Problems for BMS Basic Course "Commutative Algebra"

Prof. Dr. J. Kramer

Hand in December 6th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

7th Problem Set (30 + 10 points)

Problem 1 (10 pts)

Let A be commutative ring with 1 and M a free A-module with basis $\{b_1, \ldots, b_n\}$.

(a) Let $m_j = \sum_{k=1}^{n} a_{j,k} \cdot b_k \in M$ $(j = 1, \ldots, n)$. Show that

$$m_1 \wedge \ldots \wedge m_n = \det\left((a_{j,k})_{1 \leq j,k \leq n}\right) b_1 \wedge \cdots \wedge b_n.$$

Use this to prove that there is an A-module isomorphism

$$\bigwedge^n M \cong A.$$

(b) In general, show that for $1 \leq k \leq n$, the A-module $\bigwedge^k M$ is free of rank $\binom{n}{k}$.

Problem 2 (10 pts)

Let k be a field, $R := k[X_1, \ldots, X_n]$, and $M := R^n$. Construct R-module homomorphisms φ_j providing a projective, in fact free, resolution

$$0 \longrightarrow \bigwedge^n M \xrightarrow{\varphi_n} \bigwedge^{n-1} M \xrightarrow{\varphi_{n-1}} \ldots \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} k \longrightarrow 0$$

of k as an R-module.

Problem 3 (10 pts)

Let X be a topological space and $T(X)$ the set of open subsets of X. A *presheaf $\mathcal{F}$ of abelian groups on X* is defined as follows:

(a) To each $U \in T(X)$, there is assigned an abelian group $\mathcal{F}(U)$,

(b) for $U, V \in T(X)$ satisfying $V \subseteq U$, there exists a group homomorphism $\varrho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$,

subject to the conditions:

(i) $\mathcal{F}(\emptyset) = 0$,

(ii) $\varrho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for $U \in T(X)$,

(iii) $\varrho_{UW} = \varrho_{VW} \circ \varrho_{UV}$ for $U, V, W \in T(X)$ satisfying $W \subseteq V \subseteq U$.

Presheaves of rings on X can be defined analogously.

(a) Check that the definition of a presheaf is equivalent to saying that a presheaf of abelian groups is a functor from the category of open sets of U to the category of abelian groups.

(b) Let X be a topological space and G an abelian group. For $U \in T(X)$, define

$$\mathcal{G}(U) := \{\varphi : U \to G \,|\, \forall x \in X \,\exists V \subseteq U : V \text{ open}, x \in V, \varphi|_V = \text{const.}\}.$$

Show that $\mathcal{G}$ is a presheaf of abelian groups on X.

(c) For $U \subseteq \mathbb{R}^n$ open, let $\mathcal{C}^\infty_\mathbb{R}(U)$ be the ring of smooth real functions on U. Show that $\mathcal{C}^\infty_\mathbb{R}$ is a presheaf of rings on $\mathbb{R}^n$.

Problem 4*

Let k be a field and $U \subseteq \mathbb{A}^n(k)$ an open subset. We call a function $\varphi : U \longrightarrow k$ *regular in* $x = (x_1, \dots, x_n) \in U$ iff there is an open subset $V \subseteq U$ such that $x \in V$ and there exist polynomials $p, q \in k[X_1, \dots, X_n]$, $q \neq 0$, such that for all $y \in V$

$$\varphi(y) = \frac{p(y)}{q(y)}$$

holds. The function φ is called *regular on* U iff it is regular for all $x \in U$.

(a) Show that
$$\mathcal{O}_{\mathbb{A}^n(k)}(U) := \{\varphi : U \longrightarrow k \,|\, \varphi \text{ regular on } U\}$$
defines a presheaf of rings $\mathcal{O}_{\mathbb{A}^n(k)}$ on $\mathbb{A}^n(k)$.

(b) The natural inclusion

$$U := \mathbb{A}^1(k) \setminus \{0\} \subseteq \mathbb{A}^1(k) =: X$$

defines a homomorphism of rings

$$\varrho_{XU} : \mathcal{O}_{\mathbb{A}^1(k)}(X) \longrightarrow \mathcal{O}_{\mathbb{A}^1(k)}(U)$$

by restriction of regular functions.

Check whether ϱ_{XU} is injective or surjective in the following cases:

(i) $k = \mathbb{F}_3$,

(ii) $k = \mathbb{C}$.