Airline Network Revenue Management by Multistage Stochastic Programming

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Introduction

Airline revenue management deals with strategies for controlling the booking process within a network of flights under stochastic demand with the objective of maximizing the expected revenue. Often the booking process is controlled by seat protection levels or (so-called) bid prices.

Aims:

- Development of a scenario-based stochastic programming model for airline network revenue management;
- Approximate representation of the multivariate booking demand processes by scenario trees;
- Numerical computations for the node-based stochastic integer program (using CPLEX);
- Lagrangian decomposition of the node-based stochastic integer program; algorithm design and numerical experience.

Notation

Input data

π^n: probability of node n;
stochastic (as scenario tree):
$d^n_{i,j,k}$: passenger demand;
$\gamma^n_{i,j,k}$: cancelation rates;
deterministic:
$f^b_{i,j,k,t(n)}$: fares;
$f^c_{i,j,k,t(n)}$: refunds;
$\mathcal{C}_{l,m}$: capacity;

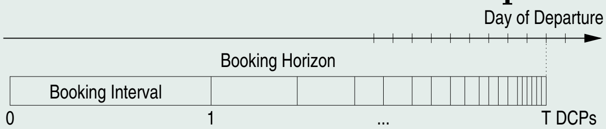
Variables

$b^n_{i,j,k}$: bookings;
$c^n_{i,j,k}$: cancelations;
$B^n_{i,j,k}$: cumulative bookings;
$C^n_{i,j,k}$: cumulative cancelations;
$P^n_{i,j,k}$: protection level;
$z^{P,n}_{i,j,k}$, $z^{d,n}_{i,j,k}$: slack variables;
$\tilde{z}^n_{i,j,k}$: auxiliary integer variables;

Indices

$t = 0, \ldots, T$: data collection points;
$i = 1, \ldots, I$: origin-destination-itin.;
$j = 1, \ldots, J$: fare classes;
$k = 1, \ldots, K$: points of sale;
$l = 1, \ldots, L$: legs;
$\mathcal{I}_l$: index set of itineraries;
$m = 1, \ldots, M(l)$: compartments;
$\mathcal{J}_m(l)$: index set of fare classes;
$n = 0, \ldots, N$: nodes;
$t(n)$: time of node n;
n_-: preceding node of node n;

Time horizon and data collection points (dcp):



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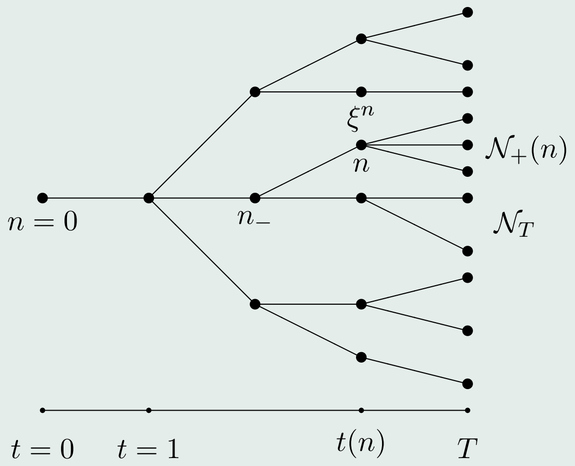
Approximation of the demand by scenario trees

The passenger demand and cancelation rate process $\{\xi_t\}_{t=0}^T = \{(d_{i,j,k,t}, \gamma_{i,j,k,t})\}_{t=0}^T$ is approximated by a finite number of scenarios forming a scenario tree.



Scenario tree with $T = 4$, $N = 22$ and 11 leaves

$n = 0$ root node, n_- unique predecessor of node n, path$(n) = \{1, \ldots, n_-, n\}$, $t(n) := |\text{path}(n)|$, $\mathcal{N}_+(n)$ set of successors to n, $\mathcal{N}_T := \{n \in \mathcal{N} : \mathcal{N}_+(n) = \emptyset\}$ set of leaves, path(n), $n \in \mathcal{N}_T$, scenario with (given) probability π^n, $\pi^n := \sum_{\nu \in \mathcal{N}_+(n)} \pi^\nu$ probability of node n, ξ^n realization of $\xi_{t(n)}$.

Generation of multivariate scenario trees

General strategy:

(i) Development of a stochastic model for the data process ξ (parametric [e.g. time series model], nonparametric [e.g. resampling from statistical data]) and generation of simulation scenarios;

(ii) Construction of a scenario tree out of the simulation scenarios by recursive scenario reduction and bundling over time such that the optimal expected revenue stays within a prescribed tolerance.

Implementation: GAMS-SCENRED

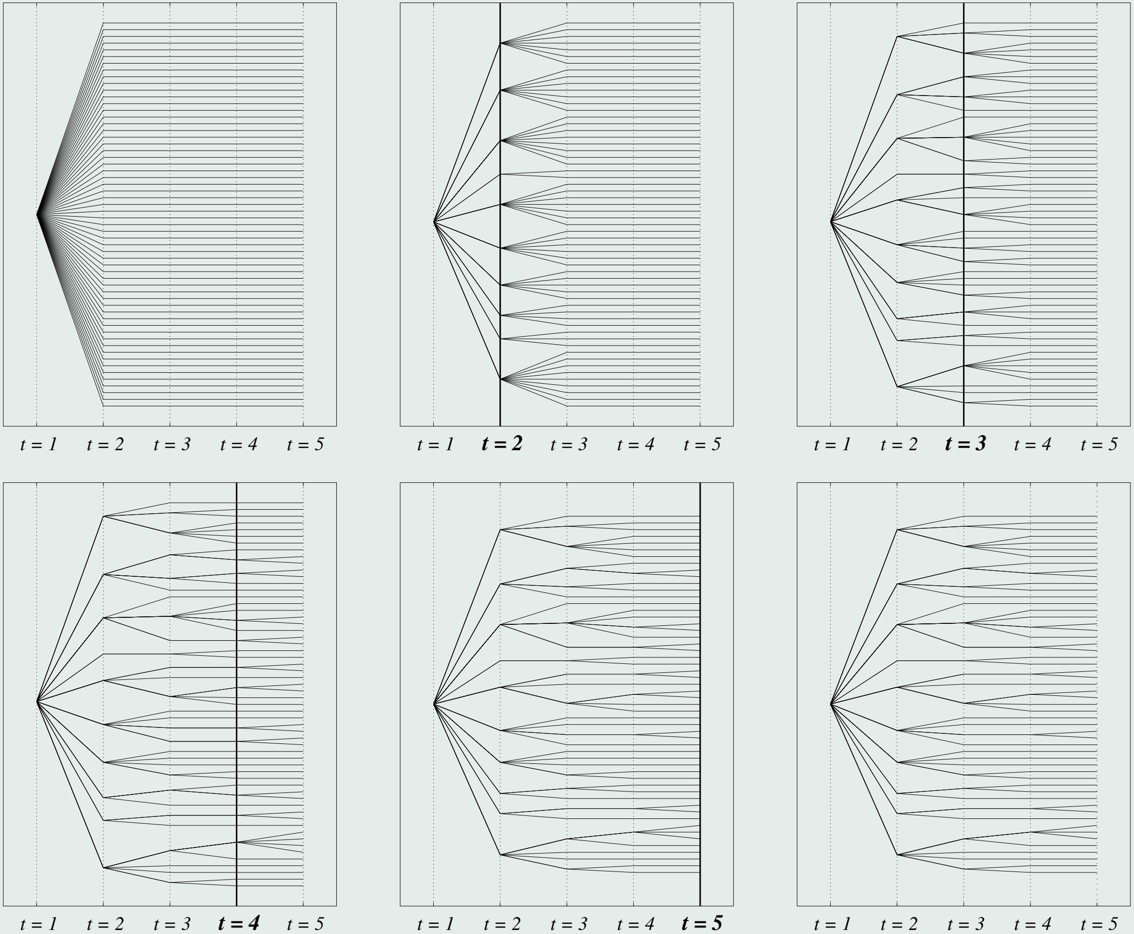


Illustration of the forward tree construction for an example including T=5 time periods starting with a scenario fan containing N=58 scenarios

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Airline network revenue management model (node representation)

Objective

$$\max_{(P^n_{i,j,k})} \left\{ \sum_{n=0}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[f^b_{i,j,k,t(n)} b^n_{i,j,k} - f^c_{i,j,k,t(n)} c^n_{i,j,k} \right] \right\}$$

Constraints

Cumulative bookings

$$B^0_{i,j,k} := \bar{B}^0_{i,j,k}; \qquad C^0_{i,j,k} := \bar{C}^0_{i,j,k}; \qquad B^n_{i,j,k} := B^{n-}_{i,j,k} + b^n_{i,j,k}$$

Cumulative cancelations

$$C^n_{i,j,k} = \lfloor \gamma^n_{i,j,k} B^n_{i,j,k} + 0.5 \rfloor$$

Cancelations

$$c^n_{i,j,k} = C^n_{i,j,k} - C^{n-}_{i,j,k}$$

Passenger demands and protection levels

$$b^n_{i,j,k} \le d^n_{i,j,k}; \quad b^n_{i,j,k} \le P^{n-}_{i,j,k} - B^{n-}_{i,j,k} + C^n_{i,j,k} \quad \text{(disjunctive constraints)}$$

Leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} P^n_{i,j,k} \le \mathcal{C}_{l,m} \qquad (n \in \mathcal{N}_{T-1})$$

Integrality and nonnegativity constraints

$$B^n_{i,j,k}, C^n_{i,j,k}, P^n_{i,j,k} \in \mathbb{Z}; \quad b^n_{i,j,k} \ge 0; \quad c^n_{i,j,k} \ge 0$$

Airline network revenue management model (final)

Objective

$$\max_{(P^n_{i,j,k})} \left\{ \sum_{n=0}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[f^b_{i,j,k,t(n)} b^n_{i,j,k} - f^c_{i,j,k,t(n)} c^n_{i,j,k} \right] \right\}$$

Constraints

Cumulative bookings

$$B^0_{i,j,k} := \bar{B}^0_{i,j,k}; \qquad C^0_{i,j,k} := \bar{C}^0_{i,j,k}; \qquad B^n_{i,j,k} := B^{n-}_{i,j,k} + b^n_{i,j,k}$$

Cumulative cancelations

$$C^n_{i,j,k} = \lfloor \gamma^n_{i,j,k} B^n_{i,j,k} + 0.5 \rfloor$$

Cancelations

$$c^n_{i,j,k} = C^n_{i,j,k} - C^{n-}_{i,j,k}$$

Passenger demands

$$b^n_{i,j,k} + z^{b,n}_{i,j,k} = d^n_{i,j,k}$$

Protection levels

$$B^n_{i,j,k} - C^n_{i,j,k} + z^{P,n}_{i,j,k} = P^{n-}_{i,j,k}$$

Number of bookings (disjunctive constraints) ($\kappa > 0$, adequately large)

$$0 \le z^{b,n}_{i,j,k} \le (1 - \tilde{z}^n_{i,j,k}) d^n_{i,j,k} \qquad 0 \le z^{P,n}_{i,j,k} \le \tilde{z}^n_{i,j,k} \kappa \qquad \tilde{z}^n_{i,j,k} \in \{0, 1\}$$

Leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} P^n_{i,j,k} \le \mathcal{C}_{l,m} \qquad (n \in \mathcal{N}_{T-1})$$

Integrality and nonnegativity constraints

$$B^n_{i,j,k}, C^n_{i,j,k}, P^n_{i,j,k} \in \mathbb{Z}; \quad b^n_{i,j,k} \ge 0; \quad c^n_{i,j,k} \ge 0$$

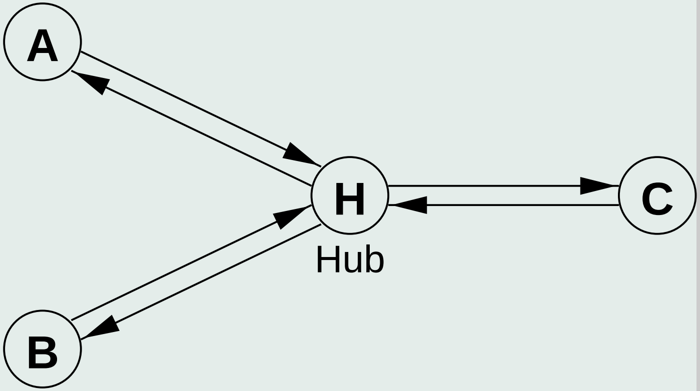
Comments:

- large scale structured integer linear program
- solvable by a standard solver (e.g. CPLEX) in reasonable time for smaller networks when neglecting integer constraints
- **Dimensions:** (S number of scenarios)
  - $4IJKN$ continuous variables,
  - $IJK(N+1-S)+2IJKN$ integer (neglected) variables,
  - $IJKN$ binary variables
  - $7IJK(N-1) + \sum_{n\in\mathcal{N}_{T-1}} \sum_{l=1}^{L} M(l)$ constraints
- Protection levels $(P^n_{i,j,k})_{n\in\mathcal{N}}$ have the same tree structure as the input data
- The (deterministic) protection levels of the first stage may be taken as a basis for the computer reservation system
- At the next dcp a new scenario tree has to be generated and the problem is resolved etc.

Airline revenue management: Numerical example

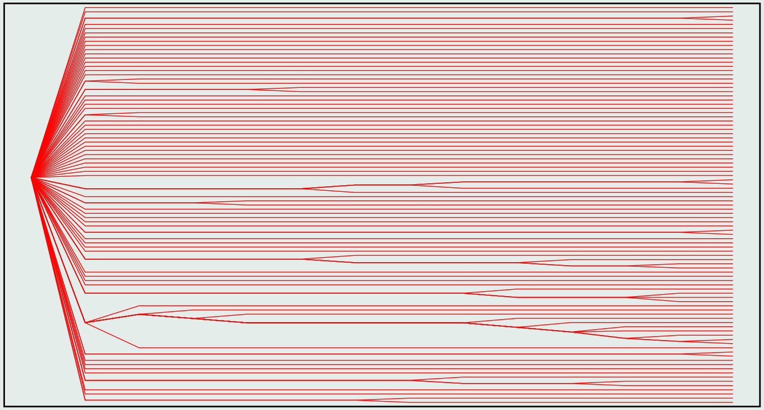
Hub-and-Spokes Network

#Legs	6
#ODIs	12
#Compartments	2
#Fare Classes	6
#POS	1
#DCPs	14



Tree and Size

#Scenarios	92
#Nodes	1017
#cont. Variables	506.016
#bin. Variables	73.224
#Constraints	513.660



Data and numerical results

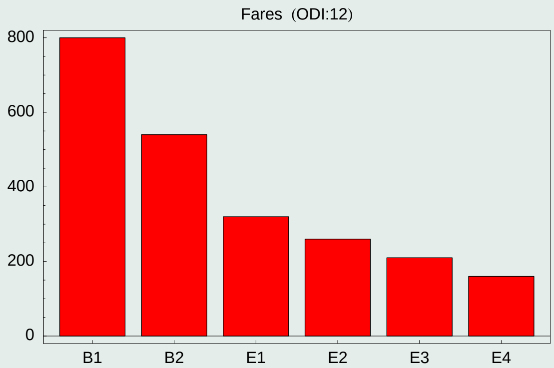
dcp	0	1	2	3	4	5	6	7	8	9	10	11	12	13
d	182	126	84	56	35	21	14	10	7	5	3	2	1	0

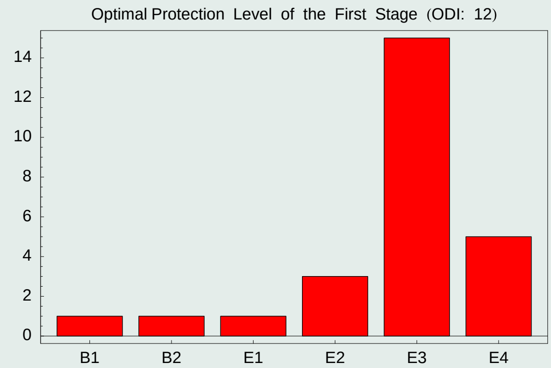
Data collection points and days to departure

The booking demand process is modeled by a non-homogeneous Poisson process. The cumulative booking requests $B_{i,j,k}(t)$ are independent and given by

$$B_{i,j,k}(t) = G_{i,j,k} \int_0^t \frac{\Gamma(a_{i,j,k} + b_{i,j,k})}{T(\Gamma(a_{i,j,k}) + \Gamma(b_{i,j,k}))} \left(\frac{\tau}{T}\right)^{a_{i,j,k}-1} \left(1 - \frac{\tau}{T}\right)^{b_{i,j,k}-1} d\tau,$$

where the integrands correspond to Beta distributions and the total number of booking requests $G_{i,j,k}$ is assumed to have a Gamma distribution.





Numerical Results (continued)

CPLEX-Results

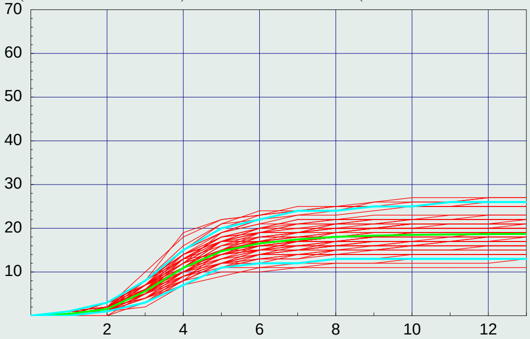
Version	8.1
MIP Gap	0.001
Solution Status	Optimal
#MIP Nodes passed	0 (root)

Computing time

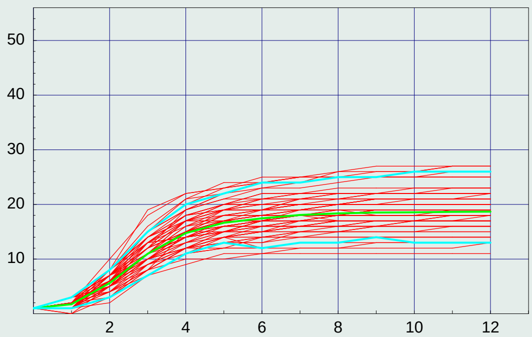
Total: 3 : 29.8 min

(Intel Celeron, 2.0 GHz, Linux)

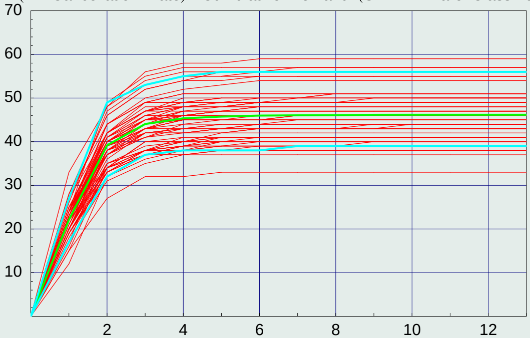
(1 − Cancellation Rate) ∗ Cumulative Demand (ODI: 12 Fare Class: 3)



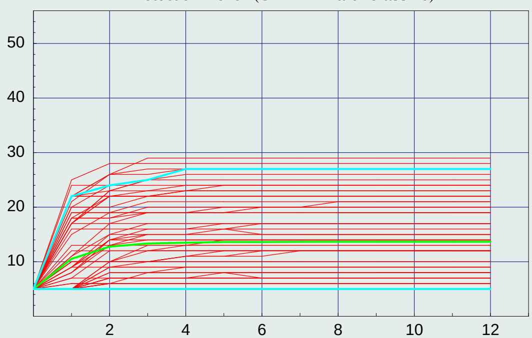
Protection Level (ODI: 12 Fare Class: 3)



(1 − Cancellation Rate) ∗ Cumulative Demand (ODI: 12 Fare Class: 6)



Protection Level (ODI: 12 Fare Class: 6)



Lagrangian decomposition in airline revenue management

Idea: Dualization of leg capacity limits

Lagrangian function Λ**:**

$$\begin{aligned}
\Lambda(\lambda, P) \; &:= \; \sum_{n=0}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left(f_{i,j,k}^{b,n} b_{i,j,k}^{n} - f_{i,j,k}^{c,n} c_{i,j,k}^{n} \right) \\
&+ \sum_{n \in \mathcal{N}_{T-1}} \pi^n \sum_{l=1}^{L} \sum_{m=1}^{M(l)} \lambda_{l,m}^{n} \left(\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} \mathcal{C}_{l,m} - P_{i,j,k}^{n} \right) \\
&= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left(\sum_{n=0}^{N} \pi^n \left(f_{i,j,k}^{b,n} b_{i,j,k}^{n} - f_{i,j,k}^{c,n} c_{i,j,k}^{n} \right) \right. \\
&\left. - \sum_{n \in \mathcal{N}_{T-1}} \pi^n \sum_{l \in \mathcal{L}_i} \sum_{m=1}^{M(l)} \delta_{j,l,m} \lambda_{l,m}^{n} P_{i,j,k}^{n} \right) + \sum_{n \in \mathcal{N}_{T-1}} \pi^n \sum_{l=1}^{L} \sum_{m=1}^{M(l)} \lambda_{l,m}^{n} \mathcal{C}_{l,m} \\
&= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \Lambda_{i,j,k}(\lambda, P_{i,j,k}) + \sum_{n \in \mathcal{N}_{T-1}} \pi^n \sum_{l=1}^{L} \sum_{m=1}^{M(l)} \lambda_{l,m}^{n} \mathcal{C}_{l,m}
\end{aligned}$$

where $\mathcal{L}_i = \{ l : i \in \mathcal{I}_l \}$ and $\delta_{j,l,m} = \begin{cases} 1 & j \in \mathcal{J}_m(l) \\ 0 & \text{otherwise} \end{cases}$

Dual function D:

$$\begin{aligned} D(\lambda) &= \sup_P \Lambda(\lambda, P) \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \sup_{P_{i,j,k}} \Lambda_{i,j,k}(\lambda, P_{i,j,k}) + \sum_{n\in\mathcal{N}_{T-1}} \pi^n \sum_{l=1}^{L}\sum_{m=1}^{M(l)} \lambda^n_{l,m}\mathcal{C}_{l,m} \end{aligned}$$

The function D is convex nondifferentiable and decomposable.

Dual problem:

$$\inf_\lambda D(\lambda)$$

The relative duality gap is small (theory by Bertsekas 82).

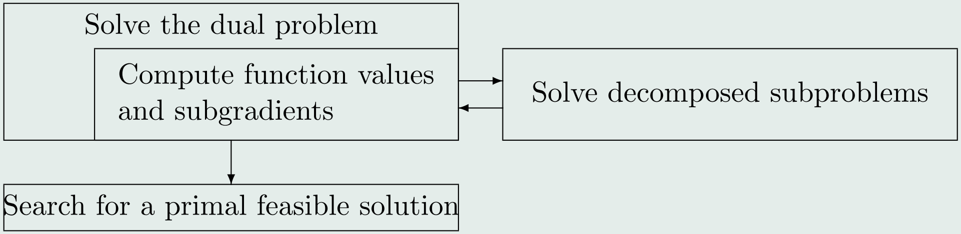
Subgradients:

$$[\partial D(\lambda)]^n_{l,m} = \pi^n \left(\mathcal{C}_{l,m} - \sum_{i\in\mathcal{I}_l} \sum_{j\in\mathcal{J}_m(l)} \sum_{k=1}^{K} P^n_{i,j,k} \right)$$

The Lagrange multipliers $\lambda^n_{l,m}$, $n \in \mathcal{N}_t$, may be interpreted as bid prices at t for leg l and compartment m. However, they are presently only available for $n \in \mathcal{N}_{T-1}$.

Dual solution algorithm

Algorithm:

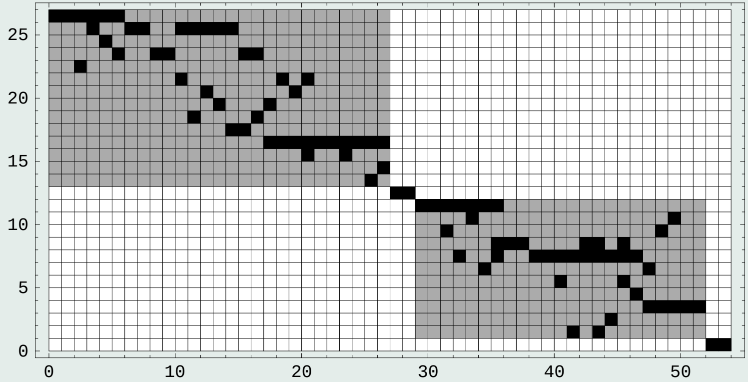


- Solution of the dual problem by a bundle subgradient method (e.g. proximal bundle method by Kiwiel or Helmberg)
- Solution of the subproblems by dynamic programming on scenario trees.
- Primal-proximal heuristic to determine a good primal feasible solution (e.g. by Daniilidis and Lemaréchal).

Numerical results (computing times):

Solving the Dual	4:52 min
Heuristic	5:16 min

A realistic mid-size airline network example



ODI-Leg-Matrix

RM problem dimensions	
#ODIs	54
#ODI-Fareclass-POS	489
#Legs	27
#Leg-Compartments	54
#DCPs	23

Tree and Size	
#Scenarios	98
#Nodes	1.441
#Variables	3.473.367
#Constraints	2.774.445
#Coupling Constr.	5.238

Conclusions and future work

We presented an approach to airline network revenue management using a scenario tree-based dynamic stochastic optimization model. The approach

- starts from a finite number of demand scenarios and their probabilities,
- requires no assumptions on the demand distributions except their decision-independence.

Stochastic programming approaches lead to solutions that are more robust with respect to perturbations of input data. However, the models have higher complexity.

Future work:

- Implementation refinements of the decomposition scheme
- Numerical test runs for mid-size networks

(URL: `www.math.hu-berlin.de/~romisch`, Email: `romisch@math.hu-berlin.de`)

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