Problems for BMS Basic Course "Commutative Algebra"

Prof. Dr. J. Kramer

Hand in Jan 17th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

11th Problem Set (40 points)

Problem 1 (10 pts)

(a) Compute the primary decomposition of the ideal (X^2, XY) in the ring $k[X,Y]$, where k is a field.

(b) Compute the primary decomposition of the principal ideal (6) in the ring $\mathbb{Z}[\sqrt{-5}]$.

Problem 2 (10 pts)

Let k be a field. Show that the ideal $\mathfrak{a} = (X,Y) \subseteq k[X,Y]$ is contained in an infinite union of prime ideals, each being different from $\mathfrak{a}$.

Problem 3 (10 pts)

Let A be a commutative ring with 1 and $\mathfrak{a} \subseteq A$ an ideal. A prime ideal $\mathfrak{p}$ is said to be associated to $\mathfrak{a}$ if there exists an $x \in A$ such that $\mathfrak{p} = \mathfrak{r}(\mathfrak{a} : x)$.
Show that $\mathfrak{a}$ is primary iff there exists precisely one prime ideal of A that is associated to this ideal $\mathfrak{a}$.

Problem 4 (10 pts)

Let A be a commutative ring with 1, $X = \operatorname{Spec}(A)$ and $U \subseteq X$ an open subset. Given an A-module M, we define

$$\widetilde{M}(U) := \{\sigma : U \longrightarrow \coprod_{\mathfrak{p} \in \operatorname{Spec}(A)} M_{\mathfrak{p}} \,|\, \sigma \text{ is locally a fraction}\};$$

where "σ is locally a fraction" means that for each $\mathfrak{p} \in U$ there exists an open neighbourhood V of $\mathfrak{p}$ in U and $m \in M$, $a \in A$, such that for all $\mathfrak{q} \in V$, $a \notin \mathfrak{q}$ and $\sigma(\mathfrak{q}) = m/a \in M_{\mathfrak{q}}$.

(a) Show that $\widetilde{M}$ defines a sheaf on X.

(b) Show that for all $f \in A$, the homomorphism of A-modules

$$M_f \longrightarrow \widetilde{M}(D(f)),$$

given by the assignment

$$\frac{m}{f^k} \mapsto \left(\sigma : \mathfrak{p} \mapsto \frac{m}{f^k} \in M_{\mathfrak{p}}\right),$$

is well-defined and injective (actually, it is even an isomorphism).