SPECTRAL ISOLATION OF BI-INVARIANT METRICS ON COMPACT LIE GROUPS

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Abstract. We show that a bi-invariant metric on a compact connected Lie group G is spectrally isolated within the class of left-invariant metrics. In fact, we prove that given a bi-invariant metric g_0 on G there is a positive integer N such that, within a neighborhood of g_0 in the class of left-invariant metrics of at most the same volume, g_0 is uniquely determined by the first N distinct non-zero eigenvalues of its Laplacian (ignoring multiplicities). In the case where G is simple, N can be chosen to be two.

RÉSUMÉ. Soit G un groupe de Lie compact et connexe, et soit g_0 une métrique bi-invariante sur G. On démontre que g_0 est isolée spectralement dans la classe des métriques invariantes à gauche: Plus précisément, il existe un entier positif N tel que, dans un voisinage de g_0 dans la classe des métriques invariantes à gauche et de volume égal ou inférieur à celui de g_0, la métrique g_0 est determinée de manière unique par les N premières valeurs propres strictement positives de son Laplacien (sans multiplicités). Si G est simple, on peut choisir $N = 2$.

1. Introduction

Given a connected closed Riemannian manifold (M, g) its **spectrum**, denoted $\mathrm{Spec}(M, g)$, is defined to be the sequence of eigenvalues, counted with multiplicities, of the associated Laplacian Δ acting on smooth functions. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral** if their spectra (counting multiplicities) agree. Inverse spectral geometry is the study of the extent to which geometric properties of a Riemannian manifold (M, g) are determined by its spectrum.

A long standing question is whether very special Riemannian manifolds – e.g., manifolds of constant curvature or symmetric spaces – may be spectrally distinguishable from other Riemannian manifolds. The strongest results are for constant curvature: Tanno showed that a round sphere of dimension at most six is uniquely determined by its spectrum among all *orientable* Riemannian manifolds [T1], and in arbitrary dimensions round metrics on spheres are at least spectrally isolated among all Riemannian metrics on spheres [T2]. In contrast,

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the first and third author have shown that in dimension 7 and higher there are isospectrally deformable metrics on spheres arbitrarily close to the standard metric [G, Sch2]. Hence, for geometries that are in some sense extremely close to being "nice" or "ideal", spectral uniqueness can fail profoundly.

While many examples exist of isospectral flat manifolds, Kuwabara [K] has proven that flat metrics are at least spectrally isolated within the space of all metrics. However, even the question of whether a flat torus may be isospectral to a non-flat manifold remains open! One cannot resolve this question by appealing to the heat invariants of a Riemannian manifold as there are examples of non-flat manifolds all of whose heat invariants vanish [Pa].

Outside of the setting of constant curvature, we are not aware of any examples of Riemannian metrics that are known to be spectrally isolated among arbitrary Riemannian metrics. Various results show that *within certain classes* of Riemannian metrics, isospectral sets are finite. Even here, many of the results involve constant curvature. For example, isospectral sets of flat tori are finite (see [W] or unpublished work of Kneser) as are isospectral sets of Riemann surfaces [McK]. As for the class of symmetric spaces, the first and third author have recently shown that any collection of mutually isospectral compact symmetric spaces is finite [GS].

This article is motivated by the question of whether one can tell from the spectrum whether a compact Riemannian manifold is symmetric. Given that this question has resisted solution even in the case of spheres, it does not appear tractable at this time to compare the spectrum of a symmetric space with that of a completely arbitrary Riemannian manifold. Instead, we ask whether symmetric spaces can be spectrally distinguished within a larger class of homogeneous Riemannian manifolds.

The compact symmetric spaces fall into two types; the type we consider are those given by bi-invariant Riemannian metrics on compact (not necessarily semisimple) Lie groups. We compare the spectrum of each such symmetric space with the spectra of arbitrary left-invariant metrics on the Lie group. As a departure point we note that the second author showed that there are no non-trivial continuous isospectral deformations of a bi-invariant metric within the class of left-invariant metrics on a compact Lie group [Sch1]. This prompts one to ask whether a bi-invariant metric on a compact Lie group G is spectrally isolated within the class of left-invariant metrics. We give an affirmative answer; in fact we obtain a significantly stronger result.

Let $\mathcal{M}_{\mathit{left}}(G)$ denote the set of left-invariant metrics on a Lie group G. This set can be canonically identified with the set of Euclidean inner products on the Lie algebra of G. The latter set can in turn be identified, after some choice of basis, with the set of positive definite symmetric $(m \times m)$-matrices, where m is the dimension of G. The canonical topology on this set of matrices gives rise to a topology on $\mathcal{M}_{\mathit{left}}(G)$ (independent of the choice of basis), and it is this topology that we consider. We call a left-invariant metric g_0 on G **spectrally isolated** in $\mathcal{M}_{\mathit{left}}(G)$ if it is locally spectrally determined within $\mathcal{M}_{\mathit{left}}(G)$; that is, there is a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ such that no $g \in \mathcal{U} \setminus \{g_0\}$ is isospectral to g_0. We prove the following:

Result. *Let g_0 be a bi-invariant metric on a compact Lie group G.*

(1) *There is a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ and a positive integer N such that if g is any metric in $\mathcal{U}$ with $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ and whose first N distinct eigenvalues (ignoring multiplicities) agree with those of g_0, then g is isometric to g_0. (See Theorem 2.3.)*
(2) *The metric g_0 is spectrally isolated in $\mathcal{M}_{left}(G)$. (See Corollary 2.4.)*
(3) *Let $\alpha_1 < \alpha_2 < \alpha_3$ be three distinct consecutive eigenvalues (ignoring multiplicities) of the associated Laplacian Δ_0. If G is* simple*, then there exists a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ such that if $g \in \mathcal{U}$ satisfies $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ and the condition that three consecutive distinct eigenvalues of Δ_g agree with α_1, α_2 and α_3 (again ignoring multiplicities), then $g = g_0$. In particular, letting $\alpha_1 = 0$, then the first two distinct non-zero eigenvalues along with the volume bound distinguish g_0 withing $\mathcal{U}$. (See Theorem 3.3.)*

The second result above is immediate from the first since the spectrum of a compact Riemannian manifold determines its volume. Hence, within the class of left-invariant metrics on a compact Lie group G, any metric $g \neq g_0$ that is isospectral to a bi-invariant metric g_0 must be sufficiently far away from g_0. In contrast, we note that the second author exhibited the first examples of continuous isospectral families of left-invariant metrics on compact simple Lie groups [Sch1]; see also [Pr].

In light of the fact that most examples of isospectral manifolds in the literature exploit metrics with "large" symmetry groups, the spectral isolation results above lend strong support to the conjecture that a bi-invariant metric on a compact Lie group G is spectrally isolated within the class of *all* metrics on G. In fact, these results lead one to speculate on whether a bi-invariant metric on a *semisimple* Lie group is uniquely determined by its spectrum.[1] In Section 3 we present strong evidence that the bi-invariant metric on a compact simple Lie group is uniquely determined by its spectrum within the class of left-invaraint metrics. In particular, we show the following.

Result. *Let g_0 be a bi-invariant metric on a compact simple Lie group G, and let $g \neq g_0$ be a left-invariant metric on G, which is isospectral to g_0. Then there is a constant $C \equiv C(g) > 1$, such that for every subspace $V \leq L^2(G)$ that is invariant under the right regular action of G, we have*

$$\frac{\mathrm{Tr}(\Delta_g \restriction V)}{\mathrm{Tr}(\Delta_0 \restriction V)} \equiv C > 1.$$

(See Proposition 3.1 for a more precise statement.)

This implies that if $g \neq g_0 \in \mathcal{M}_{left}(G)$ is isospectral to the bi-invariant metric g_0, then a very special rearrangement of the eigenvalues must occur.

[1] We must restrict our attention to semisimple Lie groups due to the existence of nontrivial pairs of isospectral flat tori (e.g., [M] and [CS]).

The outline of this paper is as follows. In Section 2 we establish the main results for bi-invariant metrics on arbitrary compact Lie groups. In Section 3 we restrict our attention to compact simple Lie groups to obtain the stronger results in this setting.

2. Proof of the main result

Following [LMNR] we introduce the notion of **eigenvalue equivalence**, which is weaker than that of isospectrality. The same notion was introduced earlier by Z.I. Szabo [Sz, p. 212], who referred to it as isotonality. We also define a notion of partial eigenvalue equivalence.

2.1. Definition. Given a compact Riemannian manifold (M, g), we define the **eigenvalue set** of (M, g) to be the ordered collection of eigenvalues of the associated Laplace operator Δ_g on functions on M, *not* counting multiplicities. We will say that two compact Riemannian manifolds are **eigenvalue equivalent** if their eigenvalue sets coincide. For N a positive integer, we will say that two compact Riemannian manifolds are **eigenvalue equivalent up to level** N if the first N elements of their eigenvalue sets coincide.

2.2. Lemma. *Let G be a compact Lie group, and let g_0 be a bi-invariant metric on G with associated Laplacian Δ_0. Let $V \leq L^2(G)$ be a finite dimensional subspace which is invariant under the right-regular representation of G on $L^2(G)$. Then there exists a positive integer N and a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ such that if $g \in \mathcal{U}$ is eigenvalue equivalent to g_0 up to level N, then*

$$\Delta_g \restriction V = \Delta_0 \restriction V.$$

Proof. First note that V, being a finite dimensional subspace of $L^2(G)$ which is invariant under the right-regular representation, contains only smooth functions. Moreover, V is a direct sum of finitely many irreducible representations of G; it is therefore enough to prove the result in the case that V is irreducible. For any $g \in \mathcal{M}_{left}(G)$ and any g-orthonormal basis $\{Y_1, \ldots, Y_n\}$ of the Lie algebra of G, the associated Laplace operator on smooth functions on G is given by

$$\Delta_g = -\sum_{j=1}^{n} (\rho_* Y_j)^2,$$

where $\rho : G \to U(L^2(G))$ is the right-regular representation of G. Thus, V is invariant under Δ_g. Since g_0 is bi-invariant, right translations in G are g_0-isometries; hence $\Delta_0 : V \to V$ commutes with the action of G on V. Irreducibility of V implies by Schur's Lemma that $\Delta_0 \restriction V$ is a multiple of the identity, say $\Delta_0 \restriction V = \lambda \, \mathrm{Id}_V$. We may choose $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon) \cap \mathrm{Spec}(\Delta_0) = \{\lambda\}$. Choose N large enough so that the Nth element of the eigenvalue set is greater than λ (and hence greater than $\lambda + \epsilon$). The hermitian operators $\Delta_g \restriction V$ on the finite dimensional vector space V depend continuously on g. Therefore, their eigenvalues also depend continuously on g. Consequently, there is a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ such that for each $g \in \mathcal{U}$ the eigenvalues of $\Delta_g \restriction V$ must lie in $(\lambda - \epsilon, \lambda + \epsilon)$. If $g \in \mathcal{U}$ is eigenvalue equivalent to g_0 up to level N, it follows that $\Delta \restriction V = \Delta_0 \restriction V = \lambda \, \mathrm{Id}_V$. $\square$

We now establish the spectral isolation of bi-invariant metrics on compact connected Lie groups. In fact, we prove a little more; namely, we replace the isospectrality condition by the much weaker condition of partial eigenvalue equivalence together with an upper volume bound.

2.3. THEOREM. *Let g_0 be a bi-invariant metric on a compact connected Lie group G. Then there is a positive integer N, depending only on g_0, and a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ such that if $g \in \mathcal{U}$ is eigenvalue equivalent to g_0 up to order N and satisfies* $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$, *then $g = g_0$.*

2.4. COROLLARY. *Let g_0 be a bi-invariant metric on a compact connected Lie group G. Then g_0 is spectrally isolated in $\mathcal{M}_{left}(G)$.*

The corollary follows from the theorem by the fact that isospectrality implies eigenvalue equivalence and equality of volumes; in fact, the volume is the first of the classical heat invariants.

Proof of Theorem 2.3. We have $G = G_{ss}T$ where G_{ss} is semisimple, T is a torus and $G_{ss} \cap T$ is finite. The Lie algebra of G is a direct sum $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{t}$, where $\mathfrak{g}_{ss}$ and $\mathfrak{t}$ are the Lie algebras of G_{ss} and T. In particular, G_{ss} and T commute, and $\mathfrak{g}_{ss}$ is g_0-orthogonal to $\mathfrak{t}$.

We first claim that there exists a positive integer N' and a neighborhood $\mathcal{U}'$ of g_0 in $\mathcal{M}_{left}(G)$ such that if $g \in \mathcal{U}'$ is eigenvalue equivalent to g_0 up to level N', then g and g_0, viewed as inner products on $\mathfrak{g}$, induce the same inner product on $\mathfrak{g}/\mathfrak{g}_{ss}$. By the inner product induced by g we mean the one obtained by identifying $\mathfrak{g}/\mathfrak{g}_{ss}$ with the g-orthogonal complement of $\mathfrak{g}_{ss}$ in $\mathfrak{g}$. To prove the claim, note that the Lie group $\overline{T} := G/G_{ss} \cong T/(G_{ss} \cap T)$ is a torus which is finitely covered by T. In particular, the Lie algebra of $\overline{T}$ is canonically identified with $\mathfrak{t}$. Let $p : G \to \overline{T}$ be the homomorphic projection. Given $g \in \mathcal{M}_{left}(G)$, denote by $\bar{g}$ the induced (flat) metric on $\overline{T}$ (i.e., the metric for which $p : (G, g) \to (\overline{T}, \bar{g})$ becomes a Riemannian submersion). Let $\mathcal{L}$ be the lattice in $\mathfrak{t}$ which is the kernel of the Lie group exponential map $\mathfrak{t} \to \overline{T}$, and let $\mathcal{L}^* \subset \mathfrak{t}^*$ be the dual lattice. For $\mu \in \mathcal{L}^*$, denote by $\|\mu\|_{\bar{g}}$ the norm of μ with respect to the dual inner product on $\mathfrak{t}^*$. Let $\nu_1, \ldots, \nu_k$ be a basis of $\mathcal{L}^*$, where $k = \dim(T)$. Write $L := k + \binom{k}{2}$, and let $\{\mu_1, \ldots, \mu_L\}$ be the set containing the vectors ν_i as well as the $\nu_i + \nu_j$ for $i \neq j$. Note that, by polarization, the norm $\| \,.\, \|_{\bar{g}}$ on $\mathfrak{t}^*$ – and hence $\bar{g}$ itself – is uniquely determined by the norms of the vectors $\mu_1, \ldots, \mu_L$. For each $s \in \{1, \ldots, L\}$ let $\bar{f}_s : \overline{T} \to U(1)$ (where $U(1)$ is the unitary group of unit complex numbers) be the associated character of $\overline{T}$. Then $\Delta_{\bar{g}} \bar{f}_s = 4\pi^2 \|\mu_s\|^2_{\bar{g}} \bar{f}_s$. Now $f_s := \bar{f}_s \circ p$ is a character on G. Since the Riemannian submersion $p : G \to \overline{T}$ has minimal fibers, f_s is an eigenfunction of Δ_g with eigenvalue $4\pi^2 \|\mu_s\|^2_{\bar{g}}$ for each $s = 1, \ldots, L$. (One can also verify this fact by direct computation.)

The one-dimensional space $F_s \leq L^2(G)$ spanned by the character f_s is invariant under the right-regular representation. Let N' be a positive integer and $\mathcal{U}'$ be a neighborhood of g_0 in $\mathcal{M}_{left}(G)$ satisfying the property from Lemma 2.2 with respect to $F_1 \oplus \ldots \oplus F_L$, and let $g \in \mathcal{U}'$ be eigenvalue equivalent to g_0 up to level N'. Then we must have $\|\mu_s\|_{\bar{g}} = \|\mu_s\|_{\bar{g}_0}$ for each $s = 1, \ldots, L$. As remarked above, this implies $\bar{g} = \bar{g}_0$. The claim follows.

In the case of the bi-invariant metric g_0, the metric $\bar{g}_0$ on $\mathfrak{t}$ coincides with the restriction of the inner product g_0 to $\mathfrak{t}$ since $\mathfrak{g}_{ss}$ and $\mathfrak{t}$ are g_0-orthogonal. However, for more general g, one has only that the differential $p_* : \mathfrak{g} \to \mathfrak{t}$ of the projection $p : G \to \overline{T}$ restricts to an inner product space isometry $p_* : (\mathfrak{g}_{ss}^{\perp_g}, g) \to (\mathfrak{t}, \bar{g})$. In particular, if $g \in \mathcal{U}'$, then it follows from the claim that

$$\text{the projection from } (\mathfrak{g}_{ss}^{\perp_g}, g) \text{ to } (\mathfrak{t}, g_0) \text{ along } \mathfrak{g}_{ss} \text{ is an isometry.} \tag{2.5}$$

For the remaining part of the argument, let $\mathfrak{g}_{ss} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ be the decomposition of $\mathfrak{g}_{ss}$ into simple Lie subalgebras. The adjoint representation of G on $\mathfrak{g}$, restricted to the invariant subspace $\mathfrak{g}_\ell$, is an irreducible representation of G for each $\ell = 1, \ldots, r$. Note that for $\ell \neq \ell'$ these representations of G are inequivalent even in the case when $\mathfrak{g}_\ell$ and $\mathfrak{g}_{\ell'}$ happen to be isomorphic as Lie algebras. By the Peter-Weyl Theorem, every irreducible representation of G occurs in the right-regular representation of G on $L^2(G)$ (with multiplicity equal to its dimension). Thus, for each $\ell = 1, \ldots, r$ we can choose a corresponding irreducible subspace $V_\ell \leq L^2(G)$, and the action of G on the subspace $V_1 \oplus \ldots \oplus V_r$ of $L^2(G)$ will then be equivalent to the adjoint representation of G acting on $\mathfrak{g}_{ss}$.

Let N'' be a positive integer and $\mathcal{U}''$ be a neighborhood of g_0 in $\mathcal{M}_{\mathit{left}}(G)$ satisfying the property from Lemma 2.2 with respect to $V_1 \oplus \ldots \oplus V_r$. We are going to show that $N := \max\{N', N''\}$ and $\mathcal{U} := \mathcal{U}' \cap \mathcal{U}''$ satisfy the property stated in the Theorem.

If g is any left-invariant metric on G and $\{U_1, \ldots, U_m\}$ is a g-orthonormal basis of $\mathfrak{g}$, then

$$\operatorname{Tr}(\Delta_g \restriction V_\ell) = -\sum_{j=1}^{m} \operatorname{Tr}\big((\operatorname{ad}_{U_j} \restriction \mathfrak{g}_\ell)^2\big). \tag{2.6}$$

Since g_0 is bi-invariant, $\mathfrak{g}_\ell$ is g_0-orthogonal to $\mathfrak{g}_{\ell'}$ for $\ell \neq \ell'$. Let n_ℓ denote the dimension of $\mathfrak{g}_\ell$, and let $n = n_1 + \ldots + n_r$ be the dimension of $\mathfrak{g}_{ss}$. Choose a g_0-orthonormal basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}_{ss}$ such that the first n_1 elements lie in $\mathfrak{g}_1$, the next n_2 elements lie in $\mathfrak{g}_2$, etc. Complete to a g_0-orthonormal basis $\mathcal{B}_0 = \{X_1, \ldots, X_n, Z_1, \ldots, Z_k\}$, where (necessarily) $Z_1, \ldots, Z_k \in \mathfrak{t}$.

Let g be a metric in $\mathcal{U}$ which is eigenvalue equivalent to g_0 up to level N and satisfies $\operatorname{vol}(g) \leq \operatorname{vol}(g_0)$. Since $g \in \mathcal{U}'$ and $N \geq N'$, statement (2.5) holds and thus there exist elements $W_i \in \mathfrak{g}_{ss}$ such that $\{Z_1 + W_1, \ldots, Z_k + W_k\}$ is a g-orthonormal basis of $\mathfrak{g}_{ss}^{\perp_g}$. Complete to a g-orthonormal basis $\mathcal{B} = \{Y_1, \ldots, Y_n, Z_1 + W_1, \ldots, Z_k + W_k\}$ of $\mathfrak{g}$ with $Y_1, \ldots, Y_k \in \mathfrak{g}_{ss}$. The change of basis matrix which expresses the elements of $\mathcal{B}$ in terms of $\mathcal{B}_0$ is given by

$$\begin{bmatrix} A & R \\ 0 & I_k \end{bmatrix},$$

where $A = (a_{ij}) \in \operatorname{Mat}_{n \times n}(\mathbb{R})$, $R = (r_{ij}) \in \operatorname{Mat}_{n \times k}(\mathbb{R})$ and I_k is the $k \times k$ identity matrix. Hence, $Y_j = \sum_{i=1}^n a_{ij} X_i$ for $j = 1, \ldots, n$ and $W_s = \sum_{i=1}^n r_{is} X_i$ for $s = 1, \ldots, k$. The condition $\operatorname{vol}(g) \leq \operatorname{vol}(g_0)$ implies that $|\det(A)| \geq 1$. Without loss of generality we assume $\det(A) > 0$ and hence $\det(A) \geq 1$.

Since g_0 is bi-invariant, there exist numbers $c_\ell > 0$ for $\ell = 1, \ldots, r$ such that the restriction of g_0 to $\mathfrak{g}_\ell$ coincides with $-c_\ell B_\ell$, where B_ℓ is the Killing form of $\mathfrak{g}_\ell$ (which in turn coincides with the restriction to $\mathfrak{g}_\ell$ of the Killing form $B : (X, Y) \mapsto \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ of $\mathfrak{g}$). In particular, by equation (2.6), we have

$$\mathrm{Tr}(\Delta_0 \restriction V_\ell) = \frac{n_\ell}{c_\ell}.$$

Since $g \in \mathcal{U} \subset \mathcal{U}''$ and $N \geq N''$, we have $\Delta_g \restriction V_\ell = \Delta_0 \restriction V_\ell$ for each $\ell = 1, \ldots, r$. In particular, for $\ell = 1$:

$$\begin{aligned}
\frac{n_1}{c_1} &= \mathrm{Tr}(\Delta_0 \restriction V_1) = \mathrm{Tr}(\Delta_g \restriction V_1) = -\sum_{j=1}^{n} \mathrm{Tr}\big((\mathrm{ad}_{Y_j} \restriction \mathfrak{g}_1)^2\big) - \sum_{s=1}^{k} \mathrm{Tr}\big((\mathrm{ad}_{Z_s+W_s} \restriction \mathfrak{g}_1)^2\big) \\
&= -\sum_{i=1}^{n_1}\sum_{j=1}^{n} a_{ij}^2 \,\mathrm{Tr}\big((\mathrm{ad}_{X_i} \restriction \mathfrak{g}_1)^2\big) - \sum_{i=1}^{n_1}\sum_{s=1}^{k} r_{is}^2 \,\mathrm{Tr}\big((\mathrm{ad}_{X_i} \restriction \mathfrak{g}_1)^2\big) \\
&= \frac{1}{c_1}\sum_{i=1}^{n_1}\sum_{j=1}^{n} a_{ij}^2 + \frac{1}{c_1}\sum_{i=1}^{n_1}\sum_{s=1}^{k} r_{is}^2,
\end{aligned}$$

where the fourth equality holds because $\mathrm{ad}_X \restriction \mathfrak{g}_1 = 0$ for $X \in \mathfrak{g}_\ell$ with $\ell \neq 1$ and because $\{X_1, \ldots, X_{n_1}\}$ is orthonormal with respect to $-c_1 B_1$. We hence obtain $n_1 = \sum_{i=1}^{n_1}\sum_{j=1}^{n} a_{ij}^2 + \sum_{i=1}^{n_1}\sum_{s=1}^{k} r_{is}^2$. Summing over the analogous equations for $\ell = 1, \ldots, r$ we conclude that

$$n = \|A\|^2 + \|R\|^2,$$

where $\| \, . \, \|$ denotes the standard Euclidean norm of matrices viewed as points in the appropriate $\mathbb{R}^N$. However, n is the minimal value (in fact, the only critical value) of the function $\mathrm{SL}(n, \mathbb{R}) \ni C \mapsto \|C\|^2 \in \mathbb{R}$ and is attained precisely on $\mathrm{SO}(n)$. It thus follows from $\det(A) \geq 1$ that $A \in \mathrm{SO}(n)$ and $R = 0$; hence $g = g_0$. □

2.7. Remark. Some of the techniques used in this section are similar to those used by Urakawa in [U].

3. A stronger spectral isolation result for simple groups

In the proof of Lemma 2.2 it was observed that if g is a left-invariant metric on a compact Lie group G, with associated Laplacian Δ_g, then any subspace $V \leq L^2(G)$ that is invariant under the right-regular representation of G is also invariant under Δ_g. With this in mind we have the following result concerning the trace of the Laplacian on compact simple Lie groups.

3.1. Proposition. *Let g_0 be a bi-invariant metric on a compact simple Lie group G, and let $g \neq g_0$ be a left-invariant metric on G which satisfies* $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$. *Then there exists a constant $C = C(g) > 1$ such that*

$$\mathrm{Tr}(\Delta_g \restriction V) = C\, \mathrm{Tr}(\Delta_0 \restriction V)$$

for every finite dimensional subspace $V \leq L^2(G)$ which is invariant under the right-regular representation ρ of G and on which G acts nontrivially.

Proof. By rescaling g and g_0 we can assume without loss of generality that g_0 coincides with $-B$ on the Lie algebra $\mathfrak{g}$ of G, where B is the Killing form on $\mathfrak{g}$. Define $h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$h(X,Y) := -\operatorname{Tr}\big(((\rho_* X) \restriction V) \circ ((\rho_* Y) \restriction V)\big).$$

Obviously h is bilinear, symmetric, and Ad_G-invariant. The map $X \mapsto (\rho_* X) \restriction V$ is nonzero because V is not a trivial representation space of G. Since $\mathfrak{g}$ is simple, this map has trivial kernel, which implies that h is positive-definite; in particular, there exists some $c > 0$ such that $h = -cB$. We now proceed similarly as in the last part of the proof of Theorem 2.3: Let $\{X_1, \dots, X_n\}$ be a g_0-orthonormal basis and $\{Y_1, \dots, Y_n\}$ be a g-orthonormal basis of $\mathfrak{g}$, and define $A = (a_{ij})$ by $Y_j = \sum_{i=1}^n a_{ij} X_i$ for $j = 1, \dots, n$. By the volume condition, $|\det(A)| \geq 1$; we can assume $\det(A) \geq 1$. Moreover, $g \neq g_0$ implies $A \notin \mathrm{SO}(n)$ and therefore $\|A\|^2 > n$. Thus,

$$\begin{aligned}\operatorname{Tr}(\Delta_g \restriction V) &= \sum_{j=1}^n h(Y_j, Y_j) = -c \sum_{j=1}^n B(Y_j, Y_j) = -c \sum_{i,j=1}^n a_{ij}^2 B(X_i, X_i) = c\|A\|^2 \\ &> cn = \sum_{i=1}^n h(X_i, X_i) = \operatorname{Tr}(\Delta_0 \restriction V).\end{aligned}$$

The proposition follows with $C = \frac{\|A\|^2}{n}$. □

3.2. Remark. Since volume is a spectral invariant the previous proposition implies the following: Let g_0 be a bi-invariant metric on a compact simple Lie group G, and suppose there exists a left-invariant metric $g \neq g_0$ on G which is isospectral to g_0. Then $\operatorname{Tr}(\Delta_g \restriction V) > \operatorname{Tr}(\Delta_0 \restriction V)$ for any finite dimensional invariant subspace V of $L^2(G)$ on which G acts nontrivially. Thus Δ_g, although isospectral to Δ_0, must have greater trace than Δ_0 on every Δ_0-eigenspace (except for the eigenvalue 0), even on each irreducible subspace. This is not a priori a contradiction because some wild reordering of eigenvalues could occur to produce this situation. Nevertheless, this seems a strong indication in support of the conjecture that a bi-invariant metric on a compact simple Lie group is globally spectrally determined among left-invariant metrics.

3.3. Theorem. *Let g_0 be a bi-invariant metric on a compact simple Lie group G. Let $\alpha_1 < \alpha_2 < \alpha_3$ be three consecutive elements of the eigenvalue set of (G, g_0). (See Definition 2.1.) Then there exists a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{left}(G)$ (depending on α_1, α_2, α_3) such that if $g \in \mathcal{U}$ satisfies $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ and if α_1, α_2 and α_3 are also consecutive elements of the eigenvalue set of (G, g), then $g = g_0$.*

3.4. Corollary. *A bi-invariant metric on a compact simple Lie group is locally determined within the set of left-invariant metrics of at most the same volume by its first two distinct non-zero eigenvalues $0 < \lambda_1 < \lambda_2$ (ignoring multiplicities).*

The corollary follows from the theorem since $0, \lambda_1, \lambda_2$ are three consecutive elements of the eigenvalue set.

Proof of Theorem 3.3. Since Δ_0 commutes with right translations in G, the α_2-eigenspace V of Δ_0 is invariant under the right-regular representation. Note that V is finite dimensional. As remarked in the proof of Lemma 2.2, the eigenvalues of $\Delta_g \restriction V$ depend continuously on g. Thus there exists a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}_{\mathit{left}}(G)$ such that for any $g \in \mathcal{U}$ the eigenvalues of Δ_g on V are contained in the interval (α_1, α_3). Let $g \in \mathcal{U}$ be a metric which satisfies $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ and the condition that α_1, α_2 and α_3 are also consecutive eigenvalues of Δ_g. Then necessarily $\Delta_g \restriction V = \alpha_2 \, \mathrm{Id}_V = \Delta_0 \restriction V$. Finally, note that G acts nontrivially on V since $\alpha_2 \neq 0$. Proposition 3.1 now implies $g = g_0$. □

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