8\frac{41}{864}$, respectively.

References

[1] Arnold V.I.: *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k, and Lagrangian singularities,*Functional Anal. and Appl. **6**, (1972), 254-272.

[2] Batyrev V.V.: *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, Journal of the European Math. Soc. **1**, (1999), 5-33.

[3] Batyrev V.V.: *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in "Integrable Systems and Algebraic Geometry", Proceedings of the Taniguchi Symposium 1997, Eds. M.-H. Saito, Y. Shimizu & K. Ueno; World Scientific, (1998), pp. 1-32.

[4] Batyrev V.V.: *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, Journal of the European Math. Soc. **1**, (1999), 5-33.

[5] Batyrev V.V., Borisov L.A.: *Mirror duality and string-theoretic Hodge numbers*, Inventiones Math. **126**, (1996), 183-203.

[6] Burkhardt H.: *Untersuchungen aus dem Gebiet der hyperelliptischen Modulfunctionen*, II-III, Math. Ann. **38**, (1890), 161-224, and **40**, (1892), 313-343.

[7] Craw A.: *An introduction to motivic integration*, preprint, math.AG/9911179.

[8] Dais D.I., Roczen M.: *On the String-Theoretic Euler Number of 3-dimensional A-D-E Singularities*, to appear in Advances in Geometry 1 (4) 2001.

[9] Deligne P.: *Théorie de Hodge* III, Publ. Math. I.H.E.S. **44**, (1975), 5-77.

[10] Denef J., Loeser F.: *Motivic integration, quotient singularities and the Mckay correspondence*, preprint, math.AG/9903187.

[11] du Val P.: *On isolated singularities of surfaces which do not affect the conditions of adjunction*, Proc. Cambridge Phil. Soc. **30**, (1934), 453-465.

[12] Giblin P. J.: *The topology of double points of rank zero on threefolds in C^4*, PhD Thesis, London University, (1967), 188 pages; a brief version of it has been published in Jour. of London Math. Soc., Second Series, Vol. **44**, (1969), 523-530.

[13] Hironaka H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. **79**, (1964), 109-326.

[14] Herszberg J.: *Classification of isolated double points of rank zero on primals in S_n*, Jour. of London Math. Soc. **32**, (1957), pp. 198-203.

[15] Knörrer H.: *Isolierte Singularitäten von Durchschnitten zweier Quadriken*, Bonner Mathematische Schriften, Bd. **117**, (1980).

[16] Roczen M.: *Some properties of the canonical resolutions of the 3-dimensional singularities A_n, D_n, E_n over a field of characteristic* $\neq 2$, in "Algebraic Geometry, Proc. of the Int. Conference in Bucharest, Aug. 1982", Lecture Notes in Math., Vol. **1056**, Springer-Verlag, (1984), pp. 297-365.

[17] Roczen M.: *Recognition of simple singularities in positive characteristic*, Math. Z. **210**, (1992), 641-654.

[18] Steenbrink J.H.M.: *Du Bois invariants of isolated complete intersection singularities*, Ann. Inst. Fourier **45**, No 5, (1997), 1367-1377.

[19] Treger R.: *Rational singularities in dimension* ≥ 2, in "Algebraic Geometry, Summer Meeting, Copenhagen, Aug. 1978", Lecture Notes in Math., Vol. **732**, Springer-Verlag, (1979), pp. 592-604.

[20] van Straten D.: *A quintic hypersurface in P^4 with 130 nodes*, Topology **32**, (1993), 857-864.