INAUDIBILITY OF SIXTH ORDER CURVATURE INVARIANTS

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Abstract. It is known that the spectrum of the Laplace operator on functions of a closed Riemannian manifold does not determine the integrals of the individual fourth order curvature invariants scal^2, $|\mathrm{ric}|^2$, $|R|^2$, which appear as summands in the second heat invariant a_2. We study the analogous question for the integrals of the sixth order curvature invariants appearing as summands in a_3. Our result is that none of them is determined individually by the spectrum, which can be shown using various examples. In particular, we prove that two isospectral nilmanifolds of Heisenberg type with three-dimensional center are locally isometric if and only if they have the same value of $|\nabla R|^2$. In contrast, any pair of isospectral nilmanifolds of Heisenberg type with centers of dimension $r > 3$ does not differ in any curvature invariant of order six, actually not in any curvature invariant of order smaller than $2r$. We also prove that this implies that for any $k \in \mathbb{N}$, there exist locally homogeneous manifolds which are not curvature equivalent but do not differ in any curvature invariant of order up to $2k$.

1. Introduction

Let (M, g) be a closed Riemannian manifold. The eigenvalue spectrum (with multiplicities) of the associated Laplace operator $\Delta_g = -\mathrm{div}_g \mathrm{grad}_g$ acting on smooth functions is classically known to determine not only the dimension and the volume of (M, g) (by Weyl's asymptotic formula), but also the so-called heat invariants $a_0(g), a_1(g), a_2(g), \ldots$. These are defined as the coefficients appearing in Minakshisundaram-Pleijel's asymptotic expansion

$$\mathrm{Tr}\big(\exp(-t\Delta_g)\big) \;\sim\; (4\pi t)^{-\dim M/2} \textstyle\sum_{q=0}^{\infty} a_q(g) t^q \quad \text{for } t \searrow 0.$$

Here,

$$a_0(g) = \mathrm{vol}(M, g),$$
$$a_1(g) = \tfrac{1}{6} \textstyle\int_M \mathrm{scal}\, dvol_g,$$
$$a_2(g) = \tfrac{1}{360} \textstyle\int_M (5\mathrm{scal}^2 - 2|\mathrm{ric}|^2 + 2|R|^2)\, dvol_g,$$

where scal, ric and R denote the scalar curvature, the Ricci tensor and the Riemannian curvature tensor of (M, g), respectively. In general, each $a_q(g)$ is known to be the integral of some curvature invariant of order $2q$ on (M, g); see, e.g., [5].

By definition, a curvature invariant is a polynomial in the coefficients of the Riemannian curvature tensor R and its covariant derivatives ∇R, $\nabla^2 R$, ..., where the coefficients are taken with respect to some orthonormal basis of the tangent space at the point under consideration, and the polynomial is required to be invariant under changes of the orthonormal basis. Following the definitions, e.g., in [10], such an invariant is called an invariant of order k if it is a sum of

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terms each of which involves a total of k derivatives of the metric tensor. Each occurrence of R or any of its contractions involves two derivatives; each occurrence of ∇ adds one more derivative. (See the proof of Proposition 4.12 below for a more explicit description.) So, for example, $|\nabla R|^2 = \langle \nabla R, \nabla R\rangle$ is a curvature invariant of order six.

It is well-known that each nonzero curvature invariant must be of even order (see [10]), and that bases for the space of curvature invariants of order two, resp. four, are given by

$$\{\mathrm{scal}\},\ \text{resp. } \{\mathrm{scal}^2, |\mathrm{ric}|^2, |R|^2, \Delta\mathrm{scal}\}.$$

Note that $\int_M \Delta\mathrm{scal} = 0$, but each of the remaining three elements of the above basis of the space of curvature invariants of order four does appear in the linear combination constituting the integrand of $a_2(g)$.

Two closed Riemannian manifolds are called *isospectral* if their Laplacians have the same eigenvalue spectra, including multiplicities. A geometric property or quantity associated with closed Riemannian manifolds is called *audible* if it is determined by the spectrum. By the above, each a_q is audible; in particular, $a_2(g) = a_2(g')$ for any isospectral manifolds (M, g), (M', g'). So the integral of $5\mathrm{scal}^2 - 2|\mathrm{ric}|^2 + 2|R|^2$ must be the same for both manifolds.

This does not hold for the individual terms in this linear combination: In [14], the second author gave the first examples of isospectral manifolds that showed that the integrals of scal^2 and $|\mathrm{ric}|^2$ are inaudible; other examples in [15] showed the same for the integral of $|R|^2$.

The aim of this paper is to prove similar results for sixth order curvature invariants. Note the following formula for $a_3(g)$ which was proved by T. Sakai in [13]:

$$\begin{aligned} a_3(g) = \tfrac{1}{45360}\int_M \big(&-142|\nabla\mathrm{scal}|^2 - 26|\nabla\mathrm{ric}|^2 - 7|\nabla R|^2 + 35\mathrm{scal}^3 - 42\mathrm{scal}|\mathrm{ric}|^2 + 42\mathrm{scal}|R|^2 \\ &- 36\mathrm{Tr}(\mathrm{Ric}^3) + 20(*) - 8(**) + 24\hat{R}\big)\,dvol_g; \end{aligned} \tag{1}$$

for the definition of the curvature invariants denoted here by $(*)$, $(**)$, $\hat{R}$ (and two more, $\mathring{R}$ and $(***)$), we refer to (2) in Section 2.

It is already known that the integral of the individual term $|\nabla\mathrm{scal}|^2$ can indeed differ in pairs of isospectral manifolds: C. Gordon and Z. Szabo constructed pairs of isospectral closed manifolds one of which has constant scalar curvature, while the other has nonconstant scalar curvature; see [8].

In this paper, we will show that for each of the individual summands in (1), there exist examples of isospectral manifolds differing in the integral of that curvature invariant. The most interesting of these is arguably $|\nabla R|^2$ which vanishes if and only if the manifold is locally symmetric. Although we do not know of any example proving inaudibility of local symmetry, we do show that the integral of $|\nabla R|^2$ is inaudible.

For a few of the sixth order curvature invariants, inaudibility will follow already from known examples of isospectral manifolds. To study the remaining ones, we will use a certain class of locally homogeneous manifolds, namely, Riemannian two-step nilmanifolds. These are quotients of two-step nilpotent Lie groups, endowed with a left invariant metric, by cocompact discrete subgroups. By local homogeneity, each curvature invariant is a constant function on such a manifold. We develop some general insight into the structure of the curvature invariants of Riemannian two-step nilmanifolds (Proposition 4.12) and give explicit formulas for the fourth and some of the sixth order curvature invariants in this setting (Lemma 4.6, Lemma 4.7). For $|\nabla R|^2$, $\hat{R}$ and $\mathring{R}$ we give only partially explicit formulas (Lemma 4.13). These formulas will, however, be sufficient to show inaudibility of $\int|\nabla R|^2$, $\int\hat{R}$ and $\int\mathring{R}$ by using isospectral pairs of nilmanifolds of Heisenberg type.

The latter constitute a special class of Riemannian two-step nilmanifolds and were introduced by A. Kaplan; the very first example of isospectral, locally nonisometric Riemannian manifolds found by C. Gordon [6] in 1993 was a pair of nilmanifolds of Heisenberg type. Within this class, we prove, in particular, the following results:

• For any pair of isospectral nilmanifolds of Heisenberg type with three-dimensional centers of the underlying Lie groups, equality of the value of (the constant function) $|\nabla R|^2$ on these manifolds is equivalent to local isometry; the same holds for $\hat{R}$ and $\mathring{R}$ (Theorem 5.7). Since isospectral, locally nonisometric pairs of this type exist, this implies inaudibility of these curvature invariants.

• A pair of isospectral nilmanifolds of Heisenberg type where the dimension of the centers of the underlying Lie groups is r can never be distinguished by the value of any curvature invariant of order $2q < 2r$ (Theorem 5.6).

• Two locally nonisometric nilmanifolds of Heisenberg type are never curvature equivalent, meaning that there is no isometry of the associated metric Lie algebras intertwining the Riemannian curvature tensors (Proposition 5.9). In particular, for any $k \in \mathbb{N}$ there exist pairs of locally homogeneous manifolds which are not curvature equivalent, but do not differ in any curvature invariant up to order $2k$ (Theorem 5.11).

This paper is organized as follows:

In Section 2, we present some background information about space of sixth order curvature invariants, introducing a commonly used basis for this space and explaining certain integral relations between the basis elements. We also observe that for some of the basis elements, it already follows from known isospectral examples that their integrals are not audible.

In Section 3, we review Riemannian two-step nilmanifolds, a method from [9] for obtaining isospectral pairs in this class, and some examples. In the case of Heisenberg type nilmanifolds, we explain the general relation between isospectral, locally nonisometric examples and the existence of nonisomorphic modules for the Clifford algebra associated with the centers (Remark 3.8).

In Section 4, we gain insight into the structure of the curvature invariants in the general two-step nilpotent setting (Proposition 4.12), give formulas for the curvature invariants of order two and four (Lemma 4.6), and also for several curvature invariants of order six (Lemma 4.7, Lemma 4.13). Those proofs which involve somewhat lengthy calculations are deferred to the Appendix. Applying the formulas, we prove inaudibility of $\int \mathrm{Tr}(\mathrm{Ric}^3)$, $\int |\nabla \mathrm{ric}|^2$, $\int (*)$, $\int (**)$, $\int (***)$ using the examples from Section 3. As an aside, we also give an example where the isospectral manifolds differ in $|\mathrm{ric}|^2$ and in $|R|^2$; although inaudibility of $\int |\mathrm{ric}|^2$ and $\int |R|^2$ was already known, this is the first such example in the class of nilmanifolds.

In Section 5 we study the structure of curvature invariants in the special class of Heisenberg type nilmanifolds. We prove inaudibility of $\int |\nabla R|^2$, $\int \hat{R}$, $\int \mathring{R}$ and the other results mentioned above (Theorem 5.7, Theorem 5.6, Proposition 5.9, Theorem 5.11).

2. Preliminaries

Let (M, g) be a closed Riemannian manifold of dimension n with Levi-Civita connection ∇. Let R be the associated Riemannian curvature tensor; our sign convention is such that

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

We denote by scal, ric, and Ric the scalar curvature, the Ricci tensor, and the Ricci operator, respectively.

It is well-known that the space of curvature invariants of order six has dimension 17 provided that $n \geq 6$ (see [10]). A basis for this space (and still a generating system in lower dimensions n)

is the following, using index notation with respect to local orthonormal bases and the Einstein summation convention:

$$\begin{gathered}\mathrm{scal}^3,\ \mathrm{scal}|\mathrm{ric}|^2,\ \mathrm{scal}|R|^2,\ \mathrm{Tr}(\mathrm{Ric}^3),\ (*) := \mathrm{ric}_{ik}\mathrm{ric}_{jl}R_{ijkl},\ (**) := \mathrm{ric}_{ij}R_{ipqr}R_{jpqr},\\ \hat{R} := R_{ijkl}R_{klpq}R_{pqij},\ \mathring{R} := R_{ikjl}R_{kplq}R_{piqj},\ |\nabla\mathrm{scal}|^2,\ |\nabla\mathrm{ric}|^2,\ |\nabla R|^2,\\ (***) := \nabla_i\mathrm{ric}_{jk}\nabla_k\mathrm{ric}_{ij},\ \mathrm{scal}\,\Delta\mathrm{scal},\ \Delta^2\mathrm{scal},\ \langle\Delta\mathrm{ric},\mathrm{ric}\rangle = -\mathrm{ric}_{ij}\nabla^2_{kk}\mathrm{ric}_{ij},\\ \langle\nabla^2\mathrm{scal},\mathrm{ric}\rangle = \big(\nabla^2_{ij}\mathrm{scal}\big)\,\mathrm{ric}_{ij},\ \langle\Delta R,R\rangle = -R_{ijkl}\nabla^2_{pp}R_{ijkl}.\end{gathered} \tag{2}$$

The integrals of seven of the invariants in this basis either vanish or can be expressed as a linear combination of integrals of certain others: First, note that (with our sign convention for Δ)

$$\begin{aligned}&\textstyle\int_M \Delta^2\mathrm{scal} = \int_M\langle\nabla\Delta\mathrm{scal},\nabla 1\rangle = 0,\\ &\textstyle\int_M \mathrm{scal}\,\Delta\mathrm{scal} = \int_M |\nabla\mathrm{scal}|^2,\\ &\textstyle\int_M\langle\Delta\mathrm{ric},\mathrm{ric}\rangle = \int_M |\nabla\mathrm{ric}|^2,\\ &\textstyle\int_M\langle\Delta R,R\rangle = \int_M |\nabla R|^2.\end{aligned} \tag{3}$$

Three more relations are give by the following proposition:

Proposition 2.1.
(i) $\int_M\langle\nabla^2\mathrm{scal},\mathrm{ric}\rangle = -\frac{1}{2}\int_M|\nabla\mathrm{scal}|^2$,
(ii) $\int_M (***) = \int_M\big(\frac{1}{4}|\nabla\mathrm{scal}|^2 - \mathrm{Tr}(\mathrm{Ric}^3) + (*)\big)$,
(iii) $\int_M \mathring{R} = \int_M\big(\frac{1}{4}|\nabla\mathrm{scal}|^2 - |\nabla\mathrm{ric}|^2 + \frac{1}{4}|\nabla R|^2 - \mathrm{Tr}(\mathrm{Ric}^3) + (*) + \frac{1}{2}(**) - \frac{1}{4}\hat{R}\big)$.

Proof. From [10], formula (2.19) we have

$$\nabla^4_{ijij}\mathrm{scal} = \Delta^2\mathrm{scal} + \tfrac{1}{2}|\nabla\mathrm{scal}|^2 + \langle\nabla^2\mathrm{scal},\mathrm{ric}\rangle.$$

From this we derive (i) by integrating and using the facts that $\int_M \Delta^2\mathrm{scal} = 0$ and, analogously, $\int_M \nabla^4_{ijij}\mathrm{scal} = 0$. For (ii), we first notice that

$$\textstyle\int_M(\nabla^2_{ij}\mathrm{ric}_{ik})\mathrm{ric}_{jk} = -\int_M\langle\nabla_j\mathrm{ric}_{ik},\nabla_i\mathrm{ric}_{jk}\rangle = -\int_M (***).$$

Moreover, formula (2.16) from [10] says

$$\big(\nabla^2_{ij}\mathrm{ric}_{ik}\big)\mathrm{ric}_{jk} = \tfrac{1}{2}\langle\nabla^2\mathrm{scal},\mathrm{ric}\rangle + \mathrm{Tr}(\mathrm{Ric}^3) - (*).$$

Therefore, we obtain (ii) by integrating this on both sides and using (i). Finally, formula (2.20) from [10] is

$$\begin{aligned}\nabla^4_{ijki}\mathrm{ric}_{jk} = {}&\tfrac{1}{2}\Delta^2\mathrm{scal} + \tfrac{1}{2}|\nabla\mathrm{scal}|^2 - 2|\nabla\mathrm{ric}|^2 + 2\langle\nabla^2\mathrm{scal},\mathrm{ric}\rangle + \langle\Delta\mathrm{ric},\mathrm{ric}\rangle + 3(***)\\ &+ 2\mathrm{Tr}(\mathrm{Ric}^3) - 2(*) + \tfrac{1}{4}\langle\Delta R,R\rangle + \tfrac{1}{2}(**) - \mathring{R} - \tfrac{1}{4}\hat{R}.\end{aligned}$$

To obtain (iii), we first integrate this on both sides and again use the facts that $\int_M \Delta^2\mathrm{scal} = 0$ and $\int_M \nabla^4_{ijki}\mathrm{ric}_{jk} = 0$. Then we use the two last equalities of (3) as well as (i) and (ii). ☐

On the other hand, note that each of the remaining ten curvature invariants does appear in formula (1) for the third heat invariant. Now, for each of the ten expressions

$$\begin{gathered}\textstyle\int_M|\nabla\mathrm{scal}|^2,\ \int_M|\nabla\mathrm{ric}|^2,\ \int_M|\nabla R|^2,\ \int_M\mathrm{scal}^3,\ \int_M\mathrm{scal}|\mathrm{ric}|^2,\ \int_M\mathrm{scal}|R|^2,\\ \textstyle\int_M\mathrm{Tr}(\mathrm{Ric}^3),\ \int_M(*),\ \int_M(**),\ \int_M\hat{R}\end{gathered} \tag{4}$$

constituting a_3 one can ask whether its integral is audible; i.e., whether it is determined by the spectrum of the Laplace operator on functions. Since a choice of basis was involved, the analogous question might of course be asked for any fixed linear combination other than that appearing in (1), such as, for example,

$$\int_M (***),\ \ \int_M \mathring{R} \tag{5}$$

from the left hand sides in Proposition 2.1. The most interesting of the above invariants is the integral over $|\nabla R|^2$: It is zero if and only if the metric is locally symmetric. Although we do not know any examples showing that local symmetry itself is inaudible, we will indeed prove that the value of $|\nabla R|^2$ is inaudible. For sake of completeness, we will prove that actually *none* of the twelve integrals just mentioned is audible.

Remark 2.2. For a few of these this is obvious already from known isospectral examples:
(i) As already mentioned in the Introduction, in [8] a pair of isospectral closed manifolds was constructed with the property that one of them had constant scalar curvature while the other did not; in particular,

$$\int_M |\nabla \mathrm{scal}|^2 \text{ is not audible.}$$

(ii) In [7], continuous families of isospectral metrics were constructed with the property that the maximal value of the scalar curvature changes during the deformation. More specifically, Example 8 of that paper gave a family of isospectral metrics $g(t)$, $t \in [0, \frac{1}{8}]$ on $M = S^5 \times (\mathbb{R}^2/\mathbb{Z}^2)$ whose volume element coincides with the standard one and whose scalar curvature at $(x, z) \in S^5 \times T^2$ depends only on $x \in S^5$ and is equal to (using Proposition 6 of [7])

$$-\frac{13}{2} + 5\cdot 4 + \frac{1}{2}\big((2-5t)x_1^2 + x_2^2 + (4+8t)x_3^2 + 4x_4^2 + (10-3t)x_5^2 + 9x_6^2 \\ + 2\sqrt{5t-40t^2}x_1x_3 - 2\sqrt{15}tx_1x_5 + 2\sqrt{3t-24t^2}x_3x_5\big).$$

The integral of the third power of this expression over $x \in S^5$ is a nonconstant function of t. More precisely, this integral turns out to be a polynomial in t with leading term $t^3\cdot(45A - 135B + (90 - 720)C)$, where $A := \int_{S^5} x_i^6\,dx = \int_{S^5} x_1^6\,dx$, $B := \int_{S^5} x_1^4x_2^2\,dx$, $C := \int_{S^5} x_1^2x_2^2x_3^2\,dx$; we have $B = 3C$ and $A = 15C$, so $45A - 135B - 630C = -360C \neq 0$. In particular,

$$\int_M \mathrm{scal}^3 \text{ is not audible.}$$

(iii) In [15], continuous families of left invariant isospectral metrics g_t on certain compact Lie groups G were constructed. By homogeneity, the functions $\mathrm{scal}(g_t)$, $|\mathrm{ric}|^2(g_t)$, $|R|^2(g_t)$ are constant on G for each fixed t. Since $a_0(g_t) = \mathrm{vol}(g_t)$ is constant in t, it follows by considering $a_1(g_t) = \frac{1}{6}\int_G \mathrm{scal}(g_t)$ that $\mathrm{scal}(g_t)$ is constant in t, too. However, as shown in [15], the term $|\mathrm{ric}|^2(g_t)$ is nonconstant in t in these examples; by considering $a_2(g_t)$ it follows that $|R|^2(g_t)$ is nonconstant in t, too. Hence, these examples show that

$$\int_M \mathrm{scal}|\mathrm{ric}|^2 \text{ and } \int_M \mathrm{scal}|R|^2 \text{ are not audible.}$$

(iv) In the following, we will show the same for the remaining eight invariants from (4) and (5). For this, we will be able to use isospectral pairs of *locally homogeneous* isospectral manifolds (more precisely, pairs of isospectral, locally non-isometric two-step nilmanifolds). In this case, each curvature invariant is a constant function on the manifold. Therefore, and since two isospectral manifolds have the same volume, proving that the integral of a certain curvature invariant is different for two given locally homogeneous isospectral manifolds amounts to showing that they differ in the (constant) value of the curvature invariant itself.

3. Isospectral two-step nilmanifolds

Let $\mathfrak{v} := \mathbb{R}^m$ and $\mathfrak{z} := \mathbb{R}^r$ be endowed with the standard euclidean inner product.

Definition 3.1. With any given linear map $j : \mathfrak{z} \ni Z \mapsto j_Z \in \mathfrak{so}(\mathfrak{v})$, we associate the following objects:

(i) The two-step nilpotent metric Lie algebra $(\mathfrak{g}(j), \langle\ ,\ \rangle)$ with underlying vector space $\mathbb{R}^{m+r} = \mathfrak{v} \oplus \mathfrak{z}$, endowed with the standard euclidean inner product $\langle\ ,\ \rangle$, and whose Lie bracket $[\ ,\ ]^j$ is defined by letting $\mathfrak{z}$ be central, $[\mathfrak{v}, \mathfrak{v}]^j \subseteq \mathfrak{z}$ and $\langle j_Z X, Y\rangle = \langle Z, [X,Y]^j\rangle$ for all $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

(ii) The two-step simply connected nilpotent Lie group $G(j)$ whose Lie algebra is $\mathfrak{g}(j)$, and the left invariant Riemannian metric $g(j)$ on $G(j)$ which coincides with the given inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}(j) = T_e G(j)$. Note that the Lie group exponential map $\exp^j : \mathfrak{g}(j) \to G(j)$ is a diffeomorphism because $G(j)$ is simply connected and nilpotent. Moreover, by the Campbell-Baker-Hausdorff formula, $\exp^j(X,Z) \cdot \exp^j(Y,W) = \exp^j(X+Y, Z+W+\frac{1}{2}[X,Y]^j)$ for all $X, Y \in \mathfrak{v}$ and $Z, W \in \mathfrak{z}$.

(iii) The subset $\Gamma(j) := \exp^j(\mathbb{Z}^m \oplus \frac{1}{2}\mathbb{Z}^r)$ of $G(j)$. If j satisfies $[\mathbb{Z}^m, \mathbb{Z}^m]^j \subset \mathbb{Z}^r$ then the Campbell-Baker-Hausdorff formula implies that $\Gamma(j)$ is a subgroup of $G(j)$; moreover, this subgroup is then discrete and cocompact.

Remark 3.2. (i) Note that *each* Riemannian two-step nilmanifold is locally isometric to some $(G(j), g(j))$: In fact, each simply connected, two-step nilpotent Lie group G, endowed with a left invariant metric g, can be viewed as some $(G(j), g(j))$. Namely, let $\mathfrak{z}$ be a linear subspace of the metric Lie algebra $(\mathfrak{g}, g_e)$ associated with (G, g) such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z} \subseteq \mathfrak{z}(\mathfrak{g})$, let $\mathfrak{v}$ be the orthogonal complement of $\mathfrak{z}$ w.r.t. g_e, and define $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by $g(j_Z X, Y) = g(Z, [X,Y])$.

(ii) As is well-known, $G(j)$ admits uniform discrete subgroups Γ if and only if there *exists* a basis of $\mathfrak{g}(j)$ such that the corresponding structure constants of $[\ ,\ ]^j$ are rational. Even if this is a case, then $\Gamma(j)$ from Definition 3.1(iii) might not be a subgroup. We will use $\Gamma(j)$ in Proposition 3.4 below and in explicit examples, while allowing other Γ in general statements.

(iii) The group $\mathrm{O}(\mathfrak{v}) \times \mathrm{O}(\mathfrak{z})$ acts on the real vector space of linear maps $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by

$$((A,B)j)(Z) = A j_{B^{-1}(Z)} A^{-1}.$$

We call j and j' *equivalent* if there exists $(A,B) \in \mathrm{O}(\mathfrak{v}) \times \mathrm{O}(\mathfrak{z})$ such that $j' = (A,B)j$. In that case, (A,B) provides a metric Lie algebra isomorphism from $(\mathfrak{g}(j), \langle\ ,\ \rangle)$ to $(\mathfrak{g}(j'), \langle\ ,\ \rangle)$. This condition is also necessary: The metric Lie algebras $(\mathfrak{g}(j), \langle\ ,\ \rangle)$ and $(\mathfrak{g}(j'), \langle\ ,\ \rangle)$ are isomorphic if and only if j and j' are equivalent (see [9]). This, in turn, is equivalent to $(G(j), g(j))$ and $(G(j'), g(j'))$ being isometric by a result from [17] concerning nilpotent Lie groups. Moreover, isometry of $(G(j), g(j))$ and $(G(j'), g(j'))$ is equivalent to local isometry of pairs of quotients $(\Gamma\backslash G(j), g(j))$, $(\Gamma'\backslash G(j'), g(j'))$ of these groups by any choice of discrete subgroups Γ, Γ', provided the quotients are endowed with the associated Riemannian quotient metrics. These quotient metrics are again denoted $g(j)$, resp. $g(j')$.

Definition 3.3.

(i) Two linear maps $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ are called *isospectral* if for each $Z \in \mathfrak{z}$, the maps $j_Z, j'_Z \in \mathfrak{so}(\mathfrak{v})$ are similar, that is, have the same eigenvalues (with multiplicities) in $\mathbb{C}$. Since each j_Z is skew-symmetric, this condition is equivalent to the following: For each $Z \in \mathfrak{z}$ there exists $A_Z \in \mathrm{O}(\mathfrak{z})$ such that $j'_Z = A_Z j_Z A_Z^{-1}$. Note that A_Z may depend on Z.

(ii) Two lattices in a euclidean vector space are called *isospectral* if the lengths of their elements, counted with multiplicities, coincide.

The following proposition is a specialized version of a result from [9]; see [16], Remark 2.5(ii) for an explanation about how to derive it from the original, more general version.

Proposition 3.4 ([9] 3.2, 3.7, 3.8)**.** *Let $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral. Assume that both $[\mathbb{Z}^m, \mathbb{Z}^m]^j$ and $[\mathbb{Z}^m, \mathbb{Z}^m]^{j'}$ are contained in $\mathbb{Z}^r$. For each $Z \in \mathbb{Z}^r$ assume that the lattices $\ker(j_Z) \cap \mathbb{Z}^m$ and $\ker(j'_Z) \cap \mathbb{Z}^m$ are isospectral. Then the compact Riemannian manifolds $(\Gamma(j)\backslash G(j), g(j))$ and $(\Gamma(j')\backslash G(j'), g(j'))$ are isospectral for the Laplace operator on functions.*

Example 3.5. Let $m := 4$, $r := 3$, and for $Z = (c_1, c_2, c_3) \in \mathfrak{z} = \mathbb{R}^3$ let j_Z, resp. j'_Z, be the endomorphism of $\mathfrak{v} = \mathbb{R}^4$ given by the matrix

$$\begin{pmatrix} 0 & -2c_1 & -2c_2 & -2c_3 \\ 2c_1 & 0 & -c_3 & c_2 \\ 2c_2 & c_3 & 0 & -c_1 \\ 2c_3 & -c_2 & c_1 & 0 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 0 & -c_1 & -c_2 & -c_3 \\ c_1 & 0 & -2c_3 & 2c_2 \\ c_2 & 2c_3 & 0 & -2c_1 \\ c_3 & -2c_2 & 2c_1 & 0 \end{pmatrix},$$

with respect to the standard basis of $\mathbb{R}^4$. This pair of maps j, j' is a special case of an example from [9]. The eigenvalues of both j_Z and j'_Z are $\{\pm i|Z|, \pm 2i|Z|\}$, each with multiplicity one if $Z \neq 0$; so j and j' are isospectral. Moreover, $\ker(j_Z) = \ker(j'_Z) = \{0\}$ for $Z \neq 0$. Therefore, all conditions from Proposition 3.4 are satisfied and $(\Gamma(j)\backslash G(j), g(j))$, $(\Gamma(j')\backslash G(j'), g(j'))$ are isospectral. In Section 4 (see Corollary 4.3), we will use this example to show inaudibility of

$$\textstyle\int_M \mathrm{Tr}(\mathrm{Ric}^3).$$

Example 3.6. Let $m := 5$, $r := 3$, and for $Z = (c_1, c_2, c_3) \in \mathfrak{z} = \mathbb{R}^3$ let j_Z, resp. j'_Z, be the endomorphism of $\mathfrak{v} = \mathbb{R}^5$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -c_3 & c_2 \\ 0 & 0 & c_3 & 0 & -c_1 \\ 0 & -c_3 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 0 \\ -c_2 & c_1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 0 & -c_3 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_3 & c_2 \\ 0 & 0 & c_3 & 0 & -c_1 \\ 0 & 0 & -c_2 & c_1 & 0 \end{pmatrix},$$

with respect to the standard basis of $\mathbb{R}^5$. In [16], it was shown that this pair of maps j, j' satisfies the conditions of Proposition 3.4, so $(\Gamma(j)\backslash G(j), g(j))$ and $(\Gamma(j')\backslash G(j'), g(j'))$ is a pair of isospectral eight-dimensional manifolds. This pair of manifolds was used in [16] to demonstrate that integrability of the geodesic flow is an inaudible property. In Section 4 (see Proposition 4.8) we will use it to prove inaudibility of

$$\textstyle\int_M |\nabla \mathrm{ric}|^2, \quad \int_M (*), \quad \int_M (**), \quad \text{and } \int_M (***).$$

Example 3.7. If $j, j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ are both of *Heisenberg type*, that is, if $j_Z^2 = j_Z'^2 = -|Z|^2 \mathrm{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$, then j and j' are obviously isospectral because the eigenvalues of both of j_Z and j'_Z then are $\pm i|Z|$, each with multiplicity $(\dim \mathfrak{v})/2$. Moreover, $\ker(j_Z) = \ker(j'_Z) = \{0\}$ for all $Z \neq 0$. Therefore, if the matrix entries of each j_{Z_α} with respect to $\{X_1, \ldots, X_m\}$ are integer, then all conditions of Proposition 3.4 are satisfied and $(\Gamma(j)\backslash G(j), g(j))$, $(\Gamma(j')\backslash G(j'), g(j'))$ are isospectral. Note that it was such a pair of manifolds which Gordon constructed in [6] as the very first example of isospectral, locally non-isometric manifolds; in the notation of Remark 3.8 below, these were the ones associated with $j = \rho^3_{(2,0)}$ and $j' = \rho^3_{(1,1)}$.

In Section 5 below we will use pairs of isospectral nilmanifolds of Heisenberg type to prove inaudibility of

$$\textstyle\int_M |\nabla R|^2, \quad \int_M \hat{R}, \quad \text{and } \int_M \mathring{R}.$$

More precisely, we will show that for any pair $N = (\Gamma\backslash G(j), g(j))$, $N' = (\Gamma'\backslash G(j'), g(j'))$ of isospectral nilmanifolds of Heisenberg type we have the equivalences

$$\textstyle\int_N |\nabla R|^2 = \int_{N'} |\nabla R|^2 \iff \int_N \hat{R} = \int_{N'} \hat{R} \iff \int_N \mathring{R} = \int_{N'} \mathring{R}, \tag{6}$$

and, in case $\dim\mathfrak{z} = 3$, that each of these equalities is equivalent to local isometry of N and N' (see Theorem 5.7). Since there do exist locally nonisometric isospectral examples with $\dim\mathfrak{z} = 3$, this will prove the desired inaudibility statements.

On the other hand, in case $\dim\mathfrak{z} > 3$ we will show that the three equalities from (6) are always true, regardless whether N and N' are locally isometric or not. Even more, the integral of *each* of the sixth order curvature invariants occurring in a_3 will coincide for isospectral pairs N, N' if $\dim\mathfrak{z} > 3$; actually, the same will hold for any curvature invariant of order strictly smaller than $2\dim\mathfrak{z}$ (see Theorem 5.6).

Remark 3.8. Locally nonisometric pairs of isospectral nilmanifolds of Heisenberg type with r-dimensional center of the underlying Lie group exist precisely for $r = \dim\mathfrak{z} \in \{3, 7, 11, 15, \dots\}$. More precisely:

(i) By the condition $j_Z^2 = -|Z|^2\mathrm{Id}_{\mathfrak{v}}$, the map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ extends to a representation of the real Clifford algebra C_r, turning $\mathfrak{v}$ into a module over C_r; the Clifford multiplication by Z is given by $j_Z : \mathfrak{v} \to \mathfrak{v}$. Each such module decomposes into copies of simple modules; see [11], p. 31. In [3] it was proved that if $\mathfrak{m}$ is a simple module over C_r, endowed with an inner product with respect to which the Clifford multiplication with each $Z \in \mathbb{R}^r$ is skew-symmetric, then there exists an orthonormal basis of $\mathfrak{m}$ with respect to which all matrix entries of the Clifford multiplications with the elements $Z_1, \dots, Z_r$ of our given orthonormal basis of $\mathbb{R}^r$ are in $\{1, 0, -1\}$.

For each $r \in \{3, 7, 11, 15, \dots\}$ there are exactly two simple real modules $\mathfrak{m}_+^r$ and $\mathfrak{m}_-^r$ over C_r up to isomorphism; see, e.g., [11], p. 32. For a given such r, these two simple C_r-modules have the same dimension d_r. They can be distinguished by the action of $\omega_r := Z_1 \cdot \ldots \cdot Z_r \in C_r$: After possibly switching names, ω_r acts on $\mathfrak{m}_+^r$ as Id and on $\mathfrak{m}_-^r$ as $-$Id. Moreover, replacing the Clifford multiplication of each $Z \in \mathbb{R}^r$ on $\mathfrak{m}_+^r$ by its negative gives a module isomorphic to $\mathfrak{m}_-^r$.

It follows by the above result from [3] that we can identify both $\mathfrak{m}_+^r$ and $\mathfrak{m}_-^r$ with $\mathbb{R}^{d_r}$ in such a way that for both modules, the Clifford multiplications with $Z_1, \dots, Z_r$ have matrix entries in $\{-1, 0, 1\}$ with respect to the standard basis of $\mathbb{R}^{d^r}$. For $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$ let $\rho^r_{(a,b)}$ denote the representation of C_r on $\mathfrak{v} := (\mathbb{R}^{d_r})^{\oplus(a+b)}$ viewed as $(\mathfrak{m}_+^r)^{\oplus a} \oplus (\mathfrak{m}_-^r)^{\oplus b}$.

For any pair (a, b), (a', b') in $\mathbb{N}_0 \times \mathbb{N}_0$ with $a + b = a' + b'$ but $\{a, b\} \neq \{a', b'\}$, consider the maps $j, j' : \mathbb{R}^r = \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}) = \mathfrak{so}(m)$, where $m := (a + b)d_r$ and where $j_Z := \rho^r_{(a,b)}(Z)$, $j'_Z := \rho^r_{(a',b')}(Z)$ for each $Z \in \mathfrak{z} = \mathbb{R}^r \subset C_r$.

Then j, j' is a pair of maps as in Example 3.7 and thus yields a pair of isospectral nilmanifolds of Heisenberg type. Moreover, these are not locally isometric. To see this, we show that j and j' are not equivalent in the sense of Remark 3.2(iii):

First note that the products $j_{Z_1} \cdot \ldots \cdot j_{Z_r} = \rho_{(a,b)}(\omega_r)$ and $j'_{Z_1} \cdot \ldots \cdot j'_{Z_r} = \rho_{(a',b')}(\omega_r)$ are equal to Id on the respective $\mathfrak{m}_+^r$ components and to $-$Id on the $\mathfrak{m}_-^r$ components of $\mathfrak{v}$. In particular,

$$(\mathrm{Tr}(j_{Z_1} \dots j_{Z_r}))^2 = ((a - b)d_r)^2 \neq ((a' - b')d_r)^2 = (\mathrm{Tr}(j'_{Z_1} \dots j'_{Z_r}))^2. \tag{7}$$

On the other hand, suppose there were $A \in \mathrm{O}(\mathfrak{v})$, $B \in \mathrm{O}(\mathfrak{z})$ such that $j'_Z = Aj_{B^{-1}Z}A^{-1}$ for all $Z \in \mathfrak{z}$. Note that $B^{-1}(Z_1) \cdot \ldots \cdot B^{-1}(Z_r) = \det(B^{-1})\omega_r$ (see [11], p. 34). Thus, we would have $j'_{Z_1} \dots j'_{Z_r} = \det(B)^{-1}Aj_{Z_1} \dots j_{Z_r}A^{-1}$, contradicting (7) since $\det(B) \in \{\pm 1\}$.

(ii) In the context of (i), the metric Lie algebras associated with $\rho^r_{(a,b)}$ and $\rho^r_{(b,a)}$ are isomorphic; an isomorphism is obviously given by $\mathfrak{v} \oplus \mathfrak{z} \ni (X, Z) \mapsto (X, -Z) \in \mathfrak{v} \oplus \mathfrak{z}$. In particular, $(G(j), g(j))$ and $(G(j'), g(j'))$ are isometric if $j = \rho^r_{(a,b)}$, $j' = \rho^r_{(a',b')}$ and $\{a, b\} = \{a', b'\}$.

(iii) Since each real module over C_r is decomposable into simple modules, it follows that for $r \in \{3, 7, 11, 15, \dots\}$ each linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ of Heisenberg type must be equivalent in the sense of Remark 3.2(iii) to one of the maps $\rho^r_{(a,b)}$ from (i). On the other hand, for $r \notin \{3, 7, 11, 15, \dots\}$, there exists only one simple module over C_r up to isomorphism (see [11], p. 32). Thus, in any pair of maps $j, j' : \mathbb{R}^r \to \mathfrak{so}(\mathfrak{v})$ of Heisenberg type with $r \notin \{3, 7, 11, \ldots\}$, j and j' are equivalent and cannot yield locally nonisometric nilmanifolds.

4. Curvature invariants of two-step nilmanifolds

We use the notation from Definition 3.1(i), (ii). We consider a fixed linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ and write, for simplicity, $[\ ,\ ] := [\ ,\ ]^j$. Let $\{X_1, \ldots, X_m\}$, resp. $\{Z_1, \ldots, Z_r\}$, denote an orthonormal basis of $\mathfrak{v}$, resp. $\mathfrak{z}$, and let ∇, R, ric denote the Levi-Civita connection, the curvature tensor, and the Ricci tensor associated with the metric $g(j)$. Recall our sign convention for R from Section 2.

Lemma 4.1. *Let $J := J(j) := \sum_{\alpha=1}^r j_{Z_\alpha}^2$. For $X, Y, U, V \in \mathfrak{v}$ and $Z, W \in \mathfrak{z}$ we have*

(i) $\nabla_X Y = \frac{1}{2}[X, Y] = \frac{1}{2}\sum_{\alpha=1}^r \langle j_{Z_\alpha} X, Y\rangle Z_\alpha \in \mathfrak{z}, \quad \nabla_X Z = \nabla_Z X = -\frac{1}{2} j_Z X \in \mathfrak{v}, \quad \nabla_Z W = 0.$

(ii) $\langle R(\mathfrak{n}_1, \mathfrak{n}_2)\mathfrak{n}_3, \mathfrak{n}_4\rangle = 0$ *whenever* $\mathfrak{n}_i \in \{\mathfrak{v}, \mathfrak{z}\}$, $i = 1, \ldots, 4$, *and either none or an odd number of the* $\mathfrak{n}_i$ *is* $\mathfrak{v}$. *Moreover,*

$$\begin{aligned}\langle R(X, U)Y, V\rangle &= \textstyle\sum_{\alpha=1}^r \big(\frac{1}{4}\langle j_{Z_\alpha} U, Y\rangle\langle j_{Z_\alpha} X, V\rangle - \frac{1}{4}\langle j_{Z_\alpha} X, Y\rangle\langle j_{Z_\alpha} U, V\rangle \\ &\qquad - \tfrac{1}{2}\langle j_{Z_\alpha} X, U\rangle\langle j_{Z_\alpha} Y, V\rangle\big), \\ \langle R(X, Y)Z, W\rangle &= \langle R(Z, W)X, Y\rangle = -\tfrac{1}{4}\langle [j_Z, j_W]X, Y\rangle, \\ \langle R(X, Z)Y, W\rangle &= \tfrac{1}{4}\langle j_W X, j_Z Y\rangle = -\tfrac{1}{4}\langle j_Z j_W X, Y\rangle.\end{aligned}$$

(iii) $\mathrm{ric}(X, Y) = \frac{1}{2}\langle JX, Y\rangle, \quad \mathrm{ric}(X, Z) = 0, \quad \mathrm{ric}(Z, W) = -\frac{1}{4}\mathrm{Tr}(j_Z j_W).$

Proof. In principle, all these formulas can be found in [4]. Alternatively, (i) follows from the Koszul formula and the definitions. From (i), one easily derives the first and third statements of (ii) and

$$\begin{aligned}\langle -\nabla_X \nabla_U Y + \nabla_{\nabla_X U} Y, V\rangle &= \tfrac{1}{4}\langle j_{[U,Y]} X, V\rangle - \tfrac{1}{4}\langle j_{[X,U]} Y, V\rangle \\ &= \tfrac{1}{4}\textstyle\sum_{\alpha=1}^r \big(\langle j_{Z_\alpha} U, Y\rangle\langle j_{Z_\alpha} X, V\rangle - \langle j_{Z_\alpha} X, U\rangle\langle j_{Z_\alpha} Y, V\rangle\big),\end{aligned}$$

from which the second statement of (ii) follows by skew-symmetrization w.r.t. X and U. Moreover,

$$\langle R(X, Z)Y, W\rangle = -\langle \nabla_X \nabla_Z Y, W\rangle = \tfrac{1}{4}\langle [X, j_Z Y], W\rangle = \tfrac{1}{4}\langle j_W X, j_Z Y\rangle.$$

Part (iii) follows directly from (i) and (ii) by taking traces and using the skew-symmetry of j_{Z_α}. □

Remark 4.2. Let $j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be isospectral to j.

(i) Since j_Z and j'_Z are similar by definition, we have $\mathrm{Tr}(j_Z^2) = \mathrm{Tr}(j_Z'^2)$ for all $Z \in \mathfrak{z}$. Thus, by polarization,

$$\mathrm{Tr}(j_Z j_W) = \mathrm{Tr}(j'_Z j'_W) \text{ for all } Z, W \in \mathfrak{z}. \tag{8}$$

(ii) In particular, by Lemma 4.1(iii), the Ricci operators associated with $g(j)$ and $g(j')$ coincide on $\mathfrak{z}$. Therefore, $\mathrm{Tr}(\mathrm{Ric}(g(j))^3)$ and $\mathrm{Tr}(\mathrm{Ric}(g(j'))^3)$ are equal if and only if $\mathrm{Tr}(J^3) = \mathrm{Tr}(J'^3)$, where $J' := \sum_{\alpha=1}^r j'^2_{Z_\alpha}$ is defined analogously as J.

Corollary 4.3. *The two isospectral manifolds from Example 3.5 differ in the value of* $\mathrm{Tr}(\mathrm{Ric}^3)$.

Proof. Here J and J' are diagonal with diagonal entries $-12, -6, -6, -6$, resp. $-3, -9, -9, -9$. In particular, $\mathrm{Tr}(J^3) = -2376 \neq -2214 = \mathrm{Tr}(J'^3)$. The statement now follows from Remark 4.2(ii). □

Definition 4.4. Let $q \in \mathbb{N}$. For each tuple $(k_1, \dots, k_{2q})$ in $\{1, \dots, q\}^{2q}$ which arises as a permutation of $(1, 1, 2, 2, \dots, q, q)$, i.e., which contains each entry exactly twice, we define the following polynomial invariants of j of order $2q$:

$$I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j) := \sum \operatorname{Tr}(j_{Z_{\alpha_{k_1}}} \dots j_{Z_{\alpha_{k_\lambda}}}) \cdot \ldots \cdot \operatorname{Tr}(j_{Z_{\alpha_{k_\mu}}} \dots j_{Z_{\alpha_{k_{2q}}}}),$$

where the sum is taken according to the Einstein summation convention: For each pair $k_i = k_j$ the sum runs over α_{k_i} once from 1 to r. So the sum has exactly r^q summands (and not r^{2q}). We also write $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}$ for $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j)$ if the context is clear. Moreover, we will usually replace the numbers k_i by other symbols; for example, $I_{\alpha\beta\alpha\beta} := I_{1212}$.

With J as defined in Lemma 4.1, we have for $q = 1$:

$$I_{\alpha\alpha} = \textstyle\sum_{\alpha=1}^r \operatorname{Tr}(j_{Z_\alpha}^2) = \operatorname{Tr}(J);$$

note that $I_{\alpha|\alpha} = 0$ since $\operatorname{Tr}(j_{Z_\alpha}) = 0$ for each α. For $q = 2$, the nonvanishing invariants of the above form are exactly

$$\begin{aligned} I_{\alpha\alpha|\beta\beta} &= \textstyle\sum_{\alpha,\beta=1}^r \operatorname{Tr}(j_{Z_\alpha}^2)\operatorname{Tr}(j_{Z_\beta}^2) = (\operatorname{Tr}(J))^2, \\ I_{\alpha\alpha\beta\beta} &= \textstyle\sum_{\alpha,\beta=1}^r \operatorname{Tr}(j_{Z_\alpha}^2 j_{Z_\beta}^2) = \operatorname{Tr}(J^2), \\ I_{\alpha\beta|\alpha\beta} &= \textstyle\sum_{\alpha,\beta=1}^r (\operatorname{Tr}(j_{Z_\alpha} j_{Z_\beta}))^2, \\ I_{\alpha\beta\alpha\beta} &= \textstyle\sum_{\alpha,\beta=1}^r \operatorname{Tr}(j_{Z_\alpha} j_{Z_\beta} j_{Z_\alpha} j_{Z_\beta}). \end{aligned}$$

Some examples for $q = 3$ (not a complete list):

$$\begin{aligned} I_{\alpha\alpha\beta\gamma\gamma\beta} &= \textstyle\sum_{\beta=1}^r \operatorname{Tr}(J j_{Z_\beta} J j_{Z_\beta}), \\ I_{\alpha\alpha\beta\gamma\beta\gamma} &= \textstyle\sum_{\beta,\gamma=1}^r \operatorname{Tr}(J j_{Z_\beta} j_{Z_\gamma} j_{Z_\beta} j_{Z_\gamma}), \\ I_{\alpha\alpha\beta\gamma|\beta\gamma} &= \textstyle\sum_{\beta,\gamma=1}^r \operatorname{Tr}(J j_{Z_\beta} j_{Z_\gamma})\operatorname{Tr}(j_{Z_\beta} j_{Z_\gamma}), \\ I_{\alpha\gamma|\beta\gamma|\alpha\beta} &= \textstyle\sum_{\alpha,\beta,\gamma=1}^r \operatorname{Tr}(j_{Z_\alpha} j_{Z_\gamma})\operatorname{Tr}(j_{Z_\beta} j_{Z_\gamma})\operatorname{Tr}(j_{Z_\alpha} j_{Z_\beta}), \\ I_{\alpha\beta\gamma|\alpha\beta\gamma} &= \textstyle\sum_{\alpha,\beta,\gamma=1}^r (\operatorname{Tr}(j_{Z_\alpha} j_{Z_\beta} j_{Z_\gamma}))^2. \end{aligned}$$

Note that it follows from skew-symmetry of the j_Z that $\operatorname{Tr}(j_{Z_\beta} j_{Z_\alpha} j_{Z_\gamma}) = -\operatorname{Tr}(j_{Z_\alpha} j_{Z_\beta} j_{Z_\gamma})$ and thus $I_{\alpha\beta\gamma|\beta\alpha\gamma} = -I_{\alpha\beta\gamma|\alpha\beta\gamma}$. The invariant $I_{\alpha\beta\gamma|\alpha\beta\gamma}$ will play a crucial role in the Heisenberg type case (see Section 5).

Remark 4.5. If j and j' are equivalent in the sense of Remark 3.2(iii) then it follows that $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j) = I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j')$ for each of the invariants from Definition 4.4.

Lemma 4.6. *For the curvature invariants* scal *(of order two) and* scal^2, $|\mathrm{ric}|^2$, $|R|^2$ *(of order four) we have:*

(i) $\mathrm{scal} = \frac{1}{4}\operatorname{Tr}(J) = \frac{1}{4} I_{\alpha\alpha}$
(ii) $\mathrm{scal}^2 = \frac{1}{16}(\operatorname{Tr}(J))^2 = \frac{1}{16} I_{\alpha\alpha|\beta\beta}$
(iii) $|\mathrm{ric}|^2 = \frac{1}{4}\operatorname{Tr}(J^2) + \frac{1}{16} I_{\alpha\beta|\alpha\beta} = \frac{1}{4} I_{\alpha\alpha\beta\beta} + \frac{1}{16} I_{\alpha\beta|\alpha\beta}$
(iv) $|R|^2 = \frac{1}{2}\operatorname{Tr}(J^2) + \frac{3}{8} I_{\alpha\beta|\alpha\beta} + \frac{1}{8} I_{\alpha\beta\alpha\beta} = \frac{1}{2} I_{\alpha\alpha\beta\beta} + \frac{3}{8} I_{\alpha\beta|\alpha\beta} + \frac{1}{8} I_{\alpha\beta\alpha\beta}$

Proof. (i), (ii), and (iii) are very easy to prove using Lemma 4.1(ii). We defer the proof of (iv) to the Appendix. □

Lemma 4.7. *Let* $(*)$, $(**)$ *be as in* (2). *Then we have*

(i) $(*) = \frac{3}{16} I_{\alpha\alpha\beta\gamma\gamma\beta} + \frac{1}{16} I_{\alpha\alpha\beta\gamma|\beta\gamma}$

(ii) $(**) = \frac{1}{8}I_{\alpha\alpha\beta\gamma\gamma\beta} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\beta\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma|\beta\gamma} + \frac{1}{32}I_{\alpha\gamma\beta\gamma|\alpha\beta}$

(iii) $|\nabla\mathrm{ric}|^2 = -\frac{1}{4}\mathrm{Tr}(J^3) + \frac{1}{8}I_{\alpha\alpha\beta\gamma\gamma\beta} - \frac{1}{8}I_{\alpha\alpha\beta\gamma|\beta\gamma} - \frac{1}{32}I_{\alpha\gamma|\beta\gamma|\alpha\beta}$
$= -\frac{1}{4}I_{\alpha\alpha\beta\beta\gamma\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\gamma\beta} - \frac{1}{8}I_{\alpha\alpha\beta\gamma|\beta\gamma} - \frac{1}{32}I_{\alpha\gamma|\beta\gamma|\alpha\beta}$

We defer the proof of Lemma 4.7 to the Appendix.

Proposition 4.8. *The two isospectral manifolds from Example 3.6 differ in each of the values of* $(*)$, $(**)$, $(***)$, *and* $|\nabla\mathrm{ric}|^2$.

Proof. Here, J and J' are diagonal with entries $-2,-2,-1,-1,-2$, resp. $-1,-1,-2,-2,-2$. In particular, $\mathrm{Tr}(J^3) = \mathrm{Tr}(J'^3)$. By an easy computation, $\mathrm{Tr}(Jj_{Z_\beta}Jj_{Z_\beta}) = -8$ for $\beta = 1,2,3$, and $\mathrm{Tr}(J'j'_{Z_1}J'j'_{Z_1}) = \mathrm{Tr}(J'j'_{Z_2}J'j'_{Z_2}) = -8$, but $\mathrm{Tr}(J'j'_{Z_3}J'j'_{Z_3}) = -10$. Therefore,

$$I_{\alpha\alpha\beta\gamma\gamma\beta}(j) = -24 \neq -26 = I_{\alpha\alpha\beta\gamma\gamma\beta}(j'). \tag{9}$$

Also, $\mathrm{Tr}(Jj_{Z_\beta}j_{Z_\gamma}) = 0$ whenever $\beta \neq \gamma$, and the same for j'; so

$$I_{\alpha\alpha\beta\gamma|\beta\gamma}(j) = \textstyle\sum_{\beta=1}^3 \mathrm{Tr}(Jj_{Z_\beta}^2)\mathrm{Tr}(j_{Z_\beta}^2) = -4\cdot 2 - 4\cdot 2 - 6\cdot 4 = I_{\alpha\alpha\beta\gamma|\beta\gamma}(j').$$

In particular, the values of $(*)$ are different for the two manifolds. The same statement for $(***)$ now follows immediately from Proposition 2.1(ii) and Remark 4.2(ii), together with the fact that $\nabla\mathrm{scal} = 0$ on both manifolds, and that $\mathrm{Tr}(J^3) = \mathrm{Tr}(J'^3)$ (see above).

Since the terms $I_{\alpha\alpha\beta\gamma\gamma\beta}$ and $I_{\alpha\alpha\beta\gamma|\beta\gamma}$ also occur in $(**)$, the statement about $(**)$ will follow once we show that the two manifolds do not differ in any of the remaining two summands of $(**)$ from Lemma 4.7(ii). We here have $j_{Z_\beta}^4 = -j_{Z_\beta}^2$ for $\beta = 1,2,3$ and $(j_{Z_\beta}j_{Z_\gamma})^2 = 0$ whenever $\beta \neq \gamma$; the same statements hold for j'. So $I_{\alpha\alpha\beta\gamma\beta\gamma}$ here happens to be $\mathrm{Tr}(-J^2) = -14 = \mathrm{Tr}(-J'^2)$ for both manifolds. Finally, note that $\mathrm{Tr}(j_{Z_\alpha}j_{Z_\beta}) = 0$ for $\alpha \neq \beta$, and the same for j'. Thus, in this example, $I_{\alpha\gamma\beta\gamma|\alpha\beta} = \sum_{\alpha,\gamma=1}^3 \mathrm{Tr}((j_{Z_\alpha}j_{Z_\gamma})^2)\mathrm{Tr}(j_{Z_\alpha}^2) = \sum_{\alpha=1}^3 \mathrm{Tr}(j_{Z_\alpha}^4)\mathrm{Tr}(j_{Z_\alpha}^2) = -2\cdot 2 - 2\cdot 2 - 4\cdot 4$, and the same for j'.

The statement about $|\nabla\mathrm{ric}|^2$ now follows immediately: By (9), the two manifolds differ in the second summand of the formula from Lemma 4.7, while the remaining summands are the same for both; for the fourth summand, this follows either from the above considerations or directly from equation (8). □

Remark 4.9. As an aside, we will use the formulas from Lemma 4.6 to give an example of a pair of isospectral nilmanifolds differing in the integrals of the fourth order curvature invariants $|\mathrm{ric}|^2$ and $|R|^2$ (see Example 4.10 below). Although these are not the first examples of isospectral manifolds with this property (see the Introduction), they are the first such examples in the category of nilmanifolds. Considering the heat invariants a_0, a_1, and a_2, note that a pair of isospectral, locally homogeneous manifolds differs in $|\mathrm{ric}|^2$ if and only it differs in $|R|^2$. In the case of two-step nilmanifolds, it follows from Lemma 4.6(iii) and Remark 4.2(i) that such a pair differs in $|\mathrm{ric}|^2$ if and only it differs in the value of $\mathrm{Tr}(J^2)$. In Example 3.5, we had $\mathrm{Tr}(J^3) \neq \mathrm{Tr}(J'^3)$. Nevertheless, the values of $\mathrm{Tr}(J^2)$ and $\mathrm{Tr}(J'^2)$ happen to coincide in that example, so we need a different one. The following is related to an example from [15], Proposition 3.6(ii) (after replacing $j_{Z_2}(t)$ from that context by $3j_{Z_2}(t/3) - i\mathrm{Id}$, evaluating at $t = 0$, resp. $t = 2$, and identifying $\mathbb{C}^3$ with $\mathbb{R}^6$).

Example 4.10. Let $m := 6$, $r := 2$, and for $Z = (c_1, c_2) \in \mathfrak{z} = \mathbb{R}^2$ let j_Z, resp. j'_Z, be the endomorphism of $\mathfrak{v} = \mathbb{R}^6$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 3c_2 & c_1{+}c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & 0 \\ -3c_2 & 0 & 0 & 0 & 0 & -c_1{+}c_2 \\ -c_1{-}c_2 & 0 & 0 & 0 & 0 & 3c_2 \\ 0 & -c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1{-}c_2 & -3c_2 & 0 & 0 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 0 & 2c_2 & c_2 & c_1{+}c_2 & 0 & 0 \\ -2c_2 & 0 & 2c_2 & 0 & c_2 & 0 \\ -c_2 & -2c_2 & 0 & 0 & 0 & -c_1{+}c_2 \\ -c_1{-}c_2 & 0 & 0 & 0 & 2c_2 & c_2 \\ 0 & -c_2 & 0 & -2c_2 & 0 & 2c_2 \\ 0 & 0 & c_1{-}c_2 & -c_2 & -2c_2 & 0 \end{pmatrix},$$

w.r.t. the standard basis of $\mathbb{R}^6$. The maps j and j' are isospectral since $j_{(c_1,c_2)}$ and $j'_{(c_1,c_2)}$ have the same characteristic polynomial $\lambda^6+(2c_1^2+21c_2^2)\lambda^4+((c_1^2+9c_2^2)^2+3c_2^4)\lambda^2+c_2^2(c_1^2+8c_2^2)^2$. Moreover, $\ker(j_{(c_1,c_2)}) = \ker(j'_{(c_1,c_2)}) = \{0\}$ if $c_2 \neq 0$; for $c_2 = 0, c_1 \neq 0$ both kernels are $\mathrm{span}\{X_2, X_5\}$. Therefore, all conditions of Proposition 3.4 are satisfied and $(\Gamma(j)\backslash G(j), g(j))$, $(\Gamma(j')\backslash G(j'), g(j'))$ are isospectral. A direct computation reveals $\mathrm{Tr}(J^2) = 630 \neq 598 = \mathrm{Tr}(J'^2)$. By Remark 4.9, this implies that the two manifolds differ in the value of $|\mathrm{ric}|^2$, and also in the value of $|R|^2$.

Proposition 4.12 below concerns the structure of curvature invariants of arbitrary order of two-step nilpotent Lie groups with left invariant metrics. This description will enable us to arrive at certain conclusions for higher order curvature invariants in a special case (see Theorem 5.6). We first need the following observation:

Remark 4.11. Using Lemma 4.1(i), (ii) repeatedly, one sees that $\langle(\nabla^p_{A_1,\ldots,A_p}R)(B,C)D,E\rangle$ with $A_1,\ldots,A_p,B,C,D,E \in \{X_1,\ldots,X_m,Z_1,\ldots,Z_r\}$ is a linear combination of terms of order $p+2$ in j which are (if not zero) of the form

$$\langle j_{Z_{\alpha_1}}\cdots j_{Z_{\alpha_i}}X_{\ell_1},X_{\ell_2}\rangle\cdot\ldots\cdot\langle j_{Z_{\alpha_j}}\cdots j_{Z_{\alpha_{p+2}}}X_{\ell_{2a-1}},X_{\ell_{2a}}\rangle. \tag{10}$$

Moreover, the multiset $\{X_{\ell_1},\ldots,X_{\ell_{2a}},Z_{\alpha_1},\ldots,Z_{\alpha_{p+2}}\}$ of vectors occurring in (10) arises from the multiset $\{A_1,\ldots,A_p,B,C,D,E\}$ by possibly enlarging it by one or several pairs of equal vectors from $\{Z_1,\ldots,Z_r\}$; the vectors from $\mathfrak{v}$ are the same in both multisets. In particular, $\langle(\nabla^p_{A_1,\ldots,A_p}R)(B,C)D,E\rangle = 0$ if the multiset $\{A_1,\ldots,A_p,B,C,D,E\}$ contains an odd number of vectors from $\mathfrak{v}$, or no such vectors at all.

Proposition 4.12. *Let $q \in \mathbb{N}$. On a two-step nilpotent Lie group $G(j)$, endowed with the left invariant metric $g(j)$, each curvature invariant of order $2q$ can be expressed as a linear combination of polynomial invariants of j of the form $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_{2q}}$ as in Definition 4.4.*

Proof. According to [12], p. 4646 (see also [1], p. 75ff.), each curvature invariant of order $2q$ is a linear combination of certain Weyl invariants of the form

$$W = \mathrm{Tr}_\sigma(\nabla^{p_1}R\otimes\ldots\otimes\nabla^{p_\nu}R), \tag{11}$$

where $\nu \in \mathbb{N}$, $p_i \in \mathbb{N}_0$ for each $i \in \{1,\ldots,\nu\}$, $p_1+\ldots+p_\nu$ is even, $2q = 2\nu+p_1+\ldots+p_\nu$, $\sigma \in S_{2N}$, $2N = 4\nu+p_1+\ldots+p_\nu$, and Tr_σ denotes the complete trace with respect to σ. The latter is defined as the sum according to the Einstein summation convention with respect to equal indices $k_i = k_j$ in the expression

$$(\nabla^{p_1}R\otimes\ldots\otimes\nabla^{p_\nu}R)(e_{s_{k_1}},\ldots,e_{s_{k_{2N}}}) = (\nabla^{p_1}R)(e_{s_{k_1}},\ldots,e_{s_{k_{p_1+4}}})\ldots(\nabla^{p_\nu}R)(e_{s_{k_{2N-p_\nu-3}}},\ldots,e_{s_{k_{2N}}}),$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of the tangent space at the point under consideration and $(k_1,\ldots,k_{2N})$ arises from $(1,1,2,2,3,3,\ldots,N,N)$ by the permutation σ.

In our case, by Remark 4.11, each summand of W in (11) is a linear combination of products of terms as in (10), so W itself is a linear combination of terms of the form

$$\langle j_{Z_{\alpha_{s_1}}}\cdots j_{Z_{\alpha_{s_c}}}X_{\ell_{u_1}},X_{\ell_{u_2}}\rangle\cdot\ldots\cdot\langle j_{Z_{\alpha_{s_d}}}\cdots j_{Z_{\alpha_{s_{2q}}}}X_{\ell_{u_{2a-1}}},X_{\ell_{u_{2a}}}\rangle, \tag{12}$$

with each s_i and each u_j occurring exactly twice. Summation over pairs of equal u_j will transform (12) into a term of the form $\mathrm{Tr}(j_{Z_{\alpha_{k_1}}}\cdots j_{Z_{\alpha_{k_\lambda}}})\cdot\ldots\cdot\mathrm{Tr}(j_{Z_{\alpha_{k_\mu}}}\cdots j_{Z_{\alpha_{k_{2q}}}})$ in which still each k_i occurs exactly twice; summation over pairs of equal indices k_i then yields $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_{2q}}$. □

We conclude this section by giving some partial results for $|\nabla R|^2$, $\hat{R}$, $\mathring{R}$ which we will use in Section 5 to prove their inaudibility:

Lemma 4.13.

(i) $|\nabla R|^2 = -\,\frac{3}{2} I_{\alpha\beta\gamma|\alpha\beta\gamma} + L_1$,

(ii) $\hat{R} = -\frac{7}{16} I_{\alpha\beta\gamma|\alpha\beta\gamma} + L_2$,

(iii) $\mathring{R} = -\frac{17}{64} I_{\alpha\beta\gamma|\alpha\beta\gamma} + L_3$,

where L_1, L_2, L_3 *are universal linear combinations of certain other* $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_6}$ *in which all occurring subtuples* $(k_1,\ldots,k_\lambda)$, $\ldots$, $(k_\mu,\ldots,k_6)$ *are of even length.*

We defer the proof of Lemma 4.13 to the Appendix.

5. Curvature invariants of Heisenberg type nilmanifolds

We continue to use the notation from Definition 3.1(i), (ii), and we now always consider linear maps $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ of Heisenberg type. Recall from Example 3.7 that this means $j_Z^2 = -|Z|^2 \mathrm{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$. By polarization, this is equivalent to

$$j_Z j_W + j_W j_Z = -2\langle Z, W\rangle \mathrm{Id}_{\mathfrak{v}} \text{ for all } Z, W \in \mathfrak{z}. \tag{13}$$

Again, let $\{X_1,\ldots,X_m\}$ and $\{Z_1,\ldots,Z_r\}$ be orthonormal bases of $\mathfrak{v}$ and $\mathfrak{z}$, respectively.

Lemma 5.1. *In the Heisenberg type case, the following holds:*

(i) $j_Z j_W = -j_W j_Z$ *for all* $Z, W \in \mathfrak{z}$ *with* $Z \perp W$.

(ii) *Let* $k \in \mathbb{N}$ *and* $(\alpha_1,\ldots,\alpha_k) \in \{1,\ldots,r\}^k$. *Let* $\ell \in \{0,\ldots,k\}$ *and* $\beta_1 < \ldots < \beta_\ell$ *be such that* $\{\beta_1,\ldots,\beta_\ell\}$ *consists precisely of those* α_i *which occur an odd number of times in* $(\alpha_1,\ldots,\alpha_k)$. *Then there exists* $c \in \{0,1\}$, *depending only on the tuple* $(\alpha_1,\ldots,\alpha_k)$, *but not on* j, *such that*
$$j_{Z_{\alpha_1}} \cdots j_{Z_{\alpha_k}} = (-1)^c j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}},$$
where in case $\ell = 0$, *the empty product* $j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}$ *is to be read as* $\mathrm{Id}_{\mathfrak{v}}$.

(iii) *If* ℓ *is a positive even number and* $\beta_1,\ldots,\beta_\ell \in \{1,\ldots,r\}$ *are pairwise different then* $\mathrm{Tr}(j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}) = 0$.

(iv) *Let* ℓ *be positive, but strictly smaller than* r, *and let* $\beta_1,\ldots,\beta_\ell \in \{1,\ldots,r\}$. *If* ℓ *is odd or if* $\beta_1,\ldots,\beta_\ell$ *are pairwise different, then* $\mathrm{Tr}(j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}) = 0$.

Proof. Part (i) is trivial by (13). For (ii), one first repeatedly uses (i) to arrange the factors in nondecreasing order w.r.t. the values of the α_i; the statement then follows from $j^2_{Z_{\alpha_i}} = -\mathrm{Id}_{\mathfrak{v}}$. If ℓ is positive and even, and $\beta_1,\ldots,\beta_\ell$ are pairwise different, then (i) and the cyclicity of the trace imply $\mathrm{Tr}(j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}) = -\mathrm{Tr}(j_{Z_{\beta_\ell}} j_{Z_{\beta_1}} \cdots j_{Z_{\beta_{\ell-1}}}) = -\mathrm{Tr}(j_{Z_1} \cdots j_{Z_{\beta_\ell}})$, hence (iii).

For proving (iv), it now suffices to consider the case that ℓ is odd. Since $\ell < r$, we can choose $\alpha \in \{1,\ldots,r\} \setminus \{\beta_1,\ldots,\beta_\ell\}$. Then, using $j^{-1}_{Z_\alpha} = -j_{Z_\alpha}$ and (i), we have $\mathrm{Tr}(j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}) = \mathrm{Tr}(j_{Z_\alpha} j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}(-j_{Z_\alpha})) = \mathrm{Tr}(j^2_{Z_\alpha} j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}}) = -\mathrm{Tr}(j_{Z_{\beta_1}} \cdots j_{Z_{\beta_\ell}})$, hence (iv). □

Corollary 5.2. *In the Heisenberg type case, the following holds:*

(i) *Any* $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_{2q}}$ *as in Definition 4.4 in which all the occurring subtuples* $(k_1,\ldots,k_\lambda)$, $\ldots$, $(k_\mu,\ldots,k_{2q})$ *are of even length can be expressed as a universal polynomial in* $m = \dim\mathfrak{v}$ *and* $r = \dim\mathfrak{z}$ *which does not depend on* j.

(ii) *If at least one of the subtuples of odd length occurring in* $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_{2q}}$ *becomes strictly shorter than* r *or equal to one after eliminating pairs of equal indices* $k_i = k_j$ *within that subtuple, then* $I_{k_1\ldots k_\lambda|\ldots|k_\mu\ldots k_{2q}} = 0$.

Proof. Let d be the length of one of the subtuples, and let $\mathrm{Tr}(j_{Z_{\alpha_1}} \dots j_{Z_{\alpha_d}})$ be the corresponding factor in one of the r^q summands occurring in the sum as which $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}$ is defined.

By Lemma 5.1(ii), $\mathrm{Tr}(j_{Z_{\alpha_1}} \dots j_{Z_{\alpha_d}})$ can be simplified to either $\pm\mathrm{Tr}(\mathrm{Id}_{\mathfrak{v}}) = \pm m$ (where the sign does not depend on j) or to a new term which involves only pairwise different Z_{α_i} and whose length $d' \leq d$ is positive and has the same parity as d.

In this latter case, if d and hence d' is even, then the new term vanishes by Lemma 5.1(iii). This proves part (i). If d is odd, then the condition of (ii) implies, a forteriori, that $d' < r$ or $d' = 1$ (note that there might be even more equal indices α_i in $(\alpha_1, \dots, \alpha_d)$ than equal indices k_i in the corresponding subtuple of $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}$). So in this case, the new term vanishes by Lemma 5.1(iv). This proves part (ii). □

Proposition 5.3. (i) *In the Heisenberg type case, each curvature invariant of order two or four and each of* $\mathrm{Tr}(\mathrm{Ric}^3)$*,* $(*)$*,* $(**)$*,* $(***)$*,* $|\nabla \mathrm{ric}|^2$ *can be expressed as a universal polynomial in* $m = \dim\mathfrak{v}$ *and* $r = \dim\mathfrak{z}$ *which does not depend on* j*.*

(ii) *Any two isospectral nilmanifolds of Heisenberg type do not differ in any of the curvature invariants mentioned in (i).*

Proof. For (i), just observe using Lemma 4.6 and Lemma 4.7 that each of these curvature invariants is a universal linear combination of terms satisfying the condition of Corollary 5.2(i). Part (ii) follows from (i) and Remark 5.4 below. □

Remark 5.4. Any two isospectral nilmanifolds of Heisenberg type share the same dimensions $m = \dim\mathfrak{v}$ and also the same dimensions $r = \dim\mathfrak{z}$.

To see this, let N and N' be two isospectral nilmanifolds of Heisenberg type, associated with $j : \mathbb{R}^r \to \mathfrak{so}(\mathbb{R}^m)$ and $j' : \mathbb{R}^{r'} \to \mathfrak{so}(\mathbb{R}^{m'})$, respectively. Then necessarily $m + r = m' + r'$ since the dimension is spectrally determined. Moreover, the two manifolds must have the same volume and the same total scalar curvature, thus $\mathrm{scal}(g(j)) = \mathrm{scal}(g(j'))$. By Lemma 4.6(i) this means $\mathrm{Tr}(J) = \mathrm{Tr}(J')$; hence $-mr = -m'r'$. Together with $m+r = m'+r'$ this implies $\{m, r\} = \{m', r'\}$. Using the classification of nilmanifolds of Heisenberg type from [2], or recalling from Remark 3.8 that $\mathbb{R}^m$ is a module over C_r and inspecting the dimensions of the simple real modules over C_r in [11], one sees $m > r$ and $m' > r'$. So indeed we have $m = m'$ and $r = r'$.

Proposition 5.5. *Let* $j, j' : \mathfrak{z} = \mathbb{R}^r \to \mathfrak{so}(\mathfrak{v}) = \mathfrak{so}(m)$ *be of Heisenberg type.*

(i) *If* $2q < 2r$ *then* $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j) = I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}(j')$ *for each of the invariants from Definition 4.4.*

(ii) *In the case* $2q = 2r$*, the only invariants from Definition 4.4 in which* j *and* j' *can possibly differ are the* $I_{k_1 \dots k_r | k_{\tau(1)} \dots k_{\tau(r)}}$*, where* $\tau \in S_r$*. Note that* $I_{k_1 \dots k_r | k_{\tau(1)} \dots k_{\tau(r)}} = \pm I_{k_1 \dots k_r | k_1 \dots k_r}$ *due to Lemma 5.1(i), depending on the sign of the permutation* τ*.*

(iii) j *and* j' *are equivalent in the sense of Remark 3.2(iii) if and only if* $I_{k_1 \dots k_r | k_1 \dots k_r}(j) = I_{k_1 \dots k_r | k_1 \dots k_r}(j')$*.*

Proof. By Corollary 5.2, j and j' cannot differ in $I_{k_1 \dots k_\lambda | \dots | k_\mu \dots k_{2q}}$ unless at least one of the subtuples $(k_1, \dots, k_\lambda)$, $\dots$, $(k_\mu, \dots, k_{2q})$ is of odd length at least $\dim\mathfrak{z} = r$, after eliminating any pairs of equal indices occurring within that subtuple. Each of the remaining (at least r) indices has to occur in one of the other subtuples (recall that each k_i occurs exactly twice in $(k_1, \dots, k_{2q})$). But this implies $2q \geq r + r$ and, in the case $2q = 2r$, that there are exactly two subtuples, both of length r. This shows (i) and (ii).

The "only if" statement of (iii) is a special case of Remark 4.5. For the converse, let j and j' be nonequivalent. By Remark 3.8(ii), (iii), it follows that $r \in \{3, 7, 11, 15, \dots\}$, and that j, j' are equivalent to certain $\rho^r_{(a,b)}$, resp. $\rho^r_{(a',b')}$ with $a+b=m=a'+b'$, but $\{a,b\}\neq\{a',b'\}$; in particular, $|a-b|\neq|a'-b'|$. Note that since r is odd, $\mathrm{Tr}(j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_r}})=0$ whenever $\alpha_1,\dots,\alpha_r$ are not pairwise distinct (recall Lemma 5.1(ii) and (iv)). Moreover, $(\mathrm{Tr}(j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_r}}))^2$ does not change under permutations of $\alpha_1,\dots,\alpha_r$ due to Lemma 5.1(i). So $I_{k_1\dots k_r|k_1\dots k_r}(j)=r!(\mathrm{Tr}(j_{Z_1}\dots j_{Z_r}))^2$, and similarly for j'. Now $I_{k_1\dots k_r|k_1\dots k_r}(j)\neq I_{k_1\dots k_r|k_1\dots k_r}(j')$ follows by (7) from Remark 3.8(i). ☐

Theorem 5.6. (i) *Any two isospectral nilmanifolds of Heisenberg type with* $\dim\mathfrak{z}=r$ *cannot differ in any curvature invariant of order* $2q<2r$.
(ii) *Any two isospectral nilmanifolds of Heisenberg type with centers of dimension strictly greater than three ($r>3$) do not differ in any of the sixth, eighth, tenth or twelfth order curvature invariants.*

Proof. (i) By Proposition 4.12, each curvature invariant of order $2q$ is a linear combination (with universal coefficients) of certain $I_{k_1\dots k_a|\dots|k_b\dots k_{2q}}$. Thus, the statement follows immediately from Remark 5.4 and Proposition 5.5(i).

(ii) For the sixth order curvature invariants, this follows directly from (i). The statement for eighth, tenth and twelfth order curvature invariants equally follows from (i) after recalling from Remark 3.8 that any two isospectral nilmanifolds of Heisenberg type with $r>3$ are either locally isometric (and the statement thus trivial) or satisfy $r\in\{7,11,15,\dots\}$, thus $r\geq 7$. ☐

Theorem 5.7. *Let N, N' be two isospectral nilmanifolds of Heisenberg type associated with Lie algebras satisfying* $r=\dim\mathfrak{z}$. *If $r=3$ then the following conditions are equivalent:*

(a) *N and N' are locally isometric.*
(b) *N and N' have the same value of $|\nabla R|^2$.*
(c) *N and N' have the same value of $\hat{R}$.*
(d) *N and N' have the same value of $\mathring{R}$.*

If $r\neq 3$, then (b), (c), (d) are true regardless of (a).

Proof. Trivially, (a) implies each of the other three statements. Moreover, if $r\notin\{3,7,11,15,\dots\}$ then (b), (c), (d) are true by Remark 3.8(iii). Let $\mathfrak{g}(j)$, $\mathfrak{g}(j')$ be the metric Lie algebras associated with N, N'. By Lemma 4.13 and Corollary 5.2(i), each of (b), (c), (d) is equivalent to

$$I_{\alpha\beta\gamma|\alpha\beta\gamma}(j)=I_{\alpha\beta\gamma|\alpha\beta\gamma}(j'). \tag{14}$$

For $r>3$, this is always true by Theorem 5.6(i). For $r=3$, (14) is equivalent to (a) by Proposition 5.5(iii) and Remark 3.2(iii). ☐

Corollary 5.8. *In any pair of isospectral, locally nonisometric manifolds of Heisenberg type associated with Lie algebras satisfying* $r=\dim\mathfrak{z}=3$, *the two manifolds differ in each of the values of $|\nabla R|^2$, $\hat{R}$, $\mathring{R}$. Since such pairs do exist (see Remark 3.8(i)), neither $\int|\nabla R|^2$ nor $\int\hat{R}$ nor $\int\mathring{R}$ is audible.*

Two locally homogeneous manifolds (M,g), (M',g') are called *curvature equivalent* (of order zero) if for $p\in M$, $p'\in M'$ there exists an euclidean isometry $F:(T_pM,g_p)\to(T_{p'}M',g'_{p'})$ which intertwines the Riemannian curvature operators; that is, $F(R(X,Y)Z)=R(F(X),F(Y))F(Z)$ for all $X,Y,Z\in T_pM$. The following result provides a certain contrast to Theorem 5.6(i):

Proposition 5.9. *Let N and N' be any two nilmanifolds of Heisenberg type (without restriction to the dimensions of the centers). If N and N' are not locally isometric, then they are not curvature equivalent.*

For the proof, the following lemma will serve as the key:

Lemma 5.10. *Let $j:\mathfrak{z}=\mathbb{R}^r\to\mathfrak{so}(\mathfrak{v})=\mathfrak{so}(m)$ be of Heisenberg type. Write $\mathfrak{g}:=\mathfrak{g}(j)$ and view R as an endomorphism of $\mathfrak{g}\wedge\mathfrak{g}$ by requiring $\langle R(A,B)C,D\rangle=\langle R(A\wedge B),C\wedge D\rangle$ for all $A,B,C,D\in\mathfrak{g}$, where the inner product on $\mathfrak{g}\wedge\mathfrak{g}$ is defined in the usual way by bilinear extension of $\langle E\wedge F,C\wedge D\rangle=\langle E,C\rangle\langle F,D\rangle-\langle E,D\rangle\langle F,C\rangle$. Write $R^{\mathfrak{v}\wedge\mathfrak{v}}:=\mathrm{Pr}_{\mathfrak{v}\wedge\mathfrak{v}}\circ R|_{\mathfrak{v}\wedge\mathfrak{v}}$, where $\mathrm{Pr}_{\mathfrak{v}\wedge\mathfrak{v}}:\mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{v}\wedge\mathfrak{v}$ denotes orthogonal projection. Then for all $q\in\mathbb{N}$ we have*

$$\mathrm{Tr}((R^{\mathfrak{v}\wedge\mathfrak{v}})^q)=(-\tfrac14)^q\big(\tfrac12 I_{k_1\dots k_q|k_1\dots k_q}-\tfrac12 I_{k_1\dots k_qk_1\dots k_q}+r(2-r+m)^q-r(2-r)^q\big).$$

Proof. As always, let $\{X_1,\dots,X_m\}$ and $\{Z_1,\dots,Z_r\}$ be orthonormal bases of $\mathfrak{v}$, resp. $\mathfrak{z}$. For $Z\in\mathfrak{z}$, we let $E_Z:=\sum_{k=1}^m X_k\wedge j_ZX_k\in\mathfrak{v}\wedge\mathfrak{v}$. Note that E_Z is defined independently of the choice of orthonormal basis in $\mathfrak{v}$. If $Z\in\mathfrak{z}$ is a unit vector then

$$\begin{aligned}|E_Z|^2&=\textstyle\sum_{k,\ell=1}^m\langle X_k\wedge j_ZX_k,X_\ell\wedge j_ZX_\ell\rangle\\&=\textstyle\sum_{k,\ell=1}^m(\langle X_k,X_\ell\rangle\langle j_ZX_k,j_ZX_\ell\rangle-\langle X_k,j_ZX_\ell\rangle\langle X_\ell,j_ZX_k\rangle)=-2\mathrm{Tr}(j_Z^2)=2m.\end{aligned}$$

Using polarization we see that $\{E_{Z_1},\dots,E_{Z_r}\}\subset\mathfrak{v}\wedge\mathfrak{v}$ is an orthogonal set of vectors of norm $\sqrt{2m}$. Define $\Phi:\mathfrak{v}\wedge\mathfrak{v}\to\mathfrak{v}\wedge\mathfrak{v}$ by $\Phi(X\wedge Y):=\sum_{\alpha=1}^r j_{Z_\alpha}X\wedge j_{Z_\alpha}Y$. Using that $\{j_{Z_\alpha}X_1,\dots,j_{Z_\alpha}X_m\}$ is again an orthogonal basis of $\mathfrak{v}$, we obtain:

$$\begin{aligned}\Phi(E_Z)&=\textstyle\sum_{\alpha=1}^r\sum_{k=1}^m(j_{Z_\alpha}X_k\wedge j_{Z_\alpha}j_ZX_k)\\&=\textstyle\sum_{\alpha=1}^r\sum_{k=1}^m(-j_{Z_\alpha}X_k\wedge j_Zj_{Z_\alpha}X_k-j_{Z_\alpha}X_k\wedge 2\langle Z,Z_\alpha\rangle X_k)=(-r+2)E_Z\end{aligned}$$

for $Z\in\mathfrak{z}$. Let $\mathrm{Pr}_{\mathcal{E}}:\mathfrak{v}\wedge\mathfrak{v}\to\mathfrak{v}\wedge\mathfrak{v}$ denote orthogonal projection to $\mathcal{E}:=\mathrm{span}\{E_{Z_1},\dots,E_{Z_r}\}$. Note that Φ is symmetric. Thus, the previous formula implies $\Phi(\mathcal{E})=\mathcal{E}$, $\Phi(\mathcal{E}^\perp)=\mathcal{E}^\perp$, and

$$\Phi\circ\mathrm{Pr}_{\mathcal{E}}=\mathrm{Pr}_{\mathcal{E}}\circ\Phi=(2-r)\mathrm{Pr}_{\mathcal{E}}. \tag{15}$$

On the other hand, for $X,U,Y,V\in\mathfrak{v}$, the formula for $\langle R(X,U)Y,V\rangle$ from Lemma 4.1(ii) easily translates into

$$\begin{aligned}\langle R(X\wedge U),Y\wedge V\rangle&=\textstyle\sum_{\alpha=1}^r\big(-\frac14\langle j_{Z_\alpha}X\wedge j_{Z_\alpha}U,Y\wedge V\rangle-\frac18\langle X\wedge U,E_{Z_\alpha}\rangle\langle E_{Z_\alpha},Y\wedge V\rangle\big)\\&=-\tfrac14\langle\Phi(X\wedge U),Y\wedge V\rangle-\tfrac18\cdot 2m\langle\mathrm{Pr}_{\mathcal{E}}(X\wedge U),Y\wedge V\rangle\end{aligned}$$

(recall that $|E_{Z_\alpha}|=\sqrt{2m}$), hence

$$R^{\mathfrak{v}\wedge\mathfrak{v}}=-\tfrac14(\Phi+m\mathrm{Pr}_{\mathcal{E}}).$$

Using (15) and $\mathrm{Tr}(\mathrm{Pr}_{\mathcal{E}})=r$ we conclude

$$\mathrm{Tr}((R^{\mathfrak{v}\wedge\mathfrak{v}})^q)=(-\tfrac14)^q\big(\mathrm{Tr}(\Phi^q)+\textstyle\sum_{p=1}^q\binom{q}{p}(2-r)^{q-p}m^pr\big)=(-\tfrac14)^q\big(\mathrm{Tr}(\Phi^q)+r((2-r+m)^q-(2-r)^q)\big).$$

The statement thus follows from

$$\begin{aligned}\mathrm{Tr}(\Phi^q)&=\textstyle\sum_{k<\ell}\langle\Phi^q(X_k\wedge X_\ell),X_k\wedge X_\ell\rangle=\frac12\sum_{k,\ell=1}^m\langle\Phi^q(X_k\wedge X_\ell),X_k\wedge X_\ell\rangle\\&=\textstyle\frac12\sum_{k,\ell=1}^m\sum_{\alpha_1,\dots,\alpha_q=1}^r\langle j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_q}}X_k\wedge j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_q}}X_\ell,X_k\wedge X_\ell\rangle\\&=\textstyle\frac12\sum_{\alpha_1,\dots,\alpha_q=1}^r\big((\mathrm{Tr}(j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_q}}))^2-\mathrm{Tr}(j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_q}}j_{Z_{\alpha_1}}\dots j_{Z_{\alpha_q}})\big)\\&=\tfrac12(I_{k_1\dots k_q|k_1\dots k_q}-I_{k_1\dots k_qk_1\dots k_q}).\end{aligned}$$

□

Proof of Proposition 5.9. Let N and N' be curvature equivalent; we are going to show that they are locally isometric. Let $F : \mathfrak{g}(j) = \mathfrak{v}\oplus\mathfrak{z} \to \mathfrak{g}(j') = \mathfrak{v}'\oplus\mathfrak{z}'$ be a euclidean isometry of the associated Lie algebras which intertwines the curvature tensors. Then F also intertwines the Ricci tensors. Note that here in the Heisenberg type case we have $\mathrm{Ric}(g(j))|_{\mathfrak{v}} = -\frac{r}{2}\mathrm{Id}_{\mathfrak{v}}$ and $\mathrm{Ric}(g(j))|_{\mathfrak{z}} = \frac{m}{4}\mathrm{Id}_{\mathfrak{z}}$ by Lemma 4.1(iii), and similarly for j'. Since F has to preserve the eigenspace associated to the negative, resp. positive eigenvalue, we have $F(\mathfrak{v}) = \mathfrak{v}'$ and $F(\mathfrak{z}) = \mathfrak{z}'$; in particular, $m = m'$ and $r = r'$. The restriction of F to $\mathfrak{v}$ now induces a linear map from $\mathfrak{v}\wedge\mathfrak{v}$ to $\mathfrak{v}'\wedge\mathfrak{v}'$ which intertwines $R(g(j))^{\mathfrak{v}\wedge\mathfrak{v}}$ and $R(g(j'))^{\mathfrak{v}'\wedge\mathfrak{v}'}$; in particular, these operators have the same trace, and so do their q-th powers for any q. Applying Lemma 5.10 in the special case $q := r$, we conclude

$$I_{k_1\dots k_r|k_1\dots k_r}(j) - I_{k_1\dots k_r k_1\dots k_r}(j) = I_{k_1\dots k_r|k_1\dots k_r}(j') - I_{k_1\dots k_r k_1\dots k_r}(j').$$

The second terms on each side of this equation coincide by Proposition 5.5(ii). Thus, the first terms have to coincide, too. By Proposition 5.5(iii) this implies that j and j' are equivalent in the sense of Remark 3.2(iii); so N and N' are indeed locally isometric. □

Together with Remark 3.8(i) and Theorem 5.6(i), the previous proposition implies:

Theorem 5.11. *For any $k \in \mathbb{N}$, there exist pairs of locally homogeneous Riemannian manifolds which are not curvature equivalent, but do not differ in any curvature invariant of order up to $2k$.*

Appendix

Proof of Remark 4.6(iv).
For $A \in \mathfrak{g}$, write $R_A : \mathfrak{g}\times\mathfrak{g} \ni (B,C) \mapsto R(A,B)C \in \mathfrak{g}$, and consider the canonical extension of $\langle\ ,\ \rangle$ to tensors of this form. We start by computing individual formulas for $\langle R_A, R_B\rangle$ because we will need them below in the proof of Lemma 4.7(ii). For $U, Y \in \mathfrak{v}$ we have, by Lemma 4.1(ii),

- $\langle R_U|_{\mathfrak{v}\times\mathfrak{v}}, R_Y|_{\mathfrak{v}\times\mathfrak{v}}\rangle = \sum_{k,\ell,a=1}^m \langle R(U,X_k)X_\ell, X_a\rangle\langle R(Y,X_k)X_\ell, X_a\rangle$
  $= \frac{1}{16}\sum_{k,\ell,a=1}^m\sum_{\beta,\gamma=1}^r \big(\langle j_{Z_\beta}U, X_a\rangle\langle j_{Z_\beta}X_k, X_\ell\rangle - \langle j_{Z_\beta}U, X_\ell\rangle\langle j_{Z_\beta}X_k, X_a\rangle$
  $- 2\langle j_{Z_\beta}U, X_k\rangle\langle j_{Z_\beta}X_\ell, X_a\rangle\big)\cdot$
  $\cdot\big(\langle j_{Z_\gamma}Y, X_a\rangle\langle j_{Z_\gamma}X_k, X_\ell\rangle - \langle j_{Z_\gamma}Y, X_\ell\rangle\langle j_{Z_\gamma}X_k, X_a\rangle - 2\langle j_{Z_\gamma}Y, X_k\rangle\langle j_{Z_\gamma}X_\ell, X_a\rangle\big)$
  $= \frac{1}{16}\sum_{\beta,\gamma=1}^r\big((1^2+1^2+2^2)\langle j_{Z_\beta}U, j_{Z_\gamma}Y\rangle\langle j_{Z_\beta}, j_{Z_\gamma}\rangle$
  $+ (1-2+1-2-2-2)\langle j_{Z_\beta}U, j_{Z_\gamma}j_{Z_\beta}j_{Z_\gamma}Y\rangle\big)$
  $= \frac{3}{8}\sum_{\beta,\gamma=1}^r\big(\langle j_{Z_\beta}j_{Z_\gamma}U, Y\rangle\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma}) + \langle j_{Z_\beta}j_{Z_\gamma}j_{Z_\beta}j_{Z_\gamma}U, Y\rangle\big)$,
- $\langle R_U|_{\mathfrak{v}\times\mathfrak{z}}, R_Y|_{\mathfrak{v}\times\mathfrak{z}}\rangle = \sum_{k=1}^m\sum_{\beta,\gamma=1}^r\langle R(U,X_k)Z_\beta, Z_\gamma\rangle\langle R(Y,X_k)Z_\beta, Z_\gamma\rangle$
  $= \frac{1}{16}\sum_{k=1}^m\sum_{\beta,\gamma=1}^r\langle[j_{Z_\beta}, j_{Z_\gamma}]U, X_k\rangle\langle[j_{Z_\beta}, j_{Z_\gamma}]Y, X_k\rangle = -\frac{1}{16}\sum_{\beta,\gamma=1}^r\langle[j_{Z_\beta}, j_{Z_\gamma}]^2U, Y\rangle$
  $= -\frac{1}{8}\sum_{\beta,\gamma=1}^r\langle j_{Z_\beta}j_{Z_\gamma}j_{Z_\beta}j_{Z_\gamma}U, Y\rangle + \frac{1}{8}\sum_{\beta=1}^r\langle j_{Z_\beta}Jj_{Z_\beta}U, Y\rangle$,
- $\langle R_U|_{\mathfrak{z}\times\mathfrak{v}}, R_Y|_{\mathfrak{z}\times\mathfrak{v}}\rangle + \langle R_U|_{\mathfrak{z}\times\mathfrak{z}}, R_Y|_{\mathfrak{z}\times\mathfrak{z}}\rangle = 2\sum_{k=1}^m\sum_{\beta,\gamma=1}^r\langle R(U,Z_\beta)Z_\gamma, X_k\rangle\langle R(Y,Z_\beta)Z_\gamma, X_k\rangle$
  $= \frac{1}{8}\sum_{k=1}^m\sum_{\beta,\gamma=1}^r\langle j_{Z_\gamma}U, j_{Z_\beta}X_k\rangle\langle j_{Z_\gamma}Y, j_{Z_\beta}X_k\rangle = \frac{1}{8}\sum_{\beta,\gamma=1}^r\langle j_{Z_\beta}j_{Z_\gamma}U, j_{Z_\beta}j_{Z_\gamma}Y\rangle$
  $= \frac{1}{8}\sum_{\beta=1}^r\langle j_{Z_\beta}Jj_{Z_\beta}U, Y\rangle$.

Hence,
(16)
$$\langle R_U, R_Y\rangle = \sum_{\beta,\gamma=1}^r\big(\tfrac{3}{8}\langle j_{Z_\beta}j_{Z_\gamma}U, Y\rangle\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma}) + \tfrac{1}{4}\langle j_{Z_\beta}j_{Z_\gamma}j_{Z_\beta}j_{Z_\gamma}U, Y\rangle\big) + \tfrac{1}{4}\sum_{\beta=1}^r\langle j_{Z_\beta}Jj_{Z_\beta}U, Y\rangle.$$

For $W \in \mathfrak{z}$ we have $R_W|_{\mathfrak{z}\times\mathfrak{z}} = 0$ and

- $|R_W|_{\mathfrak{v}\times\mathfrak{v}}|^2 + |R_W|_{\mathfrak{v}\times\mathfrak{z}}|^2 = 2\sum_{k,\ell=1}^m \sum_{\alpha=1}^r \langle R(W,X_k)X_\ell, Z_\alpha\rangle^2 = \frac{1}{8}\sum_{k,\ell=1}^m \sum_{\alpha=1}^r \langle j_W X_\ell, j_{Z_\alpha} X_k\rangle^2$
  $= \frac{1}{8}\sum_{\alpha=1}^2 |j_W j_{Z_\alpha}|^2 = \frac{1}{8}\mathrm{Tr}(Jj_W^2)$,
- $|R_W|_{\mathfrak{z}\times\mathfrak{v}}|^2 = \sum_{k,\ell=1}^m \sum_{\alpha=1}^r \langle R(W,Z_\alpha)X_k, X_\ell\rangle^2 = \frac{1}{16}\sum_{k,\ell=1}^m \sum_{\alpha=1}^r \langle [j_W, j_{Z_\alpha}]X_k, X_\ell\rangle^2$
  $= \frac{1}{16}\sum_{\alpha=1}^r |j_W j_{Z_\alpha} - j_{Z_\alpha} j_W|^2 = \frac{1}{8}\mathrm{Tr}(Jj_W^2) - \frac{1}{8}\sum_{\alpha=1}^r \mathrm{Tr}(j_W j_{Z_\alpha} j_W j_{Z_\alpha})$

and thus for $Z, W \in \mathfrak{z}$, using polarization,

$$\langle R_Z, R_W\rangle = \frac{1}{8}\mathrm{Tr}(J(j_Z j_W + j_W j_Z)) - \frac{1}{8}\sum_{\alpha=1}^r \mathrm{Tr}(j_Z j_{Z_\alpha} j_W j_{Z_\alpha}). \tag{17}$$

Moreover, $\langle R_X, R_Z\rangle = 0$ for all $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$ by Lemma 4.1(ii). Using (16) and (17), we obtain

$$|R|^2 = \sum_{k=1}^m \langle R_{X_k}, R_{X_k}\rangle + \sum_{\alpha=1}^r \langle R_{Z_\alpha}, R_{Z_\alpha}\rangle = \frac{3}{8}I_{\alpha\beta|\alpha\beta} + \frac{1}{4}I_{\alpha\beta\alpha\beta} + \frac{1}{4}I_{\alpha\alpha\beta\beta} + \frac{1}{4}I_{\alpha\alpha\beta\beta} - \frac{1}{8}I_{\alpha\beta\alpha\beta},$$

from which the statement follows. □

Proof of Lemma 4.7.
(i) First note that by Lemma 4.1 and since J is symmetric,

$$\begin{aligned}
&\sum_{k,\ell,a,b=1}^m \mathrm{ric}(X_k,X_\ell)\mathrm{ric}(X_a,X_b)\langle R(X_k,X_a)X_\ell, X_b\rangle \\
&= \frac{1}{4}\sum_{k,\ell,a,b=1}^m \langle JX_k, X_\ell\rangle\langle JX_a, X_b\rangle \cdot \\
&\quad \cdot \sum_{\beta=1}^r \big(\tfrac{1}{4}\langle j_{Z_\beta}X_a, X_\ell\rangle\langle j_{Z_\beta}X_k, X_b\rangle - \tfrac{1}{4}\langle j_{Z_\beta}X_k, X_\ell\rangle\langle j_{Z_\beta}X_a, X_b\rangle - \tfrac{1}{2}\langle j_{Z_\beta}X_k, X_a\rangle\langle j_{Z_\beta}X_\ell, X_b\rangle\big) \\
&= \frac{1}{16}\sum_{k,a=1}^m \sum_{\beta=1}^r \big(\langle j_{Z_\beta}X_a, JX_k\rangle\langle j_{Z_\beta}X_k, JX_a\rangle - \langle j_{Z_\beta}X_k, JX_k\rangle\langle j_{Z_\beta}X_a, JX_a\rangle \\
&\qquad\qquad - 2\langle j_{Z_\beta}X_k, X_a\rangle\langle j_{Z_\beta}JX_k, JX_a\rangle\big) \\
&= \frac{1}{16}\sum_{\beta=1}^r (\langle -j_{Z_\beta}J, Jj_{Z_\beta}\rangle - 0 - 2\langle j_{Z_\beta}, Jj_{Z_\beta}J\rangle) = \frac{1}{16}\sum_{\beta=1}^r 3\mathrm{Tr}(Jj_{Z_\beta}Jj_{Z_\beta}) = \frac{3}{16}I_{\alpha\alpha\beta\gamma\gamma\beta}.
\end{aligned}$$

Moreover, $\mathrm{ric}(\mathfrak{v},\mathfrak{z}) = 0$, $\langle R(\mathfrak{z},\mathfrak{z})\mathfrak{z},\mathfrak{z}\rangle = 0$, and

$$\begin{aligned}
&2\cdot\sum_{k,\ell=1}^m \sum_{\beta,\gamma=1}^r \mathrm{ric}(X_k,X_\ell)\mathrm{ric}(Z_\beta,Z_\gamma)\langle R(X_k,Z_\beta)X_\ell, Z_\gamma\rangle \\
&= \frac{1}{16}\sum_{k,\ell=1}^m \sum_{\beta,\gamma=1}^r \langle JX_k, X_\ell\rangle\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\langle j_{Z_\beta}j_{Z_\gamma}X_k, X_\ell\rangle = \frac{1}{16}\sum_{\beta,\gamma=1}^r \mathrm{Tr}(Jj_{Z_\beta}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma}) \\
&= \frac{1}{16}I_{\alpha\alpha\beta\gamma|\beta\gamma}.
\end{aligned}$$

Thus,

$$(*) = \frac{3}{16}I_{\alpha\alpha\beta\gamma\gamma\beta} + \frac{1}{16}I_{\alpha\alpha\beta\gamma|\beta\gamma}.$$

(ii) By definition of $(**)$ and by Lemma 4.1(iii),

$$(**) = \sum_{k,\ell=1}^m \frac{1}{2}\langle JX_k, X_\ell\rangle\langle R_{X_k}, R_{X_\ell}\rangle - \sum_{\alpha,\beta=1}^r \frac{1}{4}\mathrm{Tr}(j_{Z_\alpha}j_{Z_\beta})\langle R_{Z_\alpha}, R_{Z_\beta}\rangle.$$

Thus, using (16) and (17),

$$\begin{aligned}
(**) &= \frac{1}{2}\sum_{k=1}^m \Big(\sum_{\beta,\gamma=1}^r \big(\tfrac{3}{8}\langle j_{Z_\beta}j_{Z_\gamma}X_k, JX_k\rangle\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma}) + \tfrac{1}{4}\langle j_{Z_\beta}j_{Z_\gamma}j_{Z_\beta}j_{Z_\gamma}X_k, JX_k\rangle\big) \\
&\qquad + \frac{1}{4}\sum_{\beta=1}^r \langle j_{Z_\beta}Jj_{Z_\beta}X_k, JX_k\rangle\Big) \\
&\quad - \frac{1}{4}\sum_{\beta,\gamma=1}^r \mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\big(\tfrac{1}{8}\mathrm{Tr}(J(j_{Z_\beta}j_{Z_\gamma} + j_{Z_\gamma}j_{Z_\beta})) - \tfrac{1}{8}\sum_{\alpha=1}^r \mathrm{Tr}(j_{Z_\beta}j_{Z_\alpha}j_{Z_\gamma}j_{Z_\alpha})\big) \\
&= \frac{3}{16}I_{\alpha\alpha\beta\gamma|\beta\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\beta\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\gamma\beta} - \frac{1}{16}I_{\alpha\alpha\beta\gamma|\beta\gamma} + \frac{1}{32}I_{\beta\alpha\gamma\alpha|\beta\gamma} \\
&= \frac{1}{8}I_{\alpha\alpha\beta\gamma|\beta\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\beta\gamma} + \frac{1}{8}I_{\alpha\alpha\beta\gamma\gamma\beta} + \frac{1}{32}I_{\alpha\gamma\beta\gamma|\alpha\beta}.
\end{aligned}$$

(iii) For $X,Y\in\mathfrak{v}$ and $Z\in\mathfrak{z}$, we have $(\nabla_Y\mathrm{ric})|_{\mathfrak{v}\times\mathfrak{v}}=0$, $(\nabla_Y\mathrm{ric})|_{\mathfrak{z}\times\mathfrak{z}}=0$ and

$$\begin{aligned}((\nabla_Y\mathrm{ric})(Z,X))^2 &= (\tfrac12\mathrm{ric}(X,j_ZY)-\tfrac12\mathrm{ric}([Y,X],Z))^2\\ &= (\tfrac14\langle JX,j_ZY\rangle+\tfrac18\textstyle\sum_{\beta=1}^r\mathrm{Tr}(j_{Z_\beta}j_Z)\langle j_{Z_\beta}Y,X\rangle)^2, \text{ hence}\end{aligned}$$

$$\begin{aligned}|(\nabla_Y\mathrm{ric})|^2 &= 2\textstyle\sum_{\ell=1}^m\sum_{\gamma=1}^r\big(\tfrac1{16}\langle JX_\ell,j_{Z_\gamma}Y\rangle^2+\tfrac1{16}\sum_{\beta=1}^r\langle JX_\ell,j_{Z_\gamma}Y\rangle\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\langle j_{Z_\beta}Y,X_\ell\rangle\\ &\qquad+\tfrac1{64}\textstyle\sum_{\alpha,\beta=1}^r\mathrm{Tr}(j_{Z_\alpha}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\langle j_{Z_\alpha}Y,X_\ell\rangle\langle j_{Z_\beta}Y,X_\ell\rangle\big)\\ &= \textstyle\sum_{\gamma=1}^r\big(\tfrac18|Jj_{Z_\gamma}Y|^2+\tfrac18\sum_{\beta=1}^r\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\langle j_{Z_\beta}Y,Jj_{Z_\gamma}Y\rangle\\ &\qquad+\tfrac1{32}\textstyle\sum_{\alpha,\beta=1}^r\mathrm{Tr}(j_{Z_\alpha}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\langle j_{Z_\alpha}Y,j_{Z_\beta}Y\rangle\big). \text{ Thus,}\\ |(\nabla\mathrm{ric})|_{\mathfrak{v}}|^2 &= \textstyle\sum_{\gamma=1}^r\big(\tfrac18|Jj_{Z_\gamma}|^2-\tfrac18\sum_{\beta=1}^r\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\mathrm{Tr}(Jj_{Z_\gamma}j_{Z_\beta})\\ &\qquad-\tfrac1{32}\textstyle\sum_{\alpha,\beta=1}^r\mathrm{Tr}(j_{Z_\alpha}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\beta}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\alpha}j_{Z_\beta})\big)\\ &= -\tfrac18\mathrm{Tr}(J^3)-\tfrac18 I_{\alpha\alpha\beta\gamma|\beta\gamma}-\tfrac1{32}I_{\alpha\beta|\alpha\gamma|\beta\gamma},\end{aligned}$$

where $|(\nabla\mathrm{ric})|_{\mathfrak{v}}|^2$ denotes $\sum_{k=1}^m|\nabla_{X_k}\mathrm{ric}|^2$. For $X,Y\in\mathfrak{v}$ and $W\in\mathfrak{z}$, we have $(\nabla_W\mathrm{ric})|_{\mathfrak{v}\times\mathfrak{z}}=0$, $(\nabla_W\mathrm{ric})|_{\mathfrak{z}\times\mathfrak{v}}=0$, $(\nabla_W\mathrm{ric})|_{\mathfrak{z}\times\mathfrak{z}}=0$, and

$$((\nabla_W\mathrm{ric})(X,Y))^2=(-\tfrac12\mathrm{ric}(j_WX,Y)-\tfrac12\mathrm{ric}(X,j_WY))^2=(\tfrac14\langle Jj_WX,Y\rangle+\tfrac14\langle JX,j_WY\rangle)^2, \text{ hence}$$

$$\begin{aligned}|\nabla_W\mathrm{ric}|^2 &= \tfrac1{16}\textstyle\sum_{k,\ell=1}^m\big(\langle Jj_WX_k,X_\ell\rangle^2+\langle JX_k,j_WX_\ell\rangle^2+2\langle Jj_WX_k,X_\ell\rangle\langle JX_k,j_WX_\ell\rangle\big)\\ &= \tfrac18|Jj_W|^2-\tfrac18\langle Jj_W,j_WJ\rangle. \text{ Thus,}\\ |(\nabla\mathrm{ric})|_{\mathfrak{z}}|^2 &= \textstyle\sum_{\beta=1}^r\big(\tfrac18|Jj_{Z_\beta}|^2+\tfrac18\mathrm{Tr}(Jj_{Z_\beta}Jj_{Z_\beta})\big)=-\tfrac18\mathrm{Tr}(J^3)+\tfrac18 I_{\alpha\alpha\beta\gamma\gamma\beta},\end{aligned}$$

where $|(\nabla\mathrm{ric})|_{\mathfrak{z}}|^2$ denotes $\sum_{\alpha=1}^r|\nabla_{Z_\alpha}\mathrm{ric}|^2$. So,

$$|\nabla\mathrm{ric}|^2=|(\nabla\mathrm{ric})|_{\mathfrak{v}}|^2+|(\nabla\mathrm{ric})|_{\mathfrak{z}}|^2=-\tfrac14\mathrm{Tr}(J^3)-\tfrac18 I_{\alpha\alpha\beta\gamma|\beta\gamma}-\tfrac1{32}I_{\alpha\beta|\alpha\gamma|\beta\gamma}+\tfrac18 I_{\alpha\alpha\beta\gamma\gamma\beta}.$$

□

Proof of Lemma 4.13.
(i) The various contributions

$$\textstyle\sum\langle(\nabla_AR)(B,C)D,E\rangle^2 \tag{18}$$

to $|\nabla R|^2$, where each of A,B,C,D,E runs through either the orthonormal basis $\{X_1,\dots,X_m\}$ of $\mathfrak{v}$ or the orthonormal basis $\{Z_1,\dots,Z_r\}$ of $\mathfrak{z}$, can by Remark 4.11 be nonzero only in the cases where $\mathfrak{v}$ occurs an even number of times. If $\mathfrak{v}$ occurs exactly twice then, again by Remark 4.11, each $\langle(\nabla_AR)(B,C)D,E\rangle$ is a linear combination of terms of the type $\langle j_{Z_{\alpha_1}}j_{Z_{\alpha_2}}j_{Z_{\alpha_3}}X_{\ell_1}X_{\ell_2}\rangle$, where $(X_{\ell_1},X_{\ell_2},Z_{\alpha_1},Z_{\alpha_2},Z_{\alpha_3})$ is just some permutation of (A,B,C,D,E). The sum in (18) will thus be a linear combination of sums of the type

$$\textstyle\sum\langle j_{Z_{\alpha_{s_1}}}j_{Z_{\alpha_{s_2}}}j_{Z_{\alpha_{s_3}}}X_{\ell_{u_1}},X_{\ell_{u_2}}\rangle\langle j_{Z_{\alpha_{s_4}}}j_{Z_{\alpha_{s_5}}}j_{Z_{\alpha_{s_6}}}X_{\ell_{u_1}},X_{\ell_{u_2}}\rangle,$$

where (s_1,s_2,s_3) and (s_4,s_5,s_6) are the same up to permutation, and the summation is done w.r.t. pairs of equal indices s_i and u_j. Summation over equal pairs of u_j yields

$$-\textstyle\sum\mathrm{Tr}(j_{Z_{\alpha_{s_6}}}j_{Z_{\alpha_{s_5}}}j_{Z_{\alpha_{s_4}}}j_{Z_{\alpha_{s_1}}}j_{Z_{\alpha_{s_2}}}j_{Z_{\alpha_{s_3}}}),$$

which equals $-I_{s_6s_5s_4s_1s_2s_3}$, one of our invariants from Definition 4.4 in which only subtuples of even length occur (in this case, only one subtuple, and this one of length six). Hence, sums as in (18) with exactly two occurrences of $\mathfrak{v}$ contribute only to the term L_1 from the assertion. Therefore it remains to consider sums as in (18) with exactly four occurrences of $\mathfrak{v}$.

Due to the symmetries of R, the contribution of such sums is equal to

$$\sum\langle(\nabla_W R)(X,Y)U,V\rangle^2 + 4\sum\langle(\nabla_X R)(W,Y)U,V\rangle^2, \tag{19}$$

where both sums are taken over $X,Y,U,V \in \{X_1,\dots,X_m\}$, $W \in \{Z_1,\dots,Z_r\}$. For the first term in (19), we note using Lemma 4.1(i), (ii) and the skew-symmetry of the maps j_W, j_{Z_α} that $\langle(\nabla_W R)(X,Y)U,V\rangle$ is the sum of the following twelve summands:

(a) $-\langle\nabla_W\nabla_X\nabla_Y U,V\rangle = -\frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}X,V\rangle\langle j_{Z_\beta}Y,U\rangle$,
(b) $\langle\nabla_W\nabla_Y\nabla_X U,V\rangle = \frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}Y,V\rangle\langle j_{Z_\beta}X,U\rangle$,
(c) $\langle\nabla_W\nabla_{[X,Y]}U,V\rangle = \frac{1}{4}\sum_\beta\langle j_W j_{Z_\beta}U,V\rangle\langle j_{Z_\beta}X,Y\rangle$,
(d) $\langle\nabla_{\nabla_W X}\nabla_Y U,V\rangle = \frac{1}{8}\sum_\beta\langle j_{Z_\beta}j_W X,V\rangle\langle j_{Z_\beta}Y,U\rangle$,
(e) $-\langle\nabla_Y\nabla_{\nabla_W X}U,V\rangle = -\frac{1}{8}\sum_\beta\langle j_{Z_\beta}j_W X,U\rangle\langle j_{Z_\beta}Y,V\rangle$,
(f) $-\langle\nabla_{[\nabla_W X,Y]}U,V\rangle = -\frac{1}{4}\sum_\beta\langle j_{Z_\beta}j_W X,Y\rangle\langle j_{Z_\beta}U,V\rangle$,
(g) $\langle\nabla_X\nabla_{\nabla_W Y}U,V\rangle = \frac{1}{8}\sum_\beta\langle j_{Z_\beta}j_W Y,U\rangle\langle j_{Z_\beta}X,V\rangle$,
(h) $-\langle\nabla_{\nabla_W Y}\nabla_X U,V\rangle = -\frac{1}{8}\sum_\beta\langle j_{Z_\beta}j_W Y,V\rangle\langle j_{Z_\beta}X,U\rangle$,
(i) $-\langle\nabla_{[X,\nabla_W Y]}U,V\rangle = \frac{1}{4}\sum_\beta\langle j_W j_{Z_\beta}X,Y\rangle\langle j_{Z_\beta}U,V\rangle$,
(j) $\langle\nabla_X\nabla_Y\nabla_W U,V\rangle = -\frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}Y,U\rangle\langle j_{Z_\beta}X,V\rangle$,
(k) $-\langle\nabla_Y\nabla_X\nabla_W U,V\rangle = \frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}X,U\rangle\langle j_{Z_\beta}Y,V\rangle$,
(l) $-\langle\nabla_{[X,Y]}\nabla_W U,V\rangle = -\frac{1}{4}\sum_\beta\langle j_{Z_\beta}j_W U,V\rangle\langle j_{Z_\beta}X,Y\rangle$,

where the sums are taken over $\beta \in \{1,\dots,r\}$. Now $\langle(\nabla_W R)(X,Y)U,V\rangle^2$ is the square of the sum of the twelve terms. The square of each single one of them will just lead to a contribution to L_1: For example, the square of the term in (a) is $\frac{1}{64}$ times

$$\textstyle\sum_{\beta,\gamma}\langle j_W j_{Z_\beta}X,V\rangle\langle j_{Z_\beta}Y,U\rangle\langle j_W j_{Z_\gamma}X,V\rangle\langle j_{Z_\gamma}Y,U\rangle,$$

which after summation over $X,Y,U,V \in \{X_1,\dots,X_m\}$ gives $-\sum_{\beta,\gamma}\operatorname{Tr}(j_W j_W j_{Z_\beta} j_{Z_\gamma})\operatorname{Tr}(j_{Z_\beta} j_{Z_\gamma})$; summation over $W \in \{Z_1,\dots,Z_r\}$ thus yields $-I_{\alpha\alpha\beta\gamma|\beta\gamma}$, another invariant in which only subtuples of even lengths (here, four and two) occur.

Next, consider the product of the terms in (a) and (b) which is $-\frac{1}{64}$ times

$$\textstyle\sum_{\beta,\gamma}\langle j_W j_{Z_\beta}X,V\rangle\langle j_{Z_\beta}Y,U\rangle\langle j_W j_{Z_\gamma}Y,V\rangle\langle j_{Z_\gamma}X,U\rangle.$$

One easily checks that this leads to an invariant with just one subtuple of length six. The technical reason is that here, there is no way to group the four factors into subsets which would not be linked to each other by the occurrence of any common vectors from $\{X,Y,U,V\}$.

The only pairings of different terms from (a)–(l) above where this does not happen are the following twelve:

$$((a)\text{ or }(d)) \longleftrightarrow ((g)\text{ or }(j)),\quad ((b)\text{ or }(h)) \longleftrightarrow ((e)\text{ or }(k)),\quad ((c)\text{ or }(l)) \longleftrightarrow ((f)\text{ or }(i)), \tag{20}$$

For example, the product of the terms in (a) and (g) is

$$-\tfrac{1}{64}\textstyle\sum_{\beta,\gamma}\langle j_W j_{Z_\beta}X,V\rangle\langle j_{Z_\beta}Y,U\rangle\langle j_{Z_\gamma}j_W Y,U\rangle\langle j_{Z_\gamma}X,V\rangle \tag{21}$$

which after summation over X,Y,U,V becomes $-\frac{1}{64}\sum_{\beta,\gamma}\operatorname{Tr}(j_{Z_\gamma}j_W j_{Z_\beta})\operatorname{Tr}(j_{Z_\beta}j_{Z_\gamma}j_W)$. Summation over W finally yields $-\frac{1}{64}I_{\gamma\alpha\beta|\beta\gamma\alpha} = -\frac{1}{64}I_{\alpha\beta\gamma|\alpha\beta\gamma}$. Similarly, the product of the terms in (a) and (j) is

$$\tfrac{1}{64}\textstyle\sum_{\beta,\gamma}\langle j_W j_{Z_\beta}X,V\rangle\langle j_{Z_\beta}Y,U\rangle\langle j_W j_{Z_\gamma}Y,U\rangle\langle j_{Z_\gamma}X,V\rangle.$$

which gives $-\frac{1}{64}\operatorname{Tr}(j_{Z_\gamma}j_W j_{Z_\beta})\operatorname{Tr}(j_{Z_\gamma}j_W j_{Z_\beta}) = -\frac{1}{64}I_{\alpha\beta\gamma|\alpha\beta\gamma}$ again. For each of the pairings from (20), note that whenever the two terms to be paired differ in sign, they also differ in the order of

j_W and j_{Z_β} in their first factors. Just as we saw for $(a) \leftrightarrow (g)$ and $(a) \leftrightarrow (j)$, this leads each time to a negative multiple of $I_{\alpha\beta\gamma|\alpha\beta\gamma}$. Altogether, we obtain

$$2 \cdot \big(4 \cdot (-\tfrac{1}{64}) + 4 \cdot (-\tfrac{1}{64}) + 4 \cdot (-\tfrac{1}{16})\big) I_{\alpha\beta\gamma|\alpha\beta\gamma} = -\tfrac{3}{4} I_{\alpha\beta\gamma|\alpha\beta\gamma}$$

as the contribution to $|\nabla R|^2$ of the first summand in (19), apart from its contributions to L_1.

For the second summand in (19), we compute that $\langle(\nabla_X R)(W,Y)U,V\rangle$ is the sum of

$$\begin{array}{lrl}
\text{(a')} & -\langle\nabla_X\nabla_W\nabla_Y U,V\rangle = & 0,\\
\text{(b')} & \langle\nabla_X\nabla_Y\nabla_W U,V\rangle = & -\frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}Y,U\rangle\langle j_{Z_\beta}X,V\rangle,\\
\text{(c')} & \langle\nabla_X\nabla_{[W,Y]}U,V\rangle = & 0,\\
\text{(d')} & \langle\nabla_{\nabla_X W}\nabla_Y U,V\rangle = & \frac{1}{8}\sum_\beta\langle j_{Z_\beta} j_W X,V\rangle\langle j_{Z_\beta}Y,U\rangle,\\
\text{(e')} & -\langle\nabla_Y\nabla_{\nabla_X W}U,V\rangle = & -\frac{1}{8}\sum_\beta\langle j_{Z_\beta} j_W X,U\rangle\langle j_{Z_\beta}Y,V\rangle,\\
\text{(f')} & -\langle\nabla_{[\nabla_X W,Y]}U,V\rangle = & -\frac{1}{4}\sum_\beta\langle j_{Z_\beta} j_W X,Y\rangle\langle j_{Z_\beta}U,V\rangle,\\
\text{(g')} & \langle\nabla_W\nabla_{\nabla_X Y}U,V\rangle = & \frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}U,V\rangle\langle j_{Z_\beta}X,Y\rangle,\\
\text{(h')} & -\langle\nabla_{\nabla_X Y}\nabla_W U,V\rangle = & -\frac{1}{8}\sum_\beta\langle j_{Z_\beta} j_W U,V\rangle\langle j_{Z_\beta}X,Y\rangle,\\
\text{(i')} & -\langle\nabla_{[W,\nabla_X Y]}U,V\rangle = & 0,\\
\text{(j')} & \langle\nabla_W\nabla_Y\nabla_X U,V\rangle = & \frac{1}{8}\sum_\beta\langle j_W j_{Z_\beta}Y,V\rangle\langle j_{Z_\beta}X,U\rangle,\\
\text{(k')} & -\langle\nabla_Y\nabla_W\nabla_X U,V\rangle = & 0,\\
\text{(l')} & -\langle\nabla_{[W,Y]}\nabla_X U,V\rangle = & 0.
\end{array}$$

Similarly as above, the only pairings which do not just contribute to L_1 now are

$$(b') \longleftrightarrow (d'), \quad (e') \longleftrightarrow (j'), \quad (f') \longleftrightarrow ((g') \text{ or } (h')).$$

Again, each of these pairings gives a negative multiple of $I_{\alpha\beta\gamma|\alpha\beta\gamma}$. All in all, we obtain

$$4 \cdot 2 \cdot \big(-\tfrac{1}{64} - \tfrac{1}{64} + 2 \cdot (-\tfrac{1}{32})\big) I_{\alpha\beta\gamma|\alpha\beta\gamma} = -\tfrac{3}{4} I_{\alpha\beta\gamma|\alpha\beta\gamma}$$

as the contribution to $|\nabla R|^2$ of the second summand in (19), apart from its contributions to L_1. The statement now follows by $-\frac{3}{4} - \frac{3}{4} = -\frac{3}{2}$.

(ii) The various contributions

$$\sum\langle R(A,B)C,D\rangle\langle R(C,D)E,F\rangle\langle R(E,F)A,B\rangle \tag{22}$$

to $\hat{R}$, where each of A,B,C,D,E,F runs through either the orthonormal basis $\{X_1,\dots,X_m\}$ of $\mathfrak{v}$ or the orthonormal basis $\{Z_1,\dots,Z_r\}$ of $\mathfrak{z}$, can by Lemma 4.1(ii) be nonzero only in the cases where each of the tuples

$$(A,B,C,D), \quad (C,D,E,F), \quad (E,F,A,B) \tag{23}$$

contains either two or four vectors from $\mathfrak{v}$.

If each of them contains exactly two vectors from $\mathfrak{v}$, then each summand in (22) is, again by Lemma 4.1(ii), a linear combination of products of three terms of the form $\langle j_{Z_{\alpha_1}} j_{Z_{\alpha_2}} X_{\ell_1}, X_{\ell_2}\rangle$. The sum in (22) will thus be a linear combination of sums of the type

$$\sum\langle j_{Z_{\alpha_{s_1}}} j_{Z_{\alpha_{s_2}}} X_{\ell_{u_1}}, X_{\ell_{u_2}}\rangle\langle j_{Z_{\alpha_{s_3}}} j_{Z_{\alpha_{s_4}}} X_{\ell_{u_3}}, X_{\ell_{u_4}}\rangle\langle j_{Z_{\alpha_{s_5}}} j_{Z_{\alpha_{s_6}}} X_{\ell_{u_5}}, X_{\ell_{u_6}}\rangle,$$

with each s_i and each u_j occurring exactly twice. Here, summation over equal pairs of u_j will obviously always lead to invariants $I_{k_1\dots k_\lambda|\dots|k_\mu\dots k_6}$ in which all subtuples are of even length (6, or 4 and 2, or three times 2). Hence, such sums will contribute only to the term L_2 from the assertion.

So let at least one of the tuples from (23) consists of vectors in $\mathfrak{v}$. If $A, B, C, D \in \mathfrak{v}$ then either E, F must both be in $\mathfrak{v}$ or both in $\mathfrak{z}$. Therefore, the contributions of sums as in (22) where at least one of the tuples from (23) consists of vectors in $\mathfrak{v}$ is equal to

$$\sum\langle R(X,Y)U,V\rangle\langle R(U,V)S,T\rangle\langle R(S,T)X,Y\rangle + 3\sum\langle R(X,Y)U,V\rangle\langle R(U,V)Z,W\rangle\langle R(Z,W)X,Y\rangle, \tag{24}$$

where the first sum is taken over $X, Y, U, V, S, T \in \{X_1, \ldots, X_m\}$ and the second sum over $X, Y, U, V \in \{X_1, \ldots, X_m\}$ and $Z, W \in \{Z_1, \ldots, Z_r\}$. For the first term in (24), we note that

$$\begin{aligned} &\langle R(X,Y)U,V\rangle\langle R(U,V)S,T\rangle\langle R(S,T)X,Y\rangle = \\ &\textstyle\sum_{\alpha,\beta,\gamma=1}^r \big(\frac14\langle j_{Z_\alpha}Y,U\rangle\langle j_{Z_\alpha}X,V\rangle - \frac14\langle j_{Z_\alpha}X,U\rangle\langle j_{Z_\alpha}Y,V\rangle - \frac12\langle j_{Z_\alpha}X,Y\rangle\langle j_{Z_\alpha}U,V\rangle\big) \\ &\qquad\cdot\big(\tfrac14\langle j_{Z_\beta}V,S\rangle\langle j_{Z_\beta}U,T\rangle - \tfrac14\langle j_{Z_\beta}U,S\rangle\langle j_{Z_\beta}V,T\rangle - \tfrac12\langle j_{Z_\beta}U,V\rangle\langle j_{Z_\beta}S,T\rangle\big) \\ &\qquad\cdot\big(\tfrac14\langle j_{Z_\gamma}T,X\rangle\langle j_{Z_\gamma}S,Y\rangle - \tfrac14\langle j_{Z_\gamma}S,X\rangle\langle j_{Z_\gamma}T,Y\rangle - \tfrac12\langle j_{Z_\gamma}S,T\rangle\langle j_{Z_\gamma}X,Y\rangle\big) \end{aligned} \tag{25}$$

for $X, Y, U, V, S, T \in \mathfrak{v}$. For $a, b, c \in \{1, 2, 3\}$ denote by $(a * b * c)$ the sum over α, β, γ of the a-th summand in the first line, the b-th summand in the second, and the c-th summand of third line of (25). Then $(3 * 3 * 3)$ obviously yields, after summation over $X, Y, U, V, S, T \in \{X_1, \ldots, X_m\}$, a multiple of $I_{\alpha\gamma|\beta\gamma|\alpha\beta}$, and thus contributes only to L_2. Five of the other $(a * b * c)$ (for example, $(1 * 1 * 1)$) lead to multiples of certain $I_{s_1\ldots s_6}$ in which the only subtuple is of length six (the reason being, just as we noted in the proof of (i), that there is no way to group the six factors into subsets which would not be linked to each other by the occurrence any common vectors from $\{X, Y, U, V, S, T\}$); these again contribute only to L_2. The only products which instead lead to a multiple of $I_{\alpha\beta\gamma|\alpha\beta\gamma}$ are $(1 * 1 * 2)$, $(1 * 2 * 1)$, $(2 * 1 * 1)$, and $(2 * 2 * 2)$. For example,

$$(1 * 1 * 2) = -\tfrac{1}{64}\textstyle\sum_{\alpha,\beta,\gamma}\langle j_{Z_\alpha}Y,U\rangle\langle j_{Z_\beta}U,T\rangle\langle j_{Z_\gamma}T,Y\rangle\langle j_{Z_\alpha}X,V\rangle\langle j_{Z_\beta}V,S\rangle\langle j_{Z_\gamma}S,X\rangle$$

which after summation gives $-\frac{1}{64}\sum_{\alpha,\beta,\gamma}\mathrm{Tr}(j_{Z_\alpha}j_{Z_\beta}j_{Z_\gamma})\mathrm{Tr}(j_{Z_\alpha}j_{Z_\beta}j_{Z_\gamma}) = -\frac{1}{64}I_{\alpha\beta\gamma|\alpha\beta\gamma}$. The result is the same for each of the three other products just mentioned. So we obtain

$$4 \cdot (-\tfrac{1}{64})I_{\alpha\beta\gamma|\alpha\beta\gamma} = -\tfrac{1}{16}I_{\alpha\beta\gamma|\alpha\beta\gamma}$$

as the contribution to $\hat{R}$ of the first summand in (24), apart from its contributions to L_2.

For the second summand in (24), we compute

$$\begin{aligned} &\langle R(X,Y)U,V\rangle\langle R(U,V)Z,W\rangle\langle R(Z,W)X,Y\rangle = \\ &\textstyle\sum_{\alpha=1}^r \big(\frac14\langle j_{Z_\alpha}Y,U\rangle\langle j_{Z_\alpha}X,V\rangle - \frac14\langle j_{Z_\alpha}X,U\rangle\langle j_{Z_\alpha}Y,V\rangle - \frac12\langle j_{Z_\alpha}X,Y\rangle\langle j_{Z_\alpha}U,V\rangle\big) \\ &\qquad\cdot\tfrac{1}{16}\langle[j_Z,j_W]U,V\rangle\langle[j_Z,j_W]X,Y\rangle. \end{aligned} \tag{26}$$

The first two summands from the first line, multiplied with the factors from the second line, will, after summation, yields multiples of certain $I_{s_1\ldots s_6}$ in which the only subtuple is of length six; this gives a contribution to L_2. The remaining term is

$$-\tfrac{1}{32}\textstyle\sum_\alpha\langle j_{Z_\alpha}X,Y\rangle\langle j_{Z_\alpha}U,V\rangle\langle[j_Z,j_W]U,V\rangle\langle[j_Z,j_W]X,Y\rangle$$

which after summation over X, Y, U, V gives $-\frac{1}{32}\sum_\alpha(\mathrm{Tr}(j_{Z_\alpha}[j_Z,j_W]))^2$; using skew-symmetry of the maps involved, this simplifies to $-\frac18\sum_\alpha(\mathrm{Tr}(j_{Z_\alpha}j_Zj_W))^2$. Summation over $Z, W \in \{Z_1, \ldots, Z_r\}$ thus gives $-\frac18 I_{\alpha\beta\gamma|\alpha\beta\gamma}$. Hence, we obtain

$$3 \cdot (-\tfrac18)I_{\alpha\beta\gamma|\alpha\beta\gamma}$$

as the contribution to $\hat{R}$ of the second summand in (24), apart from its contributions to L_2. The statement now follows by $-\frac{1}{16} - \frac38 = -\frac{7}{16}$.

(iii) Although it would be possible to prove (iii) directly, similarly to the above proofs for (i) and (ii), we prefer to use the results of (i), (ii) together with those from Lemma 4.7 and the integral relation from Proposition 2.1(iii). If $G(j)$ admits a compact quotient, then it follows from local homogeneity and Proposition 2.1(iii) that

$$\mathring{R} = -|\nabla \mathrm{ric}|^2 + \tfrac{1}{4}|\nabla R|^2 - \mathrm{Tr}(\mathrm{Ric}^3) + (*) + \tfrac{1}{2}(**) - \tfrac{1}{4}\hat{R}.$$

From Lemma 4.1(iii) one easily derives $\mathrm{Tr}(\mathrm{Ric}^3) = \frac{1}{8}I_{\alpha\alpha\beta\beta\gamma\gamma} - \frac{1}{64}I_{\alpha\beta|\beta\gamma|\gamma\alpha}$; thus, $\mathrm{Tr}(\mathrm{Ric}^3)$ contributes to L_3 only. Now by (i), (ii) and Lemma 4.7, the right hand side is indeed of the form

$$\tfrac{1}{4}\cdot(-\tfrac{3}{2})I_{\alpha\beta\gamma|\alpha\beta\gamma} - \tfrac{1}{4}\cdot(-\tfrac{7}{16})I_{\alpha\beta\gamma|\alpha\beta\gamma} + L_3 = -\tfrac{17}{64}I_{\alpha\beta\gamma|\alpha\beta\gamma} + L_3,$$

where L_3 is a linear combination of invariants in which only subtuples of even length occur. So we have proved the statement of (iii) in the case that $G(j)$ admits a compact quotient.

The statement in the general case now follows by continuity. In fact, any $G(j)$ for which j is a rational map w.r.t. the standard rational structures on $\mathfrak{z} = \mathbb{R}^r$ and $\mathfrak{so}(\mathfrak{v}) = \mathfrak{so}(m)$ does admit a compact quotient, and the rational maps are dense in the space of all linear maps $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$. □

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