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Operations Research Letters 40 (2012) 313–318

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Operations Research Letters

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SDDP for multistage stochastic linear programs based on spectral risk measures

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ARTICLE INFO

Article history:
Received 8 February 2012
Received in revised form
29 March 2012
Accepted 10 April 2012
Available online 15 May 2012

Keywords:
Spectral risk measure
Stochastic programming
Risk-averse optimization
Decomposition algorithms
Monte Carlo sampling

ABSTRACT

We consider risk-averse formulations of multistage stochastic linear programs. For these formulations, based on convex combinations of spectral risk measures, risk-averse dynamic programming equations can be written. As a result, the Stochastic Dual Dynamic Programming (SDDP) algorithm can be used to obtain approximations of the corresponding risk-averse recourse functions. This allows us to define a risk-averse nonanticipative feasible policy for the stochastic linear program. Formulas for the cuts that approximate the recourse functions are given. In particular, we show that some cut coefficients have analytic formulas.

1. Introduction

Multistage stochastic programs play a central role when developing optimization models under stochastic uncertainty in engineering, transportation, finance, and energy. Furthermore, since measuring, bounding, or minimizing the risk of decisions becomes more and more important in applications, risk-averse formulations of such optimization models are needed and have to be solved. Several risk-averse model variants allow for a reformulation as a classical multistage model, as in [6,8] and the present paper. From a mathematical point of view, multistage stochastic optimization methods represent infinite-dimensional models in spaces of random vectors satisfying certain moment conditions and contain high-dimensional integrals. Hence, their numerical solution is a challenging task. Each solution approach consists at least of two ingredients: (i) numerical integration methods for computing the expectation functionals and (ii) algorithms for solving the resulting finite-dimensional optimization models.

The favorite approach for (i) is to generate possible scenarios (i.e., realizations) of the random vector involved and to use them as 'grid points' for the numerical integration. Scenario generation can be done by Monte Carlo, quasi-Monte Carlo, or optimal quantization methods (see [5,18] for overviews and [3, Part III] for further information). Scenarios for multistage stochastic programs have to be tree structured to model the increasing chain of σ-fields. Existing stability and convergence results such as those in [11,10,12,21] provide approaches and conditions implying the convergence of such schemes, in particular, for the deterministic first-stage solutions. Hence, they justify rolling horizon approaches based on repeated solving of multistage models; see [9], for instance.

The algorithms employed for (ii) depend on structural properties of the basic optimization model and on the inherent structure induced by the scenario tree approximation (see the survey [19] on decomposition methods).

Some algorithmic approaches incorporate the scenario generation method (i) as an algorithmic step of the solution method. Such approaches are, for example, stochastic decomposition methods for multistage models (see [20]), approximate dynamic programming (see [17]), and Stochastic Dual Dynamic Programming (SDDP), initiated in [13], revisited in [16,22], and also studied in the present paper.

We consider risk-averse formulations of multistage stochastic linear programs of the form

$$
\begin{aligned}
&\inf_{x_1,\dots,x_T} d_1^\top x_1 + \theta_1 \mathbb{E}\left[\sum_{t=2}^{T} d_t^\top x_t\right] + \sum_{t=2}^{T} \theta_t \rho_\phi\left(-\sum_{k=2}^{t} d_k^\top x_k\right) \\
&C_t x_t = \xi_t - D_t x_{t-1}, \\
&x_t \geq 0, \quad x_t \text{ is } \mathcal{F}_t\text{-measurable}, t = 1, \dots, T,
\end{aligned} \tag{1}
$$

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doi:10.1016/j.orl.2012.04.006

where x_0 is given, parameters d_t, C_t, D_t are deterministic, $(\xi_t)_{t=1}^T$ is a stochastic process, $\mathcal{F}_t$ is the sigma-algebra $\mathcal{F}_t := \sigma(\xi_j, j \le t)$, $(\theta_t)_{t=1}^T$ are nonnegative weights summing to 1, and ρ_ϕ is a *spectral risk measure* [1] or *distortion risk measure* [14,15] depending on a risk spectrum $\phi \in L_1([0,1])$. In the above formulation, we have assumed that the (one-period) spectral risk measure takes as argument a random income and that the trajectory of the process is known until the first stage. We assume *relatively complete recourse* for (1), which means that, for any feasible sequence of decisions $(x_1,\ldots,x_t)$ to any t-stage scenario $(\xi_1,\xi_2,\ldots,\xi_t)$, there exists a sequence of feasible decisions $(x_{t+1},\ldots,x_T)$ with probability 1. A non-risk-averse model amounts to taking $\theta_1 = 1$ and $\theta_t = 0$ for $t = 2,\ldots,T$. A more general risk-averse formulation for multistage stochastic programs is considered in [8]. For these models, dynamic programming (DP) equations are written in [8] and an SDDP algorithm is detailed to obtain approximations of the corresponding recourse functions in the form of cuts. The main contribution of this paper is to provide analytic formulas for some cut coefficients, independent of the sampled scenarios, that can be useful for implementation. We also specialize the SDDP algorithm and especially the computation of the cuts for the particular risk-averse model (1).

We start by setting down some notation.

- e will denote a column vector of all 1s;
- for $x, y \in \mathbb{R}^n$, the vector $x \circ y \in \mathbb{R}^n$ is defined by $(x \circ y)(i) = x(i)y(i),\ i = 1,\ldots,n$;
- for $x \in \mathbb{R}^n$, the vector $x^+ \in \mathbb{R}^n$ is defined by $x^+(i) = \max(x(i), 0),\ i = 1,\ldots,n$;
- the available history of the process at stage t is denoted by $\xi_{[t]} := (\xi_j, j \le t)$;
- for vectors $x_1,\ldots,x_n$, the notation $x_{n_1:n_2}$ stands for the concatenation $(x_{n_1}, x_{n_1+1},\ldots,x_{n_2})$ for $1 \le n_1 \le n_2 \le n$;
- δ_{ij} is the Kronecker delta defined for i, j integers by $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

2. Risk-averse dynamic programming

Let $F_Z(x) = \mathbb{P}(Z \le x)$ be the cumulative distribution function of an essentially bounded random variable Z, and let $F_Z^{\leftarrow}(p) = \inf\{x : F_Z(x) \ge p\}$ be the generalized inverse of F_Z. Given a risk spectrum $\phi \in L_1([0,1])$, the spectral risk measure ρ_ϕ generated by ϕ is (see [1]):

$$\rho_\phi(Z) = -\int_0^1 F_Z^{\leftarrow}(p)\phi(p)dp.$$

Spectral risk measures have been used in various applications (portfolio selection by Acerbi and Simonetti [2]; insurance by Cotter and Dowd [4]). The conditional value-at-risk (CVaR) of level $0 < \varepsilon < 1$, denoted by $CVaR^\varepsilon$, is a particular spectral risk measure obtained taking $\phi(u) = \frac{1}{\varepsilon} 1_{0 \le u < \varepsilon}$ (see Acerbi [1]).

In what follows, we consider more generally a piecewise constant risk function $\phi(\cdot)$ with J jumps at $0 < p_1 < p_2 < \cdots < p_J < 1$. We set $\Delta\phi_k = \phi(p_k^+) - \phi(p_k^-) = \phi(p_k) - \phi(p_{k-1}),$ for $k = 1,\ldots,J$, with $p_0 = 0$, and we assume that

(i) $\phi(\cdot)$ is positive, (ii) $\Delta\phi_k < 0, \quad k = 1,\ldots,J$,

(iii) $\displaystyle\int_0^1 \phi(u)du = 1.$

In this context, ρ_ϕ can be expressed as a linear combination of conditional value-at-risk measures. With this choice of risk function ϕ, the spectral risk measure $\rho_\phi(Z)$ can be expressed as the optimal value of a linear program; see Acerbi and Simonetti [2]:

$$\rho_\phi(Z) = \inf_{w \in \mathbb{R}^J} \sum_{k=1}^J \Delta\phi_k [p_k w_k - \mathbb{E}[w_k - Z]^+] - \phi(1)\mathbb{E}[Z]. \tag{2}$$

Using this formulation for ρ_ϕ, dynamic programming equations are given in [8] for risk-averse formulation (1). More precisely, problem (1) can be expressed as

$$\begin{aligned} &\inf_{x_1,\, w_{2:T}} \ d_1^\top x_1 + \sum_{t=2}^T \theta_t c_1^\top w_t + \mathcal{Q}_2(x_1, \xi_{[1]}, z_1, w_2, \ldots, w_T), \\ &C_1 x_1 = \xi_1 - D_1 x_0, \quad x_1 \ge 0,\ w_t \in \mathbb{R}^J,\ t = 2,\ldots,T, \end{aligned} \tag{3}$$

with $z_1 = 0$, vector $c_1 = \Delta\phi \circ p$, and where, for $t = 2,\ldots,T$,

$$\begin{aligned} &\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, z_{t-1}, w_{t:T}) \\ &= \mathbb{E}_{\xi_t|\xi_{[t-1]}} \begin{pmatrix} \inf_{x_t, z_t} f_t(z_t, w_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, z_t, w_{t+1:T}) \\ z_t = z_{t-1} - d_t^\top x_t,\ C_t x_t = \xi_t - D_t x_{t-1},\ x_t \ge 0 \end{pmatrix}, \end{aligned} \tag{4}$$

with

$$f_t(z_t, w_t) = -(\delta_{tT}\theta_1 + \phi(1)\theta_t) z_t - \theta_t\ \Delta\phi^\top (w_t - z_t e)^+, \tag{5}$$

and $\mathcal{Q}_{T+1} \equiv 0$. Function $\mathcal{Q}_{t+1}$ represents at stage t a cost-to-go or recourse function which is risk averse. As shown in the next section, it can be approximated by cutting planes by some polyhedral function $\mathfrak{Q}_{t+1}$. These approximate recourse functions are useful for defining a feasible approximate policy obtained by solving

$$\begin{aligned} &\inf_{x_t, z_t} \ f_t(z_t, w_t) + \mathfrak{Q}_{t+1}(x_t, \xi_{[t]}, z_t, w_{t+1:T}) \\ &C_t x_t = \xi_t - D_t x_{t-1}, \quad x_t \ge 0,\ z_t = z_{t-1} - d_t^\top x_t, \end{aligned} \tag{6}$$

at stage $t = 2,\ldots,T$, knowing x_{t-1}, z_{t-1}, first-stage decision variables $w_{t:T}$, and ξ_t. First-stage decision variables x_1 and $w_{2:T}$ are the solution to (3) with $\mathcal{Q}_2$ replaced by the approximation $\mathfrak{Q}_2$.

3. Algorithmic issues

The DP equations (3)–(4) make possible the use of decomposition algorithms such as SDDP to obtain approximations of the corresponding recourse functions. When applied to DP equations (3)–(4), the convergence of this algorithm is proved in [8] under the following assumptions.

(A1) The supports of the distributions of $\xi_1,\ldots,\xi_T$ are discrete and finite.
(A2) Process (ξ_t) is interstage independent.
(A3) For $t = 1,\ldots,T$, for any feasible x_{t-1}, and for any realization $\tilde{\xi}_t$ of ξ_t, the set

$$\{x_t : x_t \ge 0,\ C_t x_t = \tilde{\xi}_t - D_t x_{t-1}\}$$

is bounded and nonempty.

In what follows, we assume that Assumptions (A1)–(A3) hold. In particular, we denote the realizations of ξ_t by $\xi_t^i,\ i = 1,\ldots,q_t < +\infty$, and set $p(t,i) = \mathbb{P}(\xi_t = \xi_t^i)$.

Since the supports of the distributions of the random vectors $\xi_2,\ldots,\xi_T$ are discrete and finite, optimization problem (1) is finite dimensional, and the evolution of the uncertain parameters over the optimization period can be represented by a scenario tree having a finite number of scenarios that can happen in the future for $\xi_2,\ldots,\xi_T$. The root node of the scenario tree corresponds to the first time step with ξ_1 deterministic.

For a given stage t, to each node of the scenario tree there corresponds an history $\xi_{[t]}$. The children nodes of a node at stage $t \ge 1$ are the nodes that can happen at stage $t+1$ if we are at this node at t. A sampled scenario $(\xi_1,\ldots,\xi_T)$ corresponds to a particular succession of nodes such that ξ_t is a possible value for the process at t and ξ_{t+1} is a child of ξ_t. A given node in the tree at stage t is identified with a scenario $(\xi_1,\ldots,\xi_t)$ going from the root node to this node.

In this context, the SDDP algorithm builds polyhedral lower bounding approximations $\mathfrak{Q}_t$ of $\mathcal{Q}_t$ for $t = 2,\ldots,T+1$. Each iteration of this algorithm is made of a forward pass followed by a backward pass. Approximation $\mathfrak{Q}_t^i$ for $\mathcal{Q}_t$ available at the end of iteration i can be expressed as a maximum of cuts (hyperplanes lying below the recourse functions) built in the backward passes:

$$\mathfrak{Q}_t^i(x_{t-1}, z_{t-1}, w_{t:T}) = \max_{j=0,1,\ldots,iH} \Bigg[-E_{t-1}^j x_{t-1} - Z_{t-1}^j z_{t-1} + \sum_{\tau=1}^{T-t+1} W_{t-1}^{j,\tau} w_{t+\tau-1} + e_{t-1}^j \Bigg], \tag{7}$$

knowing that the algorithm starts taking for $\mathfrak{Q}_t^0$ a known lower bounding affine approximation of $\mathcal{Q}_t$ while $\mathfrak{Q}_{T+1}^i \equiv 0$. In the above expression, we have assumed that H cuts are built at each iteration. If the algorithm runs for K iterations, we end up with approximate recourse functions $\mathfrak{Q}_t = \mathfrak{Q}_t^K$, $t = 2, \ldots, T+1$.

At iteration i, cuts for $\mathcal{Q}_t$, $t = 2, \ldots, T$, are built at some points $x_{t-1}^k, z_{t-1}^k, w_{t:T}^i$, $k = (i-1)H+1, \ldots, iH$, computed in the forward pass replacing the recourse functions $\mathcal{Q}_{t+1}$ by $\mathfrak{Q}_{t+1}^{i-1}$ (note that, since variables $w_{2:T}$ are first-stage decision variables, they just depend on the iteration).

More precisely, the cuts are computed for time step $T+1$ down to time step 2. For time step $T+1$, since $\mathfrak{Q}_{T+1}^i = \mathcal{Q}_{T+1} = 0$, the cuts for $\mathcal{Q}_{T+1}$ are obtained taking null vectors for $E_T^k, Z_T^k, W_T^{k,\tau}$, and e_T^k for $k = (i-1)H+1, \ldots, iH$. For $t = 2, \ldots, T$, using the lower bounding approximation $\mathfrak{Q}_{t+1}^i$ of $\mathcal{Q}_{t+1}$, we can bound from below $\mathcal{Q}_t(x_{t-1}, z_{t-1}, w_{t:T})$ by $\mathbb{E}_{\xi_t}[Q_t^i(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t)]$ with $Q_t^i(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t)$ given as the optimal value of the following linear program:

$$\inf_{x_t, z_t, v_t, \tilde{\theta}_t} -(\delta_{tT}\theta_1 + \phi(1)\theta_t)z_t - \theta_t \Delta\phi^\top v_t + \tilde{\theta}_t$$

$$v_t \ge 0, \qquad v_t \ge w_t - z_t e, \qquad x_t \ge 0,$$

$$z_t + d_t^\top x_t = z_{t-1} \tag{8a}$$

$$C_t x_t = \xi_t - D_t x_{t-1} \tag{8b}$$

$$\overrightarrow{E}_t^i x_t + \overrightarrow{Z}_t^i z_t + \tilde{\theta}_t e \ge \sum_{\tau=1}^{T-t} \overrightarrow{W}_t^{i,\tau} w_{t+\tau} + \overrightarrow{e}_t^i, \tag{8c}$$

where $\overrightarrow{E}_t^i$ (respectively, $\overrightarrow{Z}_t^i$, $\overrightarrow{W}_t^{i,\tau}$, and $\overrightarrow{e}_t^i$) is the matrix whose $(j+1)$th line is E_t^j (respectively, Z_t^j, $W_t^{j,\tau}$, and e_t^j) for $j = 0, \ldots, iH$. In the backward pass of iteration i, the above problem is solved with $(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t)$ respectively replaced by $(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j)$ for $k = (i-1)H+1, \ldots, iH$ and $j = 1, \ldots, q_t$. Let $\sigma_t^{k,j}, \tilde{\sigma}_t^{k,j}, \mu_t^{k,j}, \pi_t^{k,j}$, and $\rho_t^{k,j}$ be the (row vectors) optimal Lagrange multipliers respectively for the constraints $v_t \ge w_t^i - z_t e$, $v_t \ge 0$, (8a), (8b) and (8c) for the problem defining $Q_t^i(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j)$ for $k = (i-1)H+1, \ldots, iH$ and $j = 1, \ldots, q_t$. The following proposition provides the cuts computed for $\mathcal{Q}_t$, $t = 2, \ldots, T$, at iteration i.

Proposition 3.1 (*Optimality Cuts*)**.** *Let $\mathcal{Q}_t$, $t = 2, \ldots, T+1$, be the risk-averse recourse functions given by (4). In the backward pass of iteration i of the SDDP algorithm, the following cuts are computed for these recourse functions. For $t = T+1$, E_{t-1}^k, Z_{t-1}^k, $W_{t-1}^{k,\tau}$, and e_{t-1}^k are null for $k = (i-1)H+1, \ldots, iH$. For $t = 2, \ldots, T$ and $k = (i-1)H+1, \ldots, iH$, E_{t-1}^k is given by $\sum_{j=1}^{q_t} p(t,j)\pi_t^{k,j} D_t$, and*

$$Z_{t-1}^k = -\sum_{j=1}^{q_t} p(t,j)\mu_t^{k,j}, \qquad W_{t-1}^{k,1} = \sum_{j=1}^{q_t} p(t,j)\sigma_t^{k,j}, \tag{9}$$

$$W_{t-1}^{k,\tau} = \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\overrightarrow{W}_t^{i,\tau-1}, \quad \tau = 2, \ldots, T-t+1. \tag{10}$$

Further, e_{t-1}^k is given by

$$\sum_{j=1}^{q_t} p(t,j) \Bigg[Q_t^i(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j) - \mu_t^{k,j} z_{t-1}^k - \sigma_t^{k,j} w_t^i - \sum_{\tau=1}^{T-t} \rho_t^{k,j} \overrightarrow{W}_t^{i,\tau} w_{t+\tau}^i + \pi_t^{k,j} D_t x_{t-1}^k \Bigg].$$

Proof. Since a dual solution of the problem defining $Q_t^i(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j)$ is a subgradient of the value function for problem (8), we obtain that $Q_t^i(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t^j)$ is bounded from below by

$$Q_t^i(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j) + \mu_t^{k,j}(z_{t-1} - z_{t-1}^k) + \sigma_t^{k,j}(w_t - w_t^i) + \sum_{\tau=2}^{T-t+1} \rho_t^{k,j}\overrightarrow{W}_t^{i,\tau-1}(w_{t+\tau-1} - w_{t+\tau-1}^i) - \pi_t^{k,j} D_t(x_{t-1} - x_{t-1}^k).$$

Using the above lower bound and the fact that $\mathcal{Q}_t(x_{t-1}, z_{t-1}, w_{t:T})$ is bounded from below by $\sum_{j=1}^{q_t} p(t,j) Q_t^i(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t^j)$, we obtain the announced cuts. □

The stopping criterion is discussed in [22] for a non-risk-averse model. The definition of a sound stopping criterion for the risk-averse model from [22] (based on a nested formulation of the problem defined in terms of conditional risk mappings) is a more delicate issue, and is still open for discussion. However, since problem (1) can be expressed as a non-risk-averse problem with modified objective, variables, and constraints, in our risk-averse context the stopping criterion is a simple adaptation of the stopping criterion for the non-risk-averse case.

More specifically, in the backward pass of iteration i, for the first time step, first-stage problem (3) is solved by replacing the recourse function $\mathcal{Q}_2$ by $\mathfrak{Q}_2^i \le \mathcal{Q}_2$. As a result, the optimal value of this problem gives a lower bound $z_{\inf}$ on the optimal value of (1).

In the forward pass of iteration i, we can compute the total cost $\mathcal{C}_k$ on each scenario $k = (i-1)H+1, \ldots, iH$:

$$\mathcal{C}_k = d_1^\top x_1^k + \sum_{t=2}^{T} \theta_t c_1^\top w_t^i + \sum_{t=2}^{T} f_t(z_t^k, w_t^i). \tag{11}$$

If these H scenarios were representing all possible evolutions of $(\xi_1, \ldots, \xi_T)$, then

$$\bar{\mathcal{C}} = \frac{1}{H} \sum_{k=(i-1)H+1}^{iH} \mathcal{C}_k$$

would be an upper bound on the optimal value of (1) (recall that the approximate policy is feasible and that the objective function of (1) can be written as an expectation). Since we only have a sample of all the possible scenarios, $\bar{\mathcal{C}}$ is an estimation of an upper bound on this optimal value. Introducing the empirical standard deviation $\bar{\sigma}$ of the sample $(\mathcal{C}_1, \ldots, \mathcal{C}_H)$,

$$\bar{\sigma} = \sqrt{\frac{1}{H-1} \sum_{k=(i-1)H+1}^{iH} (\bar{\mathcal{C}} - \mathcal{C}_k)^2},$$

we can compute the $(1-\alpha)$-confidence upper bound

$$\bar{\mathcal{C}} + t_{1-\alpha,H-1}\frac{\bar{\sigma}}{\sqrt{H}} \tag{12}$$

on the approximate policy mean value, where $t_{1-\alpha,H-1}$ is the $(1-\alpha)$-quantile of the Student t-distribution with $H-1$ degrees of freedom. Since the optimal value of (1) is less than or equal to the

Step 0: **INITIALIZATION.** Set $i = 1$ (iteration number) and select confidence levels $\alpha \in (1/2, 1)$ and $\varepsilon > 0$. Take null values for $E^0_{t-1}, Z^0_{t-1}, W^{0,\tau}_{t-1}$, $t = 2, \ldots, T+1$. Take $e^0_T = 0$ and for e^0_{t-1} a lower bound on $\mathcal{Q}_t$ for $t = 2, \ldots, T$. Go to Step 1.

Step 1: **FORWARD PASS.**
Sample H scenarios $(\xi_1, \xi^k_2, \ldots, \xi^k_T)$, $k = (i-1)H + 1, \ldots, iH$.
Ct=0, Ct_Sq=0.
Solve the first stage problem

$$\begin{aligned}&\inf_{x_1,\, w_{2:T}} d_1^\top x_1 + \sum_{t=2}^{T} \theta_t c_1^\top w_t + \mathfrak{Q}^{i-1}_2(x_1, z_1, w_2, \ldots, w_T),\\ &C_1 x_1 = \xi_1 - D_1 x_0,\ x_1 \geq 0,\ w_t \in \mathbb{R}^J,\ t = 2, \ldots, T,\end{aligned}$$

and store an optimal solution $(x^*_1, w^i_{2:T})$.
For $k = (i-1)H + 1, \ldots, iH$,
Set $x^k_1 = x^*_1$.
For $t = 2, \ldots, T$,
Solve

$$\begin{aligned}&\inf_{x_t, z_t} f_t(z_t, w^i_t) + \mathfrak{Q}^{i-1}_{t+1}(x_t, z_t, w^i_{t+1:T})\\ &C_t x_t = \xi^k_t - D_t x^k_{t-1},\ x_t \geq 0,\ z_t = z^k_{t-1} - d_t^\top x_t,\end{aligned}$$

and store an optimal solution (x^k_t, z^k_t).
End For
Compute $\mathcal{C}_k$ given by (11),
Ct=Ct+$\mathcal{C}_k$, Ct_Sq=Ct_Sq+$\mathcal{C}^2_k$.
End For
$\bar{\mathcal{C}} = \frac{\text{Ct}}{H}$, $\bar{\sigma} = \sqrt{\frac{1}{H-1}(\text{Ct\_Sq} - H\bar{\mathcal{C}}^2)}$, $z_{\sup} = \bar{\mathcal{C}} + t_{1-\alpha, H-1}\frac{\bar{\sigma}}{\sqrt{H}}$. Go to Step 2.

Step 2: **BACKWARD PASS.**
For $t = T + 1$ down to 2,
For $k = (i-1)H + 1, \ldots, iH$,
If $(t = T + 1)$ then set $E^k_{t-1}, Z^k_{t-1}, W^{k,\tau}_{t-1}$, and e^k_{t-1} to 0.
Else
For $j = 1, \ldots, q_t$,
Compute $Q^i_t(x^k_{t-1}, z^k_{t-1}, w^i_{t:T}, \xi^j_t)$, i.e., solve (8) replacing $(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t)$ by $(x^k_{t-1}, z^k_{t-1}, w^i_{t:T}, \xi^j_t)$ and store a dual solution.
End For
Build a cut for $\mathcal{Q}_t$, i.e., compute $E^k_{t-1}, Z^k_{t-1}, W^{k,\tau}_{t-1}$, and e^k_{t-1} using the formulas from Proposition 3.1.
End If
End For
End For
Set $z_{\inf}$ to the optimal value of the first stage problem.
Go to Step 3.

Step 3: **STOPPING RULE.**
If $z_{\sup} - z_{\inf} \leq \varepsilon$ then stop.
Else $i \leftarrow i + 1$ and go to Step 1. **End If**

Fig. 1. SDDP algorithm with relatively complete recourse for the risk-averse interstage independent stochastic linear program (1).

approximate policy mean value, (12) gives an upper bound for the optimal value of (1) with confidence at least $1-\alpha$. Consequently, we can stop the algorithm when $\bar{\mathcal{C}} + t_{1-\alpha,H-1}\frac{\bar{\sigma}}{\sqrt{H}} - z_{\inf} \leq \varepsilon$ for some $\varepsilon > 0$.

Using the previous developments, the SDDP algorithm for solving (1) can be formulated as in Fig. 1.

We now give, for some particular choices of the first-stage variables $w^1_{2:T}$, the exact expressions (independent of the sampled scenarios) of Z^k_{t-1} and $W^{k,\tau}_{t-1}$ for every $t = 2, \ldots, T$, $k = 1, \ldots, H$, and $\tau = 1, \ldots, T - t + 1$. Though the first-stage feasible set for (3) is not bounded, it can be easily shown that the optimal values of $w_{2:T}$ are bounded (see [8], for instance). As a result, well-chosen box constraints on w_t, $t = 2, \ldots, T$ can be added (at the first stage, and that do not modify the optimal value of (3)) without changing the cut calculations (since these latter are performed for stages $t = 2, \ldots, T$, where w_t are state variables).

Let us define, for $t = 1, \ldots, T$, $x^t = (x_1, \ldots, x_t)$, $\xi^t = (\xi_1, \ldots, \xi_t)$, and let us introduce the set χ^t of admissible decisions up to time step t:

$$\begin{aligned}\chi^t = \{&x^t : \exists\, \tilde{\xi}^t \text{ realization of } \xi^t : x_\tau \geq 0\\ &\text{and } C_\tau x_\tau = \tilde{\xi}_\tau - D_\tau x_{\tau-1}, \tau = 1, \ldots, t\}.\end{aligned}$$

Since (A3) holds, the sets χ^t are compact, and, since $g^t(x^t) = \sum_{\tau=2}^t d_\tau^\top x_\tau$ is continuous, we can introduce the pairs $(C_t^u, C_t^\ell) \in \mathbb{R}^2$ defined by

$$C_t^u = \begin{cases} \max\ g^t(x^t) \\ x^t \in \chi^t, \end{cases} \qquad C_t^\ell = \begin{cases} \min\ g^t(x^t) \\ x^t \in \chi^t. \end{cases}$$

The objective of the forward pass is to build states where cuts are computed in the backward pass. At the first iteration, instead of building these states using the approximate recourse functions $\mathfrak{Q}_t^0$, we can choose arbitrary feasible states $x_{t-1}^k, z_{t-1}^k, w_t^1$, $t = 2, \ldots, T$ (which is a simple task, since relatively complete recourse holds). With this variant of the first iteration, we have *iH* cuts for $\mathfrak{Q}_t^i$ at the end of iteration *i*. If we choose first-stage variables $w_{2:T}^1$ such that (i) $w_t^1 > -C_t^\ell e$ for $t = 2, \ldots, T$ (respectively, such that (ii) $w_t^1 < -C_t^u e$ for $t = 2, \ldots, T$), then Z_{t-1}^k and $W_{t-1}^{k,\tau}$, for $k = 1, \ldots, H$, can be computed using Proposition 3.2(i) (respectively, Proposition 3.2(ii)), which follows. For instance, if the costs are positive, then item (i) is fulfilled with $w_t^1 = 0$ and item (ii) by taking for each component of w_t^1 the opposite of a strict upper bound on the worst cost.

Proposition 3.2 (*Cuts Calculation at the First Iteration*)**.** *Let us consider the risk-averse recourse functions $\mathcal{Q}_t$ given by (4). Valid cuts for $\mathcal{Q}_t$ are given by Proposition 3.1. Moreover, in the following two cases, we have closed-form expressions for Z_{t-1}^k and $W_{t-1}^{k,\tau}$ (independent of the sampled scenarios).*

(i) *If, for $t = 2, \ldots, T$, $w_t^1 > -C_t^\ell e$, then, for $t = 2, \ldots, T$, $\mathcal{P}(t)$ holds, where*

$$\mathcal{P}(t) : \begin{cases} \forall k = 1, \ldots, H, & Z_{t-1}^k = \theta_1 + \phi(0) \displaystyle\sum_{\ell=t}^T \theta_\ell, \\ \forall k = 1, \ldots, H, & W_{t-1}^{k,\tau} = -\theta_{t+\tau-1} \Delta\phi^\top, \\ \quad \tau = 1, \ldots, T-t+1. & \end{cases}$$

(ii) *If, for $t = 2, \ldots, T$, $w_t^1 < -C_t^u e$, then, for $t = 2, \ldots, T$, $\tilde{\mathcal{P}}(t)$ holds, where*

$$\tilde{\mathcal{P}}(t) : \begin{cases} \forall k = 1, \ldots, H, & Z_{t-1}^k = \theta_1 + \phi(1) \displaystyle\sum_{\ell=t}^T \theta_\ell, \\ \forall k = 1, \ldots, H, & W_{t-1}^{k,\tau} = 0. \\ \forall \tau = 1, \ldots, T-t+1, & \end{cases}$$

Proof. Let us fix $t \in \{2, \ldots, T\}$, $k \in \{1, \ldots, H\}$, and $j \in \{1, \ldots, q_t\}$. We denote by $x_t, z_t, v_t, \tilde{\theta}_t$ an optimal solution to the problem defining $Q_t^1(x_{t-1}^k, z_{t-1}^k, w_{t:T}^1, \xi_t^j)$, i.e., problem (8) written for $i = 1$, and with $(x_{t-1}, z_{t-1}, w_{t:T}, \xi_t)$ replaced by $(x_{t-1}^k, z_{t-1}^k, w_{t:T}^i, \xi_t^j)$ (the dependence of the solution with respect to k, j is suppressed, to alleviate notation).

The Karush–Kuhn–Tucker (KKT) conditions for this problem imply that

$$-\delta_{tT}\theta_1 - \phi(1)\theta_t - \mu_t^{k,j} - \sigma_t^{k,j} e - \rho_t^{k,j} \overrightarrow{Z}_t^1 = 0, \tag{13}$$

$$-\theta_t \Delta\phi^\top - \tilde{\sigma}_t^{k,j} - \sigma_t^{k,j} = 0, \tag{14}$$

$$\sigma_t^{k,j} \circ (-z_t e + w_t^1 - v_t)^\top = 0, \tag{15}$$

$$\tilde{\sigma}_t^{k,j} \circ v_t^\top = 0, \tag{16}$$

where, for $t = T$, we have set $\rho_t^{k,j} = 0$. Next, since z_t can be written as $z_t = -g^t(x^t)$ for some $x^t \in \chi^t$, in case (i), we have $z_t e \le -C_t^\ell e < w_t^1$. Further, $v_t = \max(0, w_t^1 - z_t e) = w_t^1 - z_t e > 0$. Using (14) and (16), we then get

$$\tilde{\sigma}_t^{k,j} = 0 \quad \text{and} \quad \sigma_t^{k,j} = -\theta_t \Delta\phi^\top. \tag{17}$$

Let us now first show (i) by backward induction on t. Plugging the value of $\sigma_T^{k,j}$ given in (17) into (13), we obtain

$$\begin{aligned} \mu_T^{k,j} &= -\theta_1 - \phi(1)\theta_T + \theta_T e^\top \Delta\phi \\ &= -\theta_1 + \theta_T \left(-\phi(1) + \sum_{\ell=1}^J [\phi(p_\ell) - \phi(p_{\ell-1})] \right) \\ &= -\theta_1 - \theta_T \phi(0). \end{aligned}$$

Using the above relation and (9) yields $Z_{T-1}^k = -\sum_{j=1}^{q_T} p(T,j)\mu_T^{k,j} = \theta_T\phi(0) + \theta_1$. Further, using once again (9), we obtain

$$\begin{aligned} W_{T-1}^{k,1} &= \sum_{j=1}^{q_T} p(T,j)\sigma_T^{k,j} \\ &= -\sum_{j=1}^{q_T} p(T,j)\theta_T \Delta\phi^\top = -\theta_T \Delta\phi^\top. \end{aligned} \tag{18}$$

This shows $\mathcal{P}(T)$. Let us now assume that $\mathcal{P}(t+1)$ holds for some $t \in \{2, \ldots, T-1\}$, and let us show that $\mathcal{P}(t)$ holds. First, notice that (18) still holds with T substituted with t, i.e., $W_{t-1}^{k,1} = -\theta_t \Delta\phi^\top$. Further, for $\tau = 2, \ldots, T-t+1$,

$$\begin{aligned} W_{t-1}^{k,\tau} &= \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j} \overrightarrow{W}_t^{1,\tau-1}, \quad \text{from (10)}, \\ &= -\sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\theta_{t+\tau-1} e \Delta\phi^\top, \quad \text{using } \mathcal{P}(t+1), \\ &= -\sum_{j=1}^{q_t} p(t,j)\theta_{t+\tau-1}\Delta\phi^\top \quad \text{since } \rho_t^{k,j} e = 1, \\ &= -\theta_{t+\tau-1}\Delta\phi^\top. \end{aligned}$$

Also,

$$\begin{aligned} Z_{t-1}^k &= -\sum_{j=1}^{q_t} p(t,j)\mu_t^{k,j}, \quad \text{from (9)}, \\ &= -\sum_{j=1}^{q_t} p(t,j)(-\phi(1)\theta_t + \theta_t \Delta\phi^\top e - \rho_t^{k,j} \overrightarrow{Z}_t^1), \\ &\quad \text{using (13) and (17)}, \\ &= -\sum_{j=1}^{q_t} p(t,j)(-\phi(0)\theta_t - \rho_t^{k,j} \overrightarrow{Z}_t^1), \\ &\quad \text{using the definition of } \Delta\phi, \\ &= \phi(0)\theta_t + \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j} \left(\theta_1 + \phi(0) \sum_{\ell=t+1}^T \theta_\ell \right) e, \\ &\quad \text{using } \mathcal{P}(t+1), \\ &= \theta_1 + \phi(0) \sum_{\ell=t}^T \theta_\ell \quad \text{since } \rho_t^{k,j} e = 1. \end{aligned}$$

We have thus shown $\mathcal{P}(t)$ which achieves the proof of (i).

Let us now assume that $w_t^1 < -C_t^u e$ for $t = 2, \ldots, T$, and let us show (ii). Let us fix $t \in \{2, \ldots, T\}$, $k \in \{1, \ldots, H\}$, and $j \in \{1, \ldots, q_t\}$. As before, we denote by $x_t, z_t, v_t, \tilde{\theta}_t$ an optimal solution to the problem defining $Q_t^1(x_{t-1}^k, z_{t-1}^k, w_{t:T}^1, \xi_t^j)$. In this case, $z_t e \ge -C_t^u e > w_t^1$ and $v_t = \max(0, w_t^1 - z_t e) = 0$. Using

(14) and (15), we see that

$$\tilde{\sigma}_t^{k,j} = -\theta_t \Delta\phi^\top \quad \text{and} \quad \sigma_t^{k,j} = 0. \tag{19}$$

Using (9), we get $W_{t-1}^{k,1} = 0$. We show (ii) by backward induction. For $t = T$, plugging the value of $\sigma_T^{k,j}$ into (13) gives $\mu_T^{k,j} = -\theta_1 - \phi(1)\theta_T$, which, together with (9), gives $Z_{T-1}^k = \theta_1 + \phi(1)\theta_T$. We have already proved that $W_{T-1}^{k,1} = 0$, and thus $\tilde{\mathcal{P}}(T)$ holds. Let us now assume that $\mathcal{P}(t+1)$ holds for some $t \in \{2, \ldots, T-1\}$, and let us show that $\mathcal{P}(t)$ holds. Since $\overrightarrow{W}_t^{1,\tau-1} = 0$, we obtain $W_{t-1}^{k,\tau} = \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\overrightarrow{W}_t^{1,\tau-1} = 0$ for $\tau = 2, \ldots, T-t+1$. Plugging $\sigma_t^{k,j} = 0$ into (13) and using (9) gives

$$\begin{aligned} Z_{t-1}^k &= \sum_{j=1}^{q_t} p(t,j)(\phi(1)\theta_t + \rho_t^{k,j}\overrightarrow{Z}_t^1), \\ &= \sum_{j=1}^{q_t} p(t,j)\left(\theta_1 + \phi(1)\sum_{\ell=t}^{T}\theta_\ell\right), \\ &\quad \text{using } \tilde{\mathcal{P}}(t+1) \quad \text{and} \quad \rho_t^{k,j} e = 1, \\ &= \theta_1 + \phi(1)\sum_{\ell=t}^{T}\theta_\ell. \end{aligned}$$

This shows $\tilde{\mathcal{P}}(t)$ and achieves the proof of (ii). □

Proposition 3.2 can be used as a debugging tool to check the implementation of the SDDP algorithm for risk-averse problem (1). More precisely, we can check that, in cases (i) and (ii), implementing the formulas for Z_{t-1}^k and $W_{t-1}^{k,\tau}$ given in Proposition 3.1 will give the same results as implementing the formulas from Proposition 3.2.

At stage t, if instead of ρ_ϕ in (1) we use $CVaR^{\varepsilon_t}$, problem (1) becomes

$$\begin{aligned} &\inf_{x_1,\ldots,x_T} d_1^\top x_1 + \theta_1 \mathbb{E}\left[\sum_{t=2}^{T} d_t^\top x_t\right] + \sum_{t=2}^{T}\theta_t CVaR^{\varepsilon_t}\left(-\sum_{k=2}^{t} d_k^\top x_k\right) \\ &C_t x_t = \xi_t - D_t x_{t-1}, \\ &x_t \geq 0, \quad x_t \text{ is } \mathcal{F}_t\text{-measurable}, \ t = 1, \ldots, T. \end{aligned} \tag{20}$$

For this model, we obtain a result analogous to Proposition 3.2.

Proposition 3.3. *Let us consider the risk-averse recourse functions $\mathcal{Q}_t$ for model (20) and their approximations $\mathfrak{Q}_t^i$ of form (7), obtained by applying the SDDP algorithm to the corresponding DP equations. In the following two cases, we obtain closed-form expressions for Z_{t-1}^k and $W_{t-1}^{k,\tau}$ (independent of the sampled scenarios).*

(i) *If, for $t = 2, \ldots, T$, $w_t^1 > -C_t^\ell$, then, for $t = 2, \ldots, T$, $\mathcal{P}(t)$ holds, where*

$$\mathcal{P}(t): \begin{cases} \forall k = 1, \ldots, H, & Z_{t-1}^k = \theta_1 + \sum_{\ell=t}^{T}\dfrac{\theta_\ell}{\varepsilon_\ell}, \\ \forall k = 1, \ldots, H, & W_{t-1}^{k,\tau} = \dfrac{\theta_{t+\tau-1}}{\varepsilon_{t+\tau-1}}. \\ \forall \tau = 1, \ldots, T-t+1, & \end{cases}$$

(ii) *If, for $t = 2, \ldots, T$, $w_t^1 < -C_t^u$, then, for $t = 2, \ldots, T$, $\tilde{\mathcal{P}}(t)$ holds, where*

$$\tilde{\mathcal{P}}(t): \quad \begin{array}{ll} \forall k = 1, \ldots, H, & Z_{t-1}^k = \theta_1, \quad \textit{and} \\ \forall \tau = 1, \ldots, T-t+1, & W_{t-1}^{k,\tau} = 0. \end{array}$$

Proof. The proof is similar to the proof of Proposition 3.2. □

Remark 3.4. In the particular case when the CVaR levels $\varepsilon_t = \varepsilon \in (0, 1)$ are the same at each time step, Proposition 3.3 is a particular case of Proposition 3.2 with $\phi(1) = 0$, $\phi(0) = \frac{1}{\varepsilon}$, and $\Delta\phi = -1/\varepsilon \in \mathbb{R}$.

Numerical simulations for a real-life application modeled as (20) are reported in [7].

When Assumption (A1) does not hold, as stated in [22], a feasible nonanticipative policy can still be proposed using approximate recourse functions $\mathfrak{Q}_t$ obtained applying the SDDP algorithm on a *sample average approximation* (SAA) of the original problem (1).

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