Problems for BMS Basic Course "Commutative Algebra"

Prof. Dr. J. Kramer

Hand in Jan 31st, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

13th Problem Set (40 + 10 points)

Problem 1 (10 pts)

Let B be a commutative ring with 1 and A a subring of B. If for an element $b \in B$ there exists a *monic* polynomial $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in A[X]$ such that $p(b) = 0$, then b is called *integral* over A. The set $C := \{b \in B \mid b \text{ is integral over } A\}$ is called the *integral closure* of A in B.

Show that the integral closure of A in B is a subring of B containing A.

Problem 2 (10 pts)

If the integral closure of A in B is equal to B, we say that B is integral over A. If the integral closure of A in B is equal to A, we say that A is integrally closed in B.

Let $A \subseteq B \subseteq C$ be commutative rings with 1. Let B be integral over A and C integral over B. Conclude that C is integral over A.

Problem 3 (10 pts)

Let $K = \mathbb{Q}(\sqrt{D})$ ($D \in \mathbb{Z}$, $D \equiv 0, 1 \mod 4$) be a quadratic extension. Describe the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K explicitely (for instance by giving a $\mathbb{Z}$-basis of $\mathcal{O}_K$).

Problem 4 (10 + 10 pts)

Let A be the quotient ring $\mathbb{C}[X,Y]/(Y^2 - X^3)$.

(a) Show that A is isomorphic to the subring $\mathbb{C}[T^2, T^3]$ of $\mathbb{C}[T]$.

(b) Compute the integral closure of $\mathbb{C}[T^2, T^3]$ in $\mathbb{C}[T]$.

(c*) For each $\mathfrak{m} \in \mathrm{Max}(A)$, $\mathfrak{m}/\mathfrak{m}^2$ is a $A/\mathfrak{m}$-module (i.e., a $A/\mathfrak{m}$-vector space). Compute $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.