HU Berlin WS 2007/08

Problems for BMS Basic Course "Commutative Algebra"

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Hand in Oct 25th, after the 2nd lecture 4.45 p.m.

Please sign each sheet of paper with your name and student ID

1st Problem Set (30 points)

Problem 1 (10 pts)

Let A be a commutative ring with 1. Prove the following statements:

a) $\mathfrak{p} \subseteq A$ is a prime ideal iff $A/\mathfrak{p}$ is an integral domain.

b) $\mathfrak{m} \subseteq A$ is a maximal ideal iff $A/\mathfrak{m}$ is a field.

c) There exists a maximal ideal $\mathfrak{m} \subseteq A$ (use Zorn's lemma).

Problem 2 (10 pts)

Let A be a commutative ring with 1 and let $\mathfrak{a} \subseteq A$ be an ideal.
Denote the *nilradical* $\mathfrak{n}_A$ *of* A, resp. the *radical* $\mathfrak{r}(\mathfrak{a})$ *of* $\mathfrak{a}$, by

$$\mathfrak{n}_A := \{a \in A \mid \exists n \in \mathbb{N}_{>0} : \ a^n = 0\}, \text{ resp.}$$

$$\mathfrak{r}(\mathfrak{a}) := \{a \in A \mid \exists n \in \mathbb{N}_{>0}, \ a^n \in \mathfrak{a}\}.$$

a) Prove that $\mathfrak{n}_A$ is an ideal of A.

b) Show that $\mathfrak{r}(\mathfrak{a})$ is an ideal of A.

c) Denote by $\pi : A \longrightarrow A/\mathfrak{a}$ the canonical projection. Prove that $\mathfrak{r}(\mathfrak{a}) = \pi^{-1}(\mathfrak{n}_{A/\mathfrak{a}})$.

Problem 3 (10 pts)

Let A be a commutative ring with 1 and let $\mathfrak{a} \subseteq A$ be an ideal. Show the following properties of the radical:

a) $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

b) $\mathfrak{r}(\mathfrak{a} \cdot \mathfrak{b}) = \mathfrak{r}(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{r}(\mathfrak{a}) \cap \mathfrak{r}(\mathfrak{b})$.

c) $\mathfrak{r}(\mathfrak{a}) = (1)$ iff $\mathfrak{a} = (1)$.

d) $\mathfrak{r}(\mathfrak{a} + \mathfrak{b}) = \mathfrak{r}(\mathfrak{r}(\mathfrak{a}) + \mathfrak{r}(\mathfrak{b}))$.

e) $\mathfrak{r}(\mathfrak{p}^n) = \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(A)$, $n \in \mathbb{N}_{>0}$.