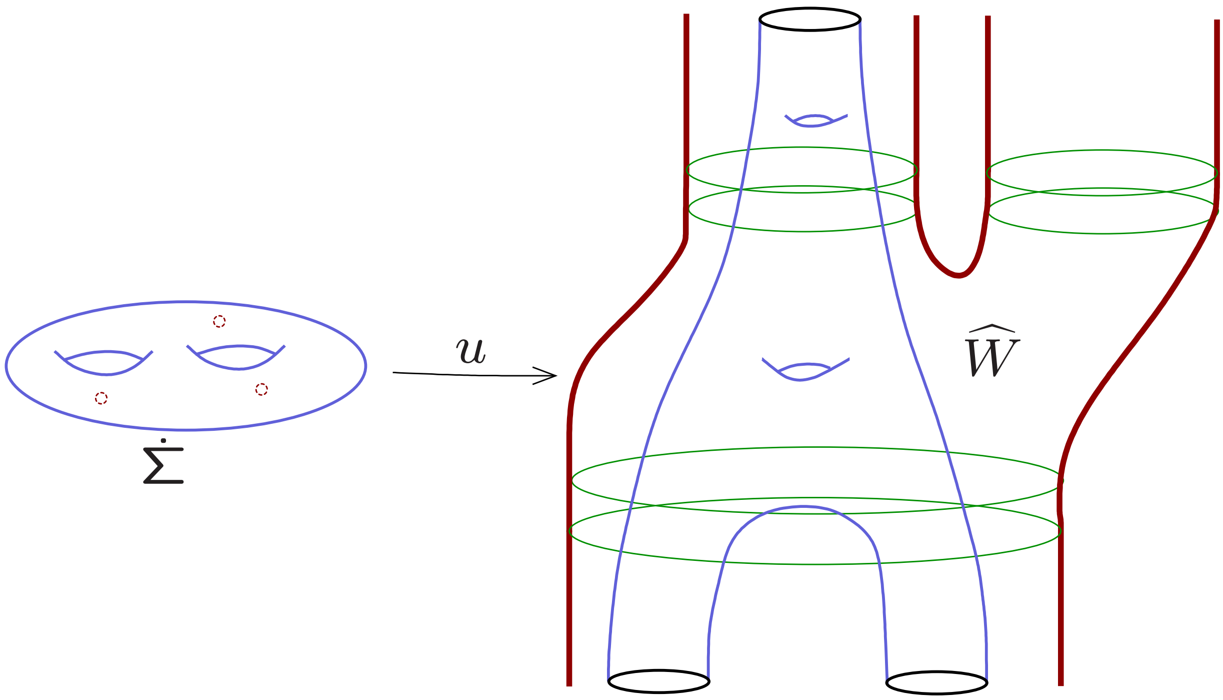
Background material 3

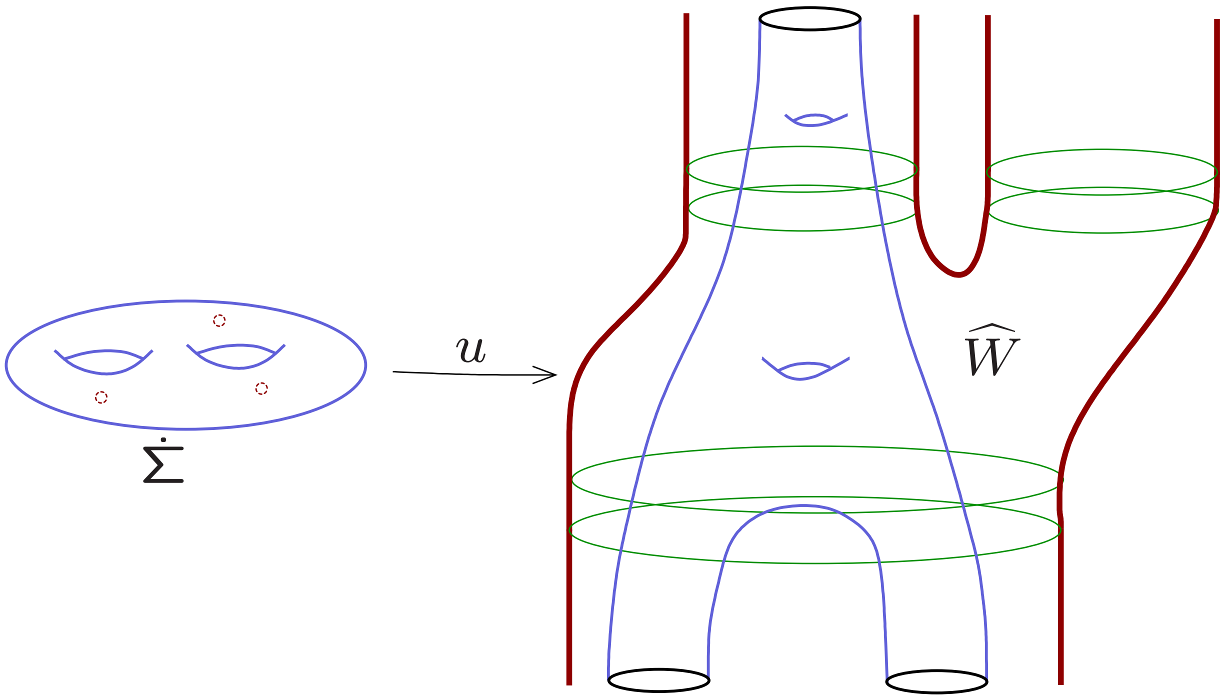
The basic goal

We want an intersection theory for asymptotically cylindrical holomorphic curves:

$$u : \dot{\Sigma} \to \widehat{W}, \qquad v : \dot{\Sigma}' \to \widehat{W}$$

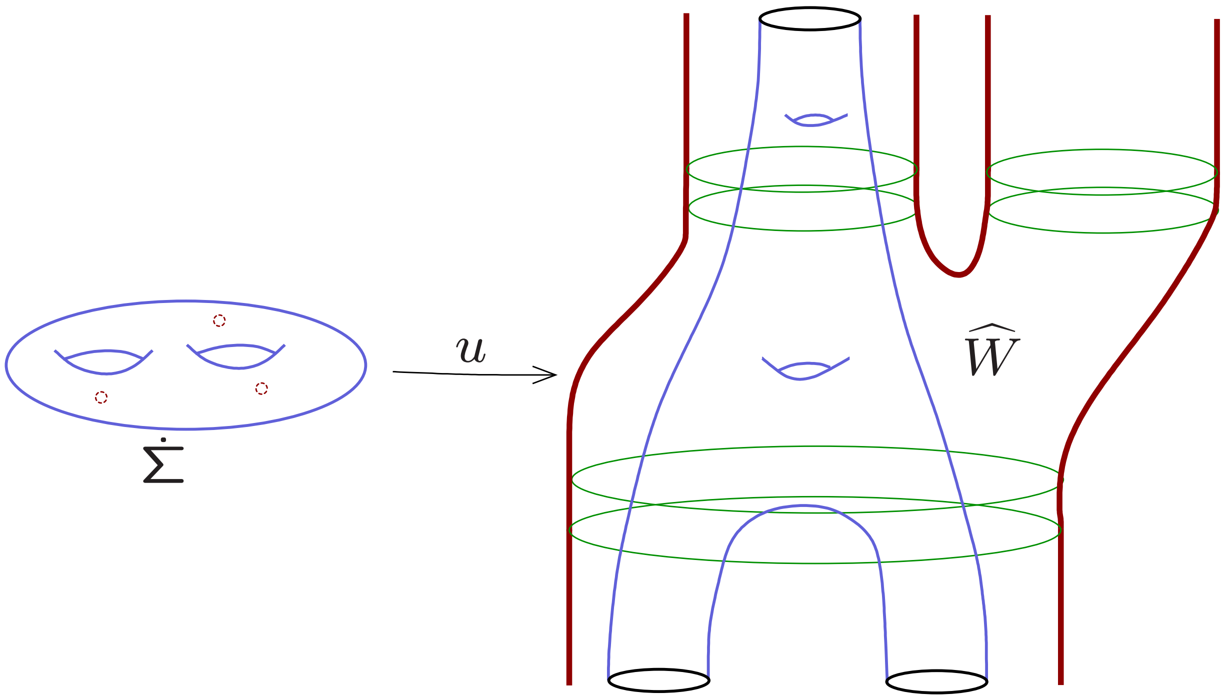


Desired properties:



Desired properties:

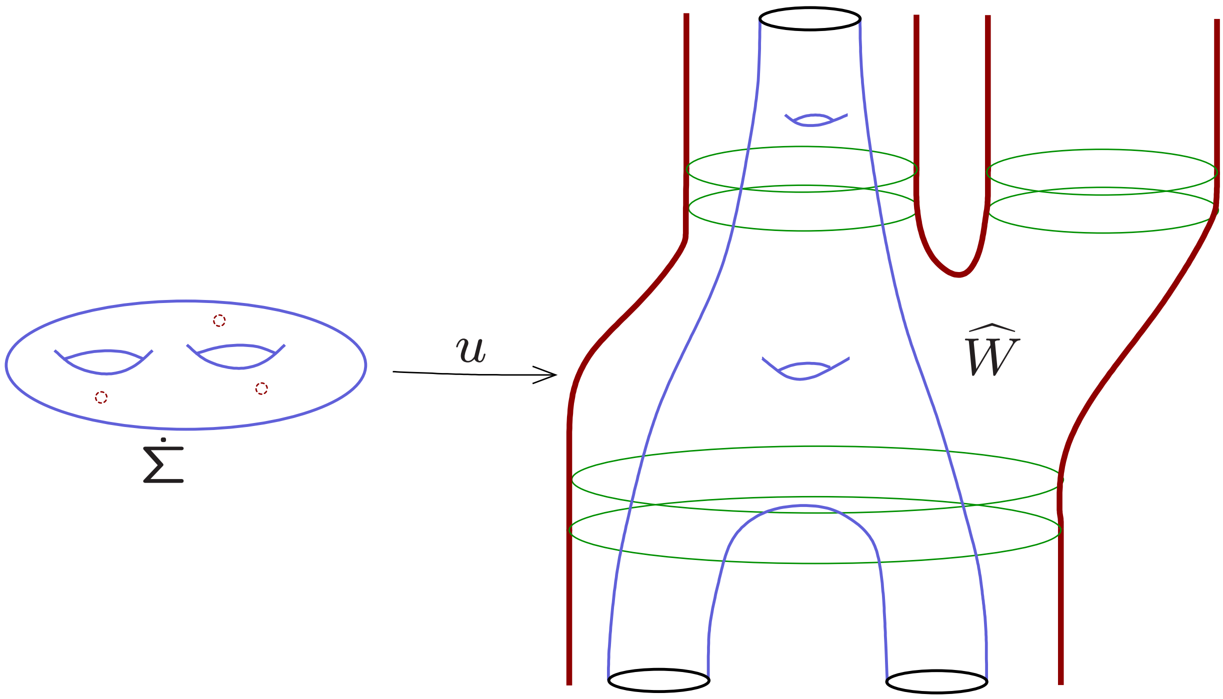
1. Homotopy-invariant sufficient conditions for u and v to be *disjoint* or *transverse*



Desired properties:

1. Homotopy-invariant sufficient conditions for u and v to be *disjoint* or *transverse*

2. Homotopy-invariant sufficient conditions for simple curves to be *embedded*



It looks promising at first...

Whenever $u(\dot{\Sigma}) \neq v(\dot{\Sigma}')$, we have

$$u \cdot v \geq \Big|\{(z, \zeta) \mid u(z) = v(\zeta)\}\Big|$$

with equality iff $u \pitchfork v$.

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Similarly, if u is simple,

$$\delta(u) \geq \frac{1}{2} \left| \{ (z, \zeta) \mid u(z) = u(\zeta), \ z \neq \zeta \} \right|$$

with equality iff u is immersed with all double points transverse.

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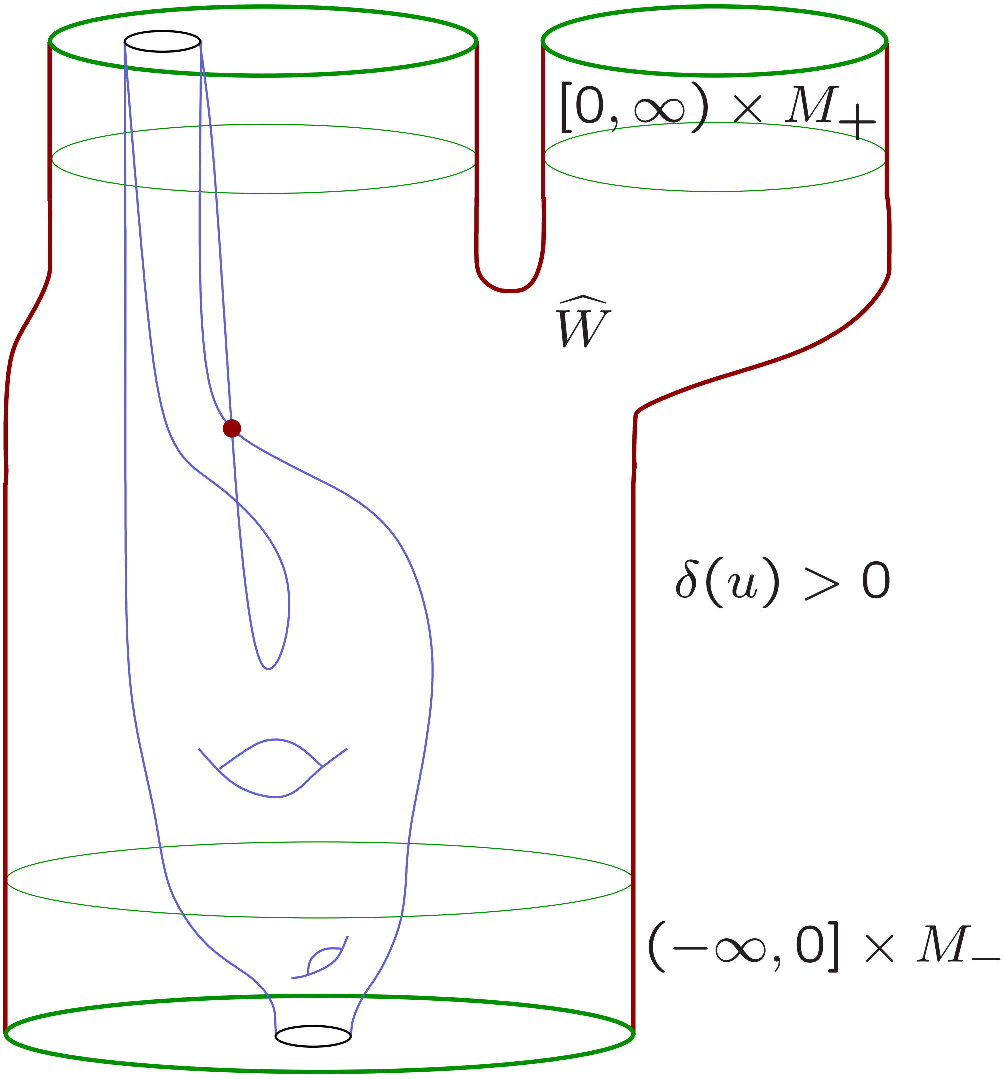
The basic problem:
Neither $u \cdot v$ nor $\delta(u)$ is *homotopy invariant*!

Disaster scenario:

Suppose $u : \dot{\Sigma} \to \widehat{W}$ has two ends approaching the same Reeb orbit...



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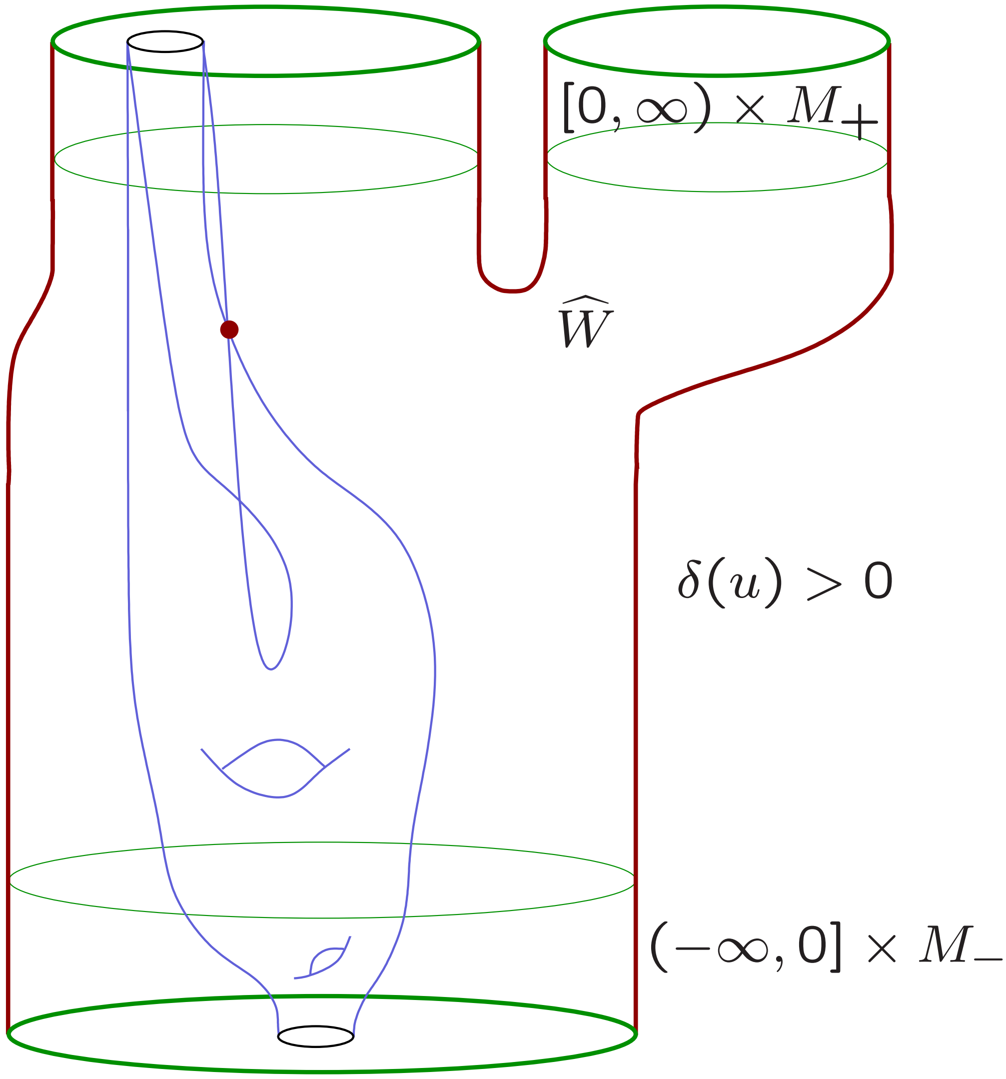

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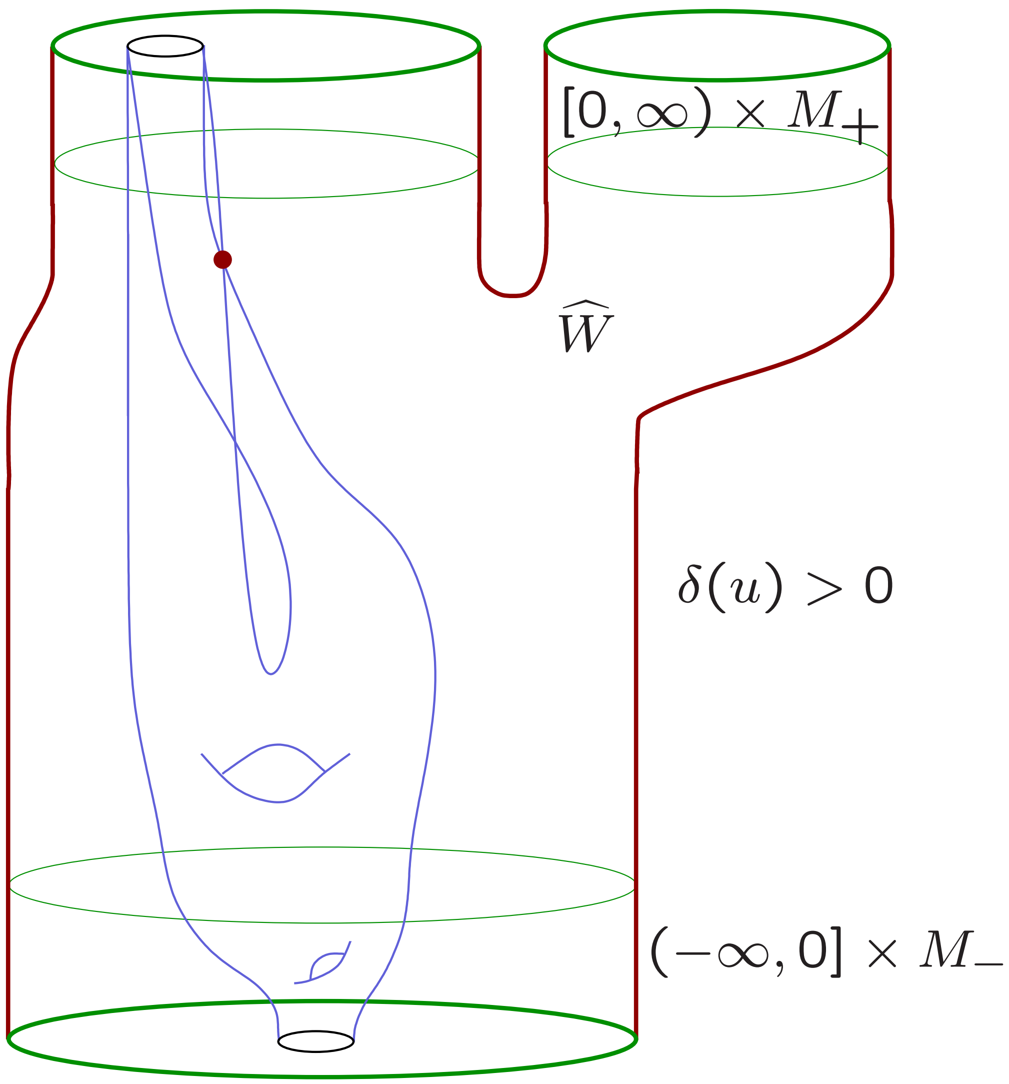

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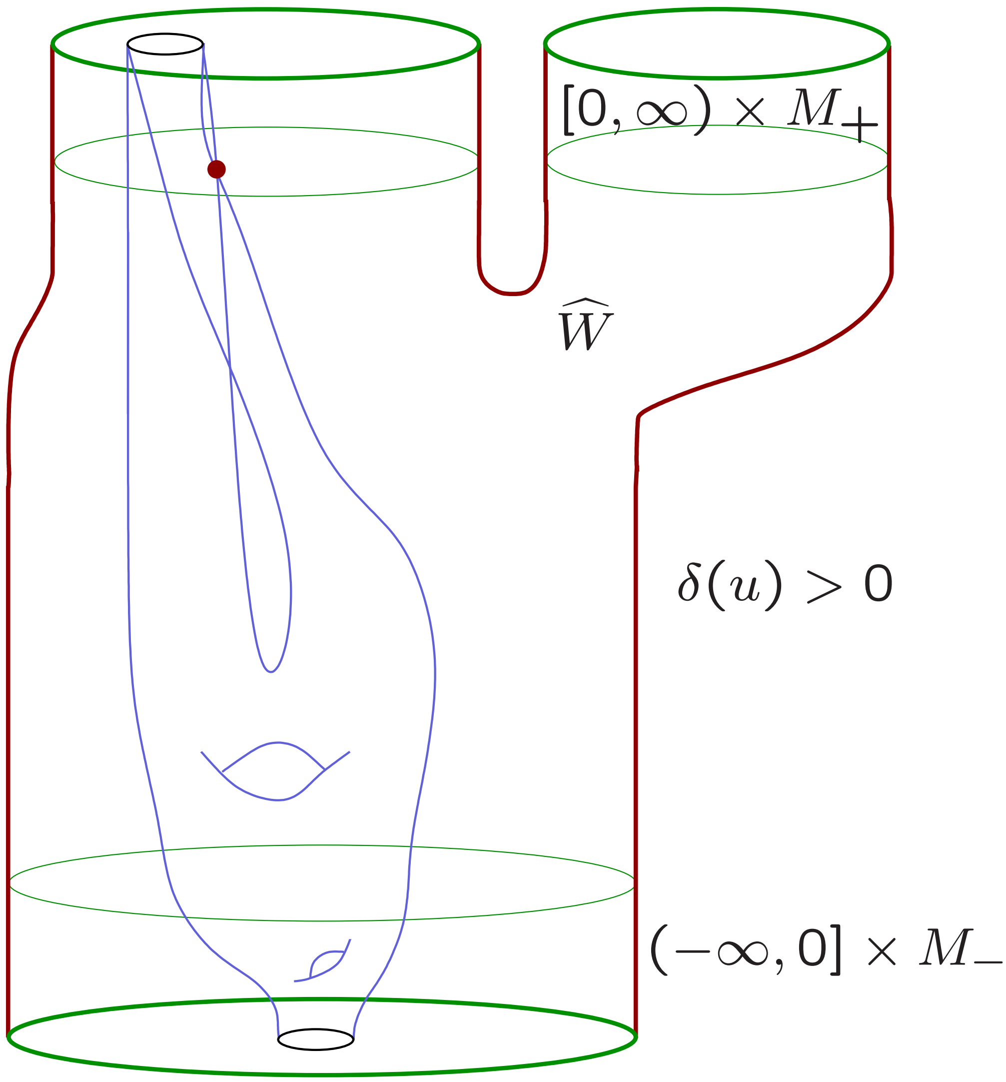

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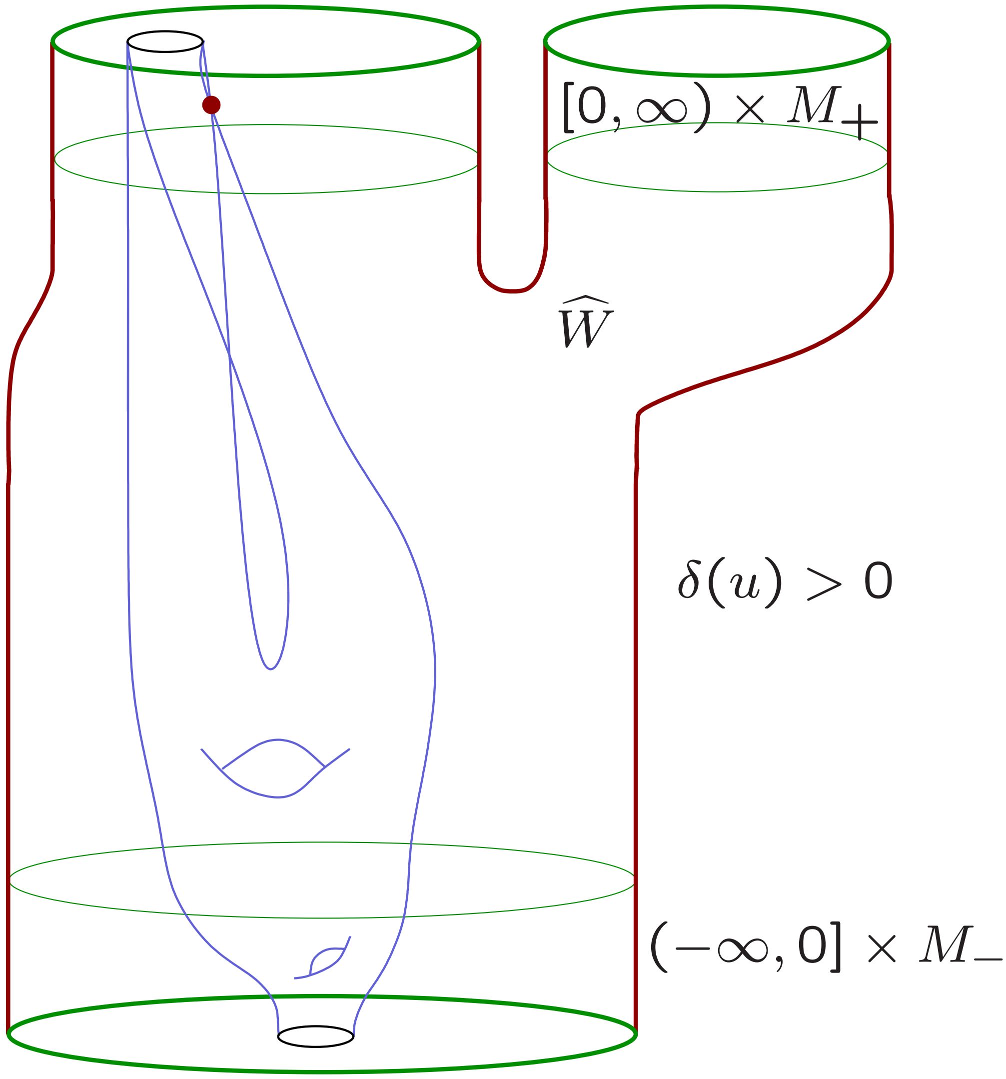

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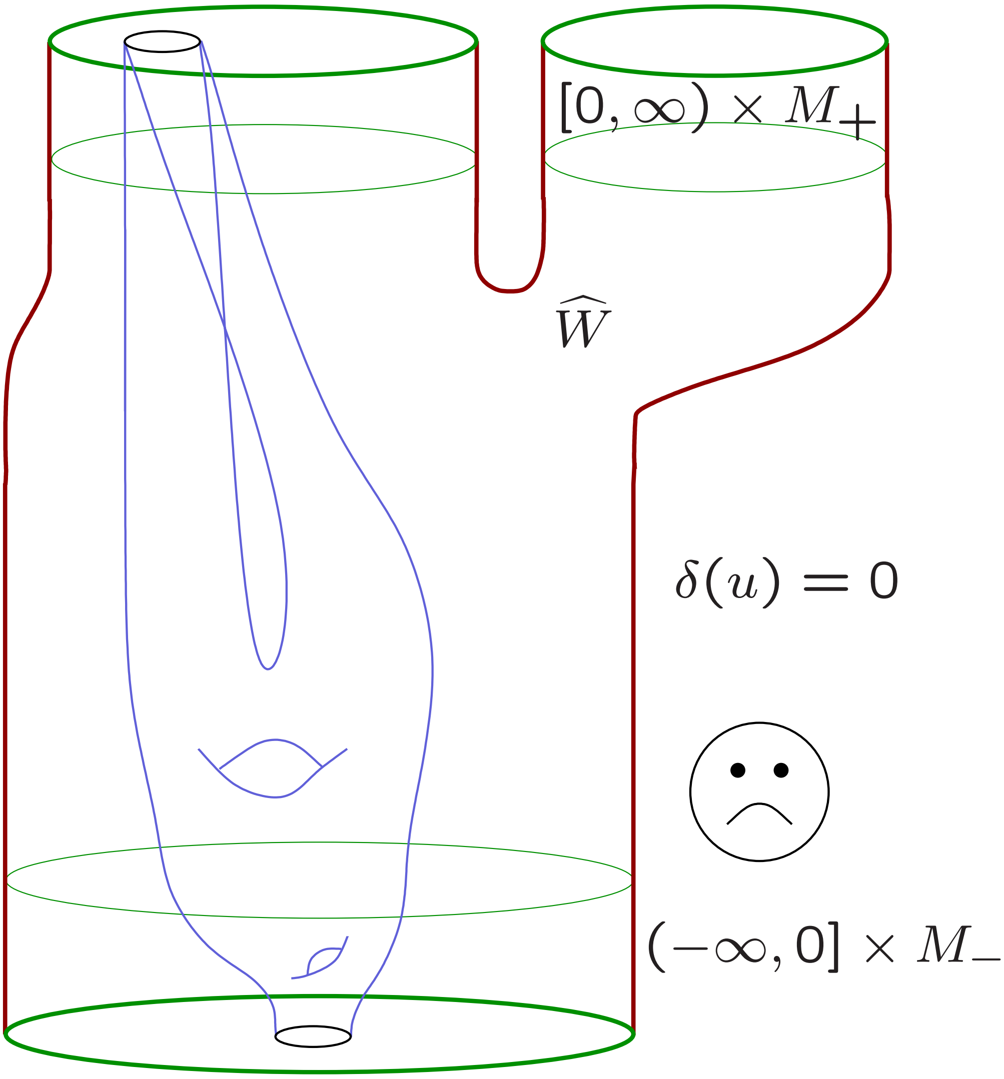

Intersections can **escape to infinity**!

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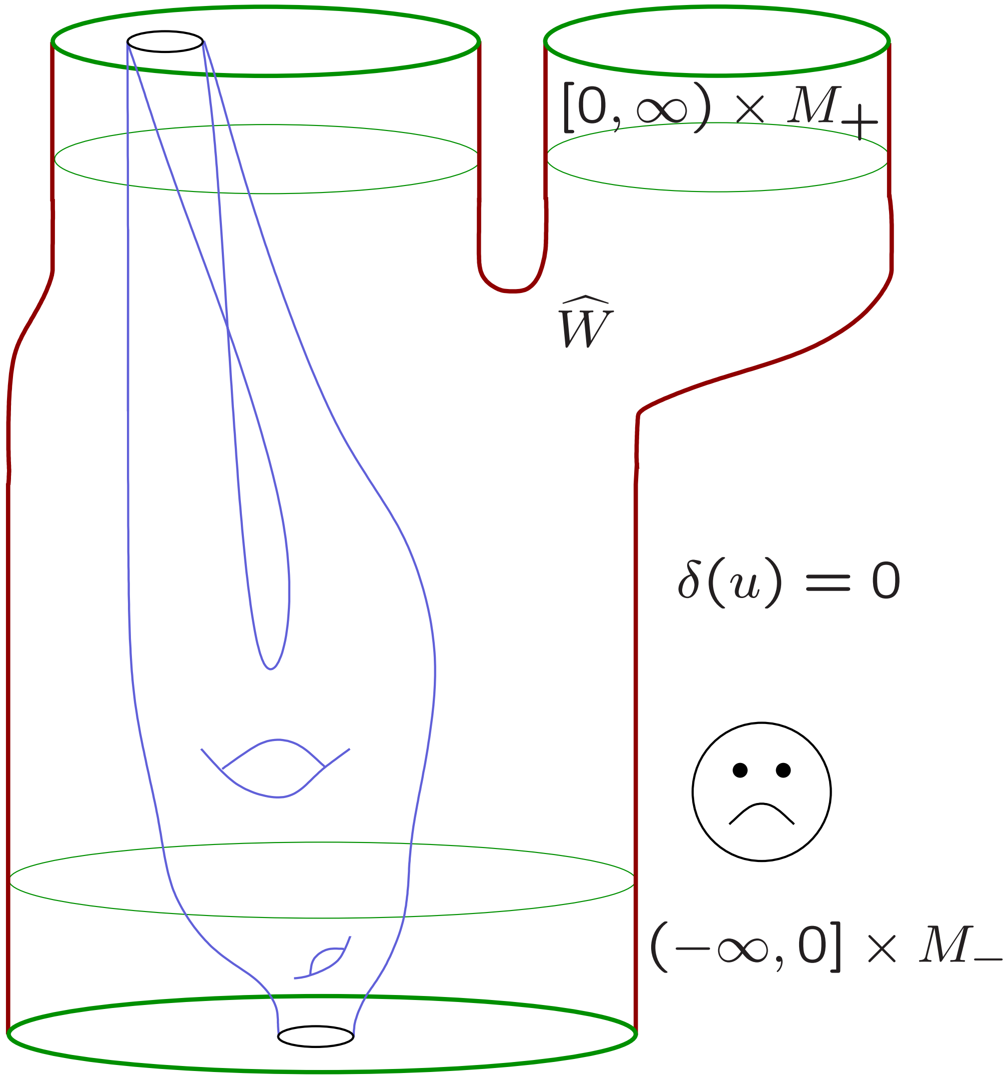


Intersections can **escape to infinity**!

Solution:
Understand asymptotic behaviour *well*.

Analogy with Morse theory

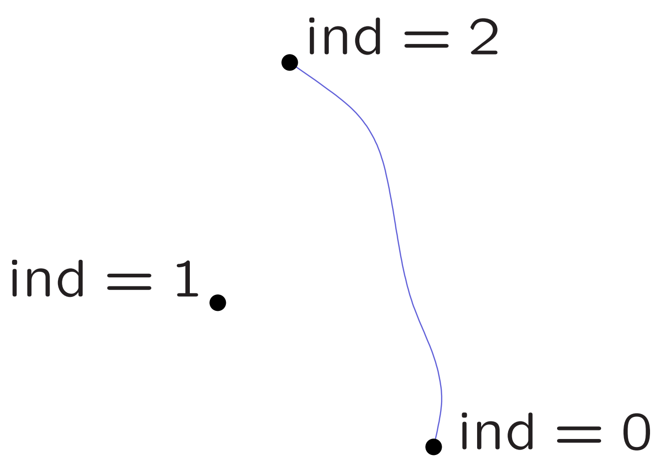
In Morse homology, one studies gradient-flow lines

$$x : \mathbb{R} \to M, \qquad \dot{x} = \nabla f(x)$$

of a function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g), where we assume each $p \in \mathrm{Crit}(f)$ is nondegenerate, i.e. the Hessian

$$\mathbf{A}_p := \nabla(\nabla f)(p) : T_pM \to T_pM$$

has trivial kernel.



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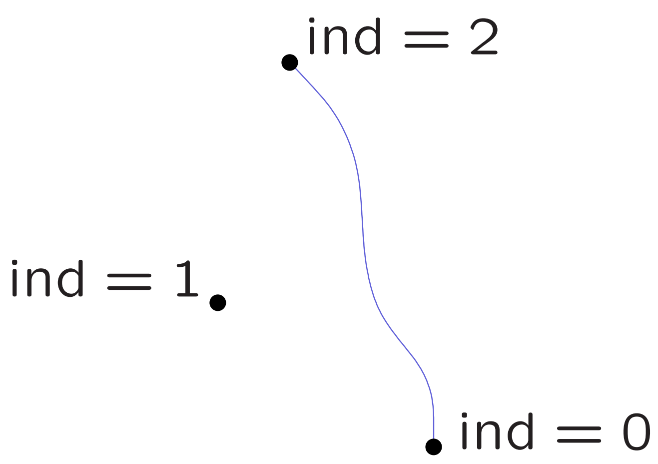

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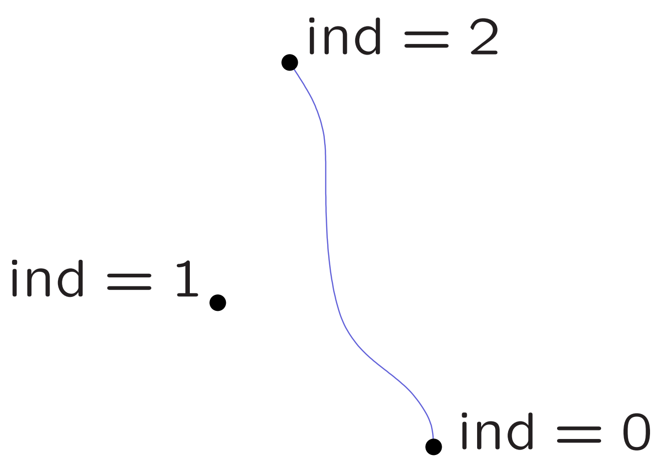

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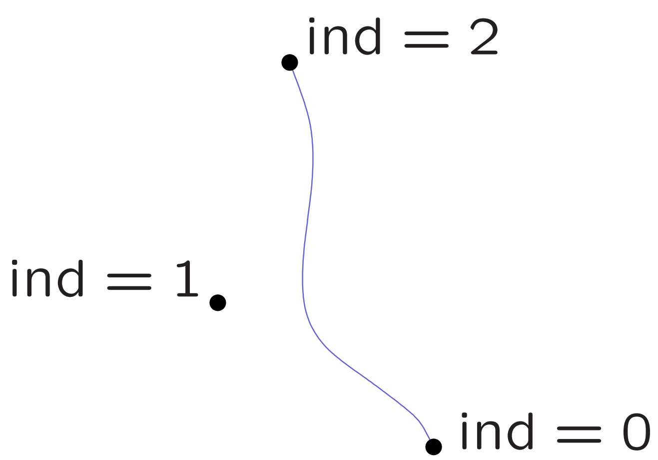

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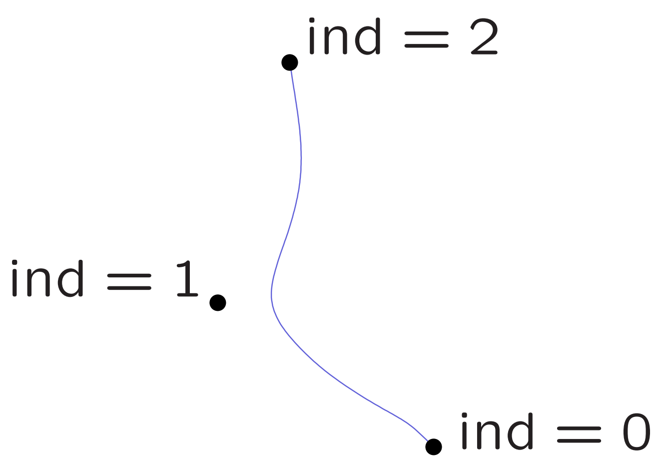

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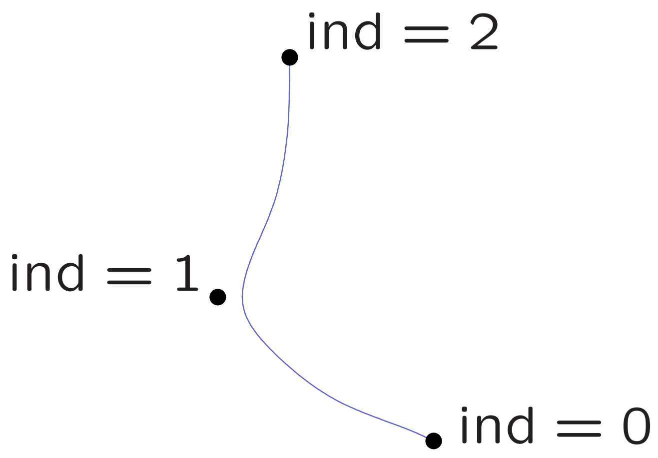

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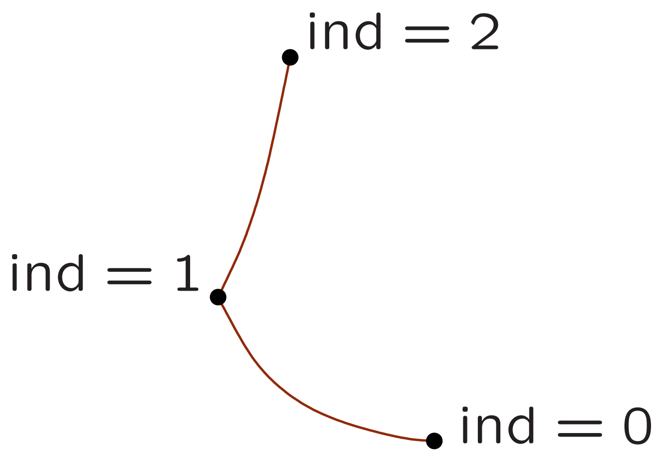

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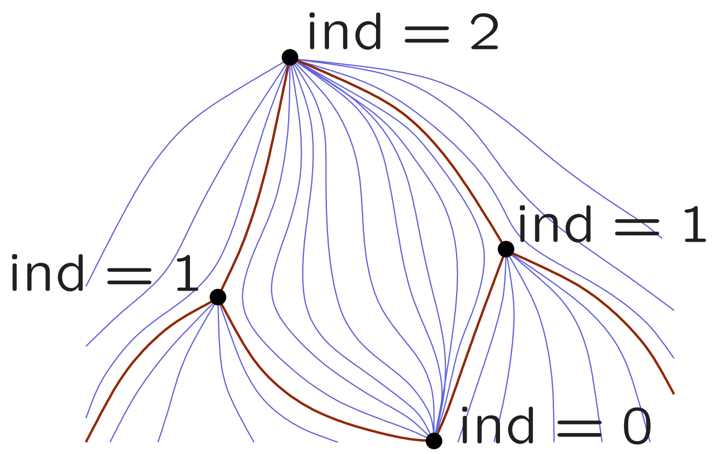

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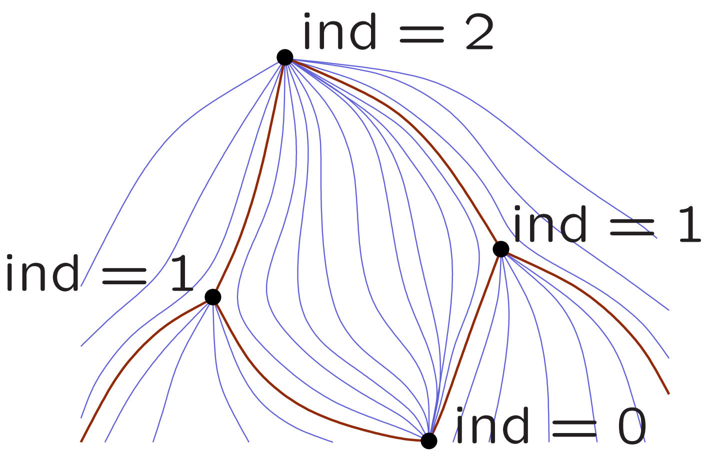

$\mathbf{A}_p$ is symmetric, so its eigenvalues are real.

Asymptotic formula for gradient-flow

Theorem

Assume $p \in \mathrm{Crit}(f)$, $h(s) \in T_pM$ is defined for s close to $\pm\infty$ and

$$x(s) = \exp_p h(s) \in M$$

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$$h(s) = e^{\lambda s}(v + r(s))$$

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"Flow lines approach critical points along *asymptotic eigenvectors*."

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$$u = (f, v) : [0, \infty) \times S^1 \to \mathbb{R} \times M$$

is J-holomorphic if and only if

$$\begin{aligned} \partial_s f - \alpha(\partial_t v) &= 0, \\ \partial_t f + \alpha(\partial_s v) &= 0, \\ \pi_\alpha \partial_s v + J\, \pi_\alpha \partial_t v &= 0, \end{aligned}$$

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Claim: the third equation can be interpreted as the *L^2-gradient flow* of the **contact action functional**

$$\Phi_\alpha : C^\infty(S^1, M) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^* \alpha,$$

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whose critical points are Reeb orbits. Its Hessian at a T-periodic Reeb orbit $\gamma : S^1 \to M$ is the L^2-symmetric operator

$$\begin{aligned} \nabla(\nabla\Phi_\alpha)(\gamma) : \Gamma(\gamma^*\xi) &\to \Gamma(\gamma^*\xi) \\ \eta &\mapsto -J\left(\nabla_t \eta - T \nabla_\eta R_\alpha\right), \end{aligned}$$

where ∇ is any symmetric connection on M.