APPENDIX A

Sobolev spaces

Contents

The purpose of this appendix is to summarize all essential facts about Sobolev spaces that are needed elsewhere in the text. Most of the results stated here may be considered standard, and many of the proofs (sometimes in much greater generality than what is stated here) can be found in the comprehensive book by Adams and Fournier [**AF03**]. Many of them can also be found in standard PDE books such as Evans [**Eva98**], though often not quite at the level of generality that we need. We will provide proofs for some of the results which are harder to find in the literature than others.

A.1. Domains in Euclidean space

A.1.1. Definitions. Throughout this section, we consider functions defined on a nonempty open subset

$$\mathcal{U} \subset \mathbb{R}^n$$

with its standard Lebesgue measure, and taking values in a finite-dimensional vector space that will usually not need to be specified, though occasionally we will assume it is $\mathbb{R}$ or $\mathbb{C}$ so that one can define products of functions. Integrals of such functions f over measurable subsets $\mathcal{V} \subset \mathcal{U}$ will be denoted by

$$\int_{\mathcal{V}} f = \int_{\mathcal{V}} f(x)\, d\mu(x).$$

The domain $\mathcal{U}$ will also sometimes have additional conditions specified such as boundedness or regularity at the boundary, though we will try not to add too many more restrictions than are really needed. The most useful assumption to impose on $\mathcal{U}$ is known as the **strong local Lipschitz condition**: if $\mathcal{U}$ is bounded, then it means simply that near every boundary point of $\mathcal{U}$, one can find smooth local coordinates in which $\mathcal{U}$ looks like the region bounded by the graph of a Lipschitz-continuous function, and in this case we call $\mathcal{U}$ a **bounded Lipschitz domain**. If $\mathcal{U}$ is unbounded, then one needs to impose extra conditions guaranteeing e.g. uniformity of Lipschitz constants, and the precise definition becomes a bit lengthy (see [**AF03**, §4.9]). For our purposes, all we really need to know about the strong local Lipschitz condition is that that it is satisfied both by bounded Lipschitz domains and by relatively tame unbounded domains such as $(0,1) \times (0,\infty) \subset \mathbb{R}^2$ which have smooth boundary with finitely many corners—the latter is the most exotic domain we have any need for in this book.

Given $p \in [1,\infty)$, we define the L^p-norm for measurable functions on $\mathcal{U} \subset \mathbb{R}^n$ by

$$\|f\|_{L^p(\mathcal{U})} := \left(\int_{\mathcal{U}} |f|^p \right)^{1/p},$$

and extend this to the case $p = \infty$ as the essential supremum,

$$\|f\|_{L^\infty(\mathcal{U})} := \operatorname{ess\,sup}_{\mathcal{U}} |f|.$$

We usually abbreviate

$$\|f\|_{L^p} := \|f\|_{L^p(\mathcal{U})}$$

when the domain $\mathcal{U}$ is understood and there is no danger of confusion. The Banach space $L^p(\mathcal{U})$ is defined as the space of equivalence classes of measurable functions (where $f \sim g$ means that f and g are equal almost everywhere) whose L^p-norms are finite. For $p = 2$, there is also a natural inner product

$$\langle f, f \rangle_{L^2} := \langle f, g \rangle_{L^2(\mathcal{U})} := \int_{\mathcal{U}} \bar{f} g,$$

giving $L^2(\mathcal{U})$ the structure of a Hilbert space. (Here we are assuming for simplicity that f and g are complex valued, but the definition can be extended in obvious ways to functions with values in more general inner product spaces.) We will often need to use **Hölder's inequality**

$$\big\| |f| \cdot |g| \big\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q} \quad \text{whenever } 1 \le p, q \le \infty \text{ and } 1 = \frac{1}{p} + \frac{1}{q},$$

which generalizes easily to

$$\text{(A.1.1)} \quad \left\| \prod_{i=1}^m |f_i| \right\|_{L^p} \le \prod_{i=1}^m \|f_i\|_{L^{p_i}} \quad \text{for } 1 \le p \le p_1, \dots, p_m \le \infty \text{ with } \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$$

for any $m \ge 2$.

Denote by

$$C_0^\infty(\mathcal{U}) \subset C^\infty(\mathcal{U})$$

the space of smooth real-valued functions with compact support in $\mathcal{U}$, and recall that for $j = 1, \dots, n$, a function f on $\mathcal{U}$ is said to have **weak partial derivative** $\partial_j f = g$ if g is another function on $\mathcal{U}$ such that the obvious "integration by parts" formula is satisfied:

$$\int_{\mathcal{U}} g\varphi = -\int_{\mathcal{U}} f \partial_j \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathcal{U}).$$

In other words, g is a partial derivative of f *in the sense of distributions* (cf. §2.5). The n-tuple of all weak first-order partial derivatives of f is often denoted by

$$\nabla f := (\partial_1 f, \dots, \partial_n f),$$

so for instance of $f : \mathcal{U} \to \mathbb{R}^m$, then one can think of the *weak gradient* ∇f as a function $\mathcal{U} \to \mathbb{R}^{mn}$. The integration by parts formula can obviously be iterated to define weak derivatives of higher order via the condition

$$\partial^\beta f = g \quad \Leftrightarrow \quad \int_{\mathcal{U}} g\varphi = (-1)^{|\beta|} \int_{\mathcal{U}} f \partial^\beta \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathcal{U}),$$

Here β is a **multiindex**, i.e. an n-tuple of nonnegative integers $(i_1, \dots, i_n)$ with **degree** $|\beta| := i_1 + \dots + i_n$, so the symbol ∂^β represents the differential operator

$$\partial^\beta := \partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n}.$$

For $k \in \mathbb{N}$ and $1 \le p \le \infty$, we now define $W^{k,p}(\mathcal{U})$ as the space of all functions in $L^p(\mathcal{U})$ that have weak partial derivatives up to order k also in $L^p(\mathcal{U})$, and define the $W^{k,p}$-norm by

$$\|f\|_{W^{k,p}} := \sum_{|\beta| \le k} \|\partial^\beta f\|_{L^p},$$

where the sum ranges over all multiindices of degrees $0, 1, \dots, k$. This has an obvious extension to $k = 0$ as $W^{0,p}(\mathcal{U}) := L^p(\mathcal{U})$. It is straightforward to show that $W^{k,p}(\mathcal{U})$ is a Banach space for every k and p, and it can always be identified with a closed subspace of some product of L^p spaces, e.g.

$$W^{1,p}(\mathcal{U}) \cong \left\{ (f_0, f_1, \dots, f_n) \in \bigoplus_{j=0}^{n} L^p(\mathcal{U}) \;\middle|\; \partial_j f_0 = f_j \text{ for } j = 1, \dots, n \right\}.$$

It follows that $W^{k,p}(\mathcal{U})$ is separable for $1 \le p < \infty$ and reflexive for $1 < p < \infty$, as a consequences of the corresponding facts about $L^p(\mathcal{U})$. In the case $p = 2$, these spaces are also often denoted by

$$H^k(\mathcal{U}) := W^{k,2}(\mathcal{U}),$$

and they admit Hilbert space structures with inner product

$$\langle f, g \rangle_{H^k} = \sum_{|\beta| \le k} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2}.$$

We denote by

$$W_0^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U}), \qquad H_0^k(\mathcal{U}) \subset H^k(\mathcal{U})$$

the closed subspaces defined as the closures of $C_0^\infty(\mathcal{U})$ with respect to the relevant norms. Since $C_0^\infty(\mathcal{U})$ is dense in $L^p(\mathcal{U})$ for $1 \le p < \infty$ (see e.g. [**LL01**, §2.19]), there is no difference between $W^{0,p}(\mathcal{U})$ and $W_0^{0,p}(\mathcal{U})$ for $p < \infty$, but in general $W_0^{k,p}(\mathcal{U}) \ne W^{k,p}(\mathcal{U})$ for $k \ge 1$, with a few notable exceptions such as the case $\mathcal{U} = \mathbb{R}^n$ (cf. Corollary A.1.2 below).

One further definition: let

$$W_{\mathrm{loc}}^{k,p}(\mathcal{U}) := \big\{ \text{functions } f \text{ on } \mathcal{U} \;\big|\; f \in W^{k,p}(\mathcal{V}) \text{ for all open subsets } \mathcal{V} \subset \mathcal{U} \text{ with compact closure } \overline{\mathcal{V}} \subset \mathcal{U} \big\},$$

and we say that a sequence $f_j \in W^{k,p}_{\text{loc}}(\mathcal{U})$ converges in $W^{k,p}_{\text{loc}}$ to $f \in W^{k,p}_{\text{loc}}(\mathcal{U})$ if the restrictions to all precompact open subsets $\mathcal{V} \subset \overline{\mathcal{V}} \subset \mathcal{U}$ converge in $W^{k,p}(\mathcal{V})$. Since $\mathcal{U}$ is always assumed open, $W^{k,p}_{\text{loc}}(\mathcal{U})$ is always larger than $W^{k,p}(\mathcal{U})$—it contains for instance all smooth functions, even those that blow up near $\partial\overline{\mathcal{U}}$—and it is not a Banach space.

A.1.2. Approximation, extension, embedding and compactness. We shall now state various results relating $W^{k,p}(\mathcal{U})$ to spaces of continuous and/or differentiable functions. For $k = 0, 1, 2, \ldots, \infty$, let $C^k(\mathcal{U})$ denote the space of functions on $\mathcal{U}$ that have continuous derivatives up to order m. Such functions and their derivatives are not required to be bounded, but one can also define for each integer $k \geq 0$ the norm

$$\|f\|_{C^k} := \|f\|_{C^k(\mathcal{U})} := \max_{|\beta| \leq k} \sup_{\mathcal{U}} |\partial^\beta f|,$$

leading to the Banach space

$$C^k_b(\mathcal{U}) := \left\{ f \in C^k(\mathcal{U}) \;\middle|\; \|f\|_{C^k} < \infty \right\}.$$

This contains a closed subspace

$$C^k(\overline{\mathcal{U}}) \subset C^k_b(\mathcal{U})$$

consisting of functions whose derivatives or orders $0, 1, \ldots, k$ are bounded and uniformly continuous on $\mathcal{U}$—the latter is true if and only if these derivatives have continuous and bounded extensions to the closure $\overline{\mathcal{U}}$. (There is some abuse of notation that one should be aware of here, e.g. $C^k(\overline{\mathbb{R}^n})$ is not the same space as $C^k(\mathbb{R}^n)$, even though $\mathbb{R}^n$ is its own closure.) Finally, for $0 < \alpha < 1$, we define the Hölder seminorm $|f|_{C^\alpha}$ of a function on $\mathcal{U}$ by

$$|f|_{C^\alpha} := |f|_{C^\alpha(\mathcal{U})} := \sup_{x \neq y \in \mathcal{U}} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

and define $C^{k,\alpha}(\overline{\mathcal{U}})$ as the Banach space of functions $f \in C^k(\overline{\mathcal{U}})$ for which the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \max_{|\beta| = k} |\partial^\beta f|_{C^\alpha}$$

is finite.

THEOREM A.1.1 ([**AF03**, §3.17, 3.22]). *For any open subset $\mathcal{U} \subset \mathbb{R}^n$, and any $k \geq 0$, $1 \leq p < \infty$, the subspace*

$$C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. Moreover, if $\mathcal{U} \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition, then the space

$$\left\{ f \in C^\infty(\mathcal{U}) \;\middle|\; f = \tilde{f}|_{\mathcal{U}} \text{ for some } \tilde{f} \in C^\infty_0(\mathbb{R}^n) \right\}$$

is also dense in $W^{k,p}(\mathcal{U})$, so in particular,

$$C^\infty(\overline{\mathcal{U}}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. □

COROLLARY A.1.2. *The space* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$ *for every* $k \geq 0$ *and* $p \in [1,\infty)$. □

THEOREM A.1.3 ([**AF03**, §5.29]). *Assume* $\mathcal{U} \subset \mathbb{R}^n$ *is an open subset satisfying the strong local Lipschitz condition. Then a function* $f \in W^{k,p}(\mathcal{U})$ *belongs to* $W_0^{k,p}(\mathcal{U})$ *if and only if the function* $\tilde{f}$ *on* $\mathbb{R}^n$ *defined to match* f *on* $\mathcal{U}$ *and* 0 *everywhere else belongs to* $W^{k,p}(\mathbb{R}^n)$. □

While it is obvious from the definitions that functions in $W_0^{k,p}(\mathcal{U})$ always admit extensions of class $W^{k,p}$ over $\mathbb{R}^n$, this is much less obvious for functions in $W^{k,p}(\mathcal{U})$ in general, and it is not true without sufficient assumptions about the regularity of $\partial\mathcal{U}$. For our purposes it suffices to consider the following case.

THEOREM A.1.4 ([**AF03**, §5.22]). *Assume* $\mathcal{U} \subset \mathbb{R}^n$ *is a bounded open subset such that* $\partial\overline{\mathcal{U}}$ *is a submanifold of class* C^m *for some* $m \in \{1,2,3,\ldots,\infty\}$. *Then there exists a linear operator* E *that maps functions defined almost everywhere on* $\mathcal{U}$ *to functions defined almost everywhere on* $\mathbb{R}^n$ *and has the following properties:*

- *For every function* f *on* $\mathcal{U}$, $Ef|_{\mathcal{U}} \equiv f$ *almost everywhere;*
- *For every nonnegative integer* $k \leq m$ *and every* $p \in [1,\infty)$, E *defines a bounded linear operator* $W^{k,p}(\mathcal{U}) \to W^{k,p}(\mathbb{R}^n)$.

□

COROLLARY A.1.5. *Suppose* $\mathcal{U},\mathcal{U}' \subset \mathbb{R}^n$ *are open subsets such that* $\mathcal{U}$ *has compact closure contained in* $\mathcal{U}'$. *If* $\mathcal{U}$ *satisfies the hypothesis of Theorem A.1.4, then the resulting extension operator* E *can be chosen such that it maps each* $W^{k,p}(\mathcal{U})$ *for* $k \leq m$ *and* $1 \leq p < \infty$ *into* $W_0^{k,p}(\mathcal{U}')$.

PROOF. Choose a smooth function $\rho : \mathcal{U}' \to [0,1]$ that has compact support and equals 1 on $\overline{\mathcal{U}}$, then replace the operator E given by Theorem A.1.4 with the operator $f \mapsto \rho \cdot Ef$. □

The next statement is an assemblage of results collectively known as the **Sobolev embedding theorem**. To understand its meaning, recall that elements of $W^{k,p}(\mathcal{U})$ are technically not functions, but rather *equivalence classes* of functions defined almost everywhere. Thus when we say e.g. that there is an inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow C^{m,\alpha}(\overline{\mathcal{U}})$, the literal meaning is that for every function f representing an element of $W^{k,p}(\mathcal{U})$, one can change the values of f in a unique way on some set of measure zero in $\mathcal{U}$ so that after this change, $f \in C^{m,\alpha}(\overline{\mathcal{U}})$. Continuity of the inclusion means that there is a bound of the form

$$\|f\|_{C^{m,\alpha}} \leq c\|f\|_{W^{k,p}}$$

for all $f \in W^{k,p}(\mathcal{U})$, where $c > 0$ is a constant which may in general depend on m, α, k, p and $\mathcal{U}$, but not on f.

THEOREM A.1.6 ([**AF03**, §4.12]). *Assume* $\mathcal{U} \subset \mathbb{R}^n$ *is an open subset satisfying the strong local Lipschitz condition,* $k \geq 1$ *is an integer and* $1 \leq p < \infty$.

(1) *If $kp > n$ and $k - n/p < 1$, then there exist continuous inclusions*

$$W^{k,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\overline{\mathcal{U}}) \quad \text{for each } \alpha \in (0, k - n/p],$$
$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}) \quad \text{for each } q \in [p, \infty].$$

(2) *If $kp < n$ and $p^* > p$ is defined by the condition*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n},$$

then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p, p^*].$$

(3) *If $kp = n$, then there exist continuous inclusions*

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p, \infty).$$

Moreover, the spaces $W_0^{k,p}(\mathcal{U})$ admit similar inclusions under no assumption on the open subset $\mathcal{U} \subset \mathbb{R}^n$.

REMARK A.1.7. The statement about $W_0^{k,p}(\mathcal{U})$ for arbitrary $\mathcal{U}$ follows immediately from the case $\mathcal{U} = \mathbb{R}^n$, since there is always a canonical continuous extension map $W_0^{k,p}(\mathcal{U}) \to W^{k,p}(\mathbb{R}^n)$. More generally, the $\mathbb{R}^n$ case implies these embeddings for $W^{k,p}(\mathcal{U})$ whenever $\mathcal{U} \subset \mathbb{R}^n$ is a domain for which bounded extension operators as in Theorem A.1.4 exist.

Under the same assumption on the domain $\mathcal{U}$, one can apply Theorem A.1.6 to successive derivatives of functions in $W^{k,p}(\mathcal{U})$ and thus obtain the following inclusions for any integer $d \geq 0$:

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\overline{\mathcal{U}}) \quad \text{if } kp > n \text{ and } 0 < \alpha \leq k - n/p < 1, \tag{A.1.2}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp > n \text{ and } p \leq q \leq \infty, \tag{A.1.3}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp < n \text{ and } p \leq q \leq p^*, \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}, \tag{A.1.4}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp = n \text{ and } p \leq q < \infty. \tag{A.1.5}$$

This last inclusion can then be composed with (A.1.2) for an arbitrarily large choice of q, giving another inclusion

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d-1,\alpha}(\overline{\mathcal{U}}) \quad \text{if } kp = n \text{ and } 0 < \alpha < 1. \tag{A.1.6}$$

REMARK A.1.8. The embedding theorem suggests that one should intuitively think of $W^{k,p}(\mathcal{U})$ as consisting of functions with "$k - n/p$ continuous derivatives," where the number $k - n/p$ may in general be a non-integer and/or negative. This provides a useful mnemonic for results about embeddings of one Sobolev space into another, such as the following.

COROLLARY A.1.9. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $1 \le p, q < \infty$, and $k, m \ge 0$ are integers satisfying*

$$k \ge m, \qquad p \le q, \qquad \text{and} \qquad k - \frac{n}{p} \ge m - \frac{n}{q}.$$

Then there exists a continuous inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$. □

By the Arzelà-Ascoli theorem, the natural inclusion

$$C^{k,\alpha'}(\overline{\mathcal{U}}) \hookrightarrow C^{k,\alpha}(\overline{\mathcal{U}})$$

for $\alpha < \alpha'$ is a compact operator whenever $\mathcal{U} \subset \mathbb{R}^n$ is bounded, i.e. sequences bounded in $C^{k,\alpha'}(\overline{\mathcal{U}})$ have convergent subsequences in $C^{k,\alpha}(\overline{\mathcal{U}})$. It follows that if $\mathcal{U} \subset \mathbb{R}^n$ in (A.1.2) is bounded and α is *strictly* less than the extremal value $k - n/p$, then the inclusion (A.1.2) is also compact. A similar statement holds for the inclusion (A.1.4) when $p \le q < p^*$, and this is known as the **Rellich-Kondrachov compactness theorem**. We summarize these as follows:

THEOREM A.1.10 ([**AF03**, §6.3]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $k \ge 1$ and $d \ge 0$ are integers and $1 \le p < \infty$.*

(1) *If $kp > n$ and $k - n/p < 1$, then the inclusions*

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\overline{\mathcal{U}}) \quad \text{for } \alpha \in (0, k - n/p),$$
$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, \infty)$$

are compact.

(2) *If $kp \le n$ and $p^* \in (p, \infty]$ is defined by the condition $1/p^* = 1/p - k/n$, then the inclusions*

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, p^*)$$

are compact.

□

REMARK A.1.11. Since the domain $\mathcal{U}$ in Theorem A.1.10 is bounded, the inclusion $W^{k+d,p} \hookrightarrow W^{d,q}$ for $kp \le n$ is also well defined and compact for $1 \le q < p$: this follows from the fact that $L^p(\mathcal{U}) \subset L^q(\mathcal{U})$ in this case since $\mathcal{U}$ has finite measure.

COROLLARY A.1.12. *If $\mathcal{U} \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then for every integer $k \ge 1$ and every $p \in [1, \infty)$, the natural inclusion*

$$W^{k,p}(\mathcal{U}) \hookrightarrow W^{k-1,p}(\mathcal{U})$$

is compact. □

The following simple estimate is commonly known as **Poincaré's inequality**.

THEOREM A.1.13 ([**AF03**, §6.30]). *Suppose $\mathcal{U} \subset \mathbb{R}^n$ is an open subset that lies between two parallel $(n-1)$-dimensional planes in $\mathbb{R}^n$. Then for each $p \in [1, \infty]$, there exists a constant $c > 0$ depending only on p such that*

$$\|f\|_{L^p(\mathcal{U})} \le c \|\nabla f\|_{L^p(\mathcal{U})} \quad \text{for all } f \in C_0^\infty(\mathcal{U}).$$

□

A.1.3. Products, compositions, and rescaling. The results of this section are especially useful for applications in *nonlinear* analysis. All of them may be viewed as corollaries of the Sobolev embedding theorem, so in particular they hold for the same class of domains $\mathcal{U} \subset \mathbb{R}^n$, and the restrictions on $\mathcal{U}$ can be dropped at the cost of replacing each space $W^{k,p}$ by $W_0^{k,p}$.

For the next result, we consider Sobolev spaces of functions valued in $\mathbb{C}$, so that pointwise products of functions are well defined almost everywhere (see Remark A.1.16 on further generalizations of this setting). We say that there is a **continuous product map**,

$$W^{k_1,p_1}(\mathcal{U}) \times \ldots \times W^{k_m,p_m}(\mathcal{U}) \to W^{k,p}(\mathcal{U}),$$

or a continuous product **pairing** in the case $m = 2$, if for every set of functions $f_i \in W^{k_i,p_i}(\mathcal{U})$ with $i = 1, \ldots, m$, the pointwise product function $f_1 \cdot \ldots \cdot f_m$ is in $W^{k,p}(\mathcal{U})$ and there is an estimate of the form

$$\|f_1 \cdot \ldots \cdot f_m\|_{W^{k,p}} \le c\|f_1\|_{W^{k_1,p_1}} \cdot \ldots \cdot \|f_m\|_{W^{k_m,p_m}}$$

for some constant $c > 0$ not depending on $f_1, \ldots, f_m$. The case $m = 2$, $k_1 = k_2 = k$ and $p_1 = p_2 = p$ is especially interesting, as the space $W^{k,p}(\mathcal{U})$ is then a **Banach algebra**. More generally, one can ask under what circumstances multiplication by functions of class $W^{k,p}$ defines a bounded linear operator on functions of class $W^{m,q}$. A hint about this comes from the world of classically differentiable functions: multiplication by C^k-smooth functions defines a continuous map $C^m \to C^m$ if and only if $k \ge m$. The corresponding answer in Sobolev spaces turns out to be that functions of class $W^{k,p}$ need to have strictly more than zero derivatives in the sense of Remark A.1.8, and at least as many derivatives as functions of class $W^{m,q}$.

THEOREM A.1.14. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, k, p, m and q satisfy the same numerical hypotheses as in Corollary A.1.9 (so in particular $W^{k,p}(\mathcal{U})$ embeds continuously into $W^{m,q}(\mathcal{U})$), and $kp > n$. Then there exists a continuous product pairing*

$$W^{k,p}(\mathcal{U}, \mathbb{C}) \times W^{m,q}(\mathcal{U}, \mathbb{C}) \to W^{m,q}(\mathcal{U}, \mathbb{C}) : (f, g) \mapsto fg.$$

COROLLARY A.1.15. *If $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition and $1 \le p < \infty$ and $k \in \mathbb{N}$ satisfy $kp > n$, then $W^{k,p}(\mathcal{U}, \mathbb{C})$ is a Banach algebra.*

REMARK A.1.16. While the results above are stated for spaces of complex-valued functions, they extend in obvious ways to functions valued in any spaces for which pointwise products can be defined, e.g. Theorem A.1.14 implies the existence of continuous product pairings

$$W^{k,p}(\mathcal{U}, \mathrm{Hom}(\mathbb{R}^a, \mathbb{R}^b)) \times W^{m,q}(\mathcal{U}, \mathbb{R}^a) \to W^{m,q}(\mathcal{U}, \mathbb{R}^b) : (f, g) \mapsto fg$$

under the same assumptions on k, p, q, m and $\mathcal{U}$.

The following preparatory lemma will be useful both for proving the product estimate and for further results below. It is an easy consequence of Theorem A.1.6 and Hölder's inequality.

LEMMA A.1.17. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $m \ge 2$ is an integer, and we are given positive numbers $p_1, \dots, p_m \ge 1$ and integers $k_1, \dots, k_m \ge 0$. Let $I := \{ i \in \{1, \dots, m\} \mid k_i p_i \le n \}$. Then for any $q \ge 1$ satisfying*

$$\sum_{i \in I} \left(\frac{1}{p_i} - \frac{k_i}{n} \right) < \frac{1}{q} \le \sum_{i=1}^m \frac{1}{p_i},$$

there is a continuous product map

$$W^{k_1,p_1}(\mathcal{U}) \times \dots \times W^{k_m,p_m}(\mathcal{U}) \to L^q(\mathcal{U}).$$

PROOF. By the generalized Hölder inequality (A.1.1), it suffices to show that for any $q \ge 1$ in the stated range, one can find numbers $q_1, \dots, q_m \in [q, \infty]$ satisfying $1/q = 1/q_1 + \dots + 1/q_m$ for which Theorem A.1.6 provides continuous inclusions

$$W^{k_i,p_i}(\mathcal{U}) \hookrightarrow L^{q_i}(\mathcal{U})$$

for each $i = 1, \dots, m$. Whenever $k_i p_i > n$, this inclusion is valid with q_i chosen freely from the interval $[p_i, \infty]$, so $1/q_i$ can then take any value subject to the constraint

$$0 \le \frac{1}{q_i} \le \frac{1}{p_i}.$$

If on the other hand $k_i p_i \le n$, then we can arrange $1/q_i$ to take any value in the range

$$\frac{1}{p_i} - \frac{k_i}{n} < \frac{1}{q_i} \le \frac{1}{p_i}.$$

Adding these up, the range of values for $\sum_i \frac{1}{q_i}$ that we can achieve in this way covers the stated interval. □

PROOF OF THEOREM A.1.14. By density of smooth functions, it suffices to prove that an estimate of the form

$$\|fg\|_{W^{m,q}} \le c \|f\|_{W^{k,p}} \|g\|_{W^{m,q}}$$

holds for all $f \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ and $g \in C^\infty(\mathcal{U}) \cap W^{m,q}(\mathcal{U})$. Equivalently, we need to show that for all f and g of this type and every multiindex α of degree $|\alpha| \le m$, there is a constant $c > 0$ independent of f and g such that

$$\|\partial^\alpha (fg)\|_{L^q} \le c \|f\|_{W^{k,p}} \|g\|_{W^{m,q}}.$$

Since f and g are smooth, we are free to use the product rule in computing $\partial^\alpha(fg)$, which will then be a linear combination of terms of the form $\partial^\beta f \cdot \partial^\gamma g$ where $|\alpha| = |\beta| + |\gamma|$, hence we have reduced the problem to proving a bound

$$\|\partial^\beta f \cdot \partial^\gamma g\|_{L^q} \le c \|f\|_{W^{k,p}} \|g\|_{W^{m,q}}$$

for every pair of multiindices β, γ with $|\beta| + |\gamma| \le m$. Since $\partial^\beta f \in W^{k-|\beta|,p}(\mathcal{U})$ and $\partial^\gamma f \in W^{m-|\gamma|,q}(\mathcal{U})$, the result follows if we can assume that for every pair of integers $a, b \ge 0$ satisfying $a + b \le m$, there exists a continuous product pairing

$$W^{k-a,p}(\mathcal{U}) \times W^{m-b,q}(\mathcal{U}) \to L^q(\mathcal{U}). \tag{A.1.7}$$

If $(k-a)p > n$, then $W^{k-a,p} \hookrightarrow L^\infty$ and (A.1.7) is immediate since $W^{m-b,q} \hookrightarrow L^q(\mathcal{U})$. For the remaining cases, we shall apply Lemma A.1.17, noting that the condition $1/q \le 1/p + 1/q$ is trivially satisfied.

If $(m-b)q > n$ but $(k-a)p \le n$, then the hypotheses of the lemma are satisfied if and only if

$$\frac{1}{p} - \frac{k-a}{n} < \frac{1}{q}.$$

Since $\frac{1}{p} - \frac{k}{n} \le \frac{1}{q} - \frac{m}{n}$ by assumption, we have

$$\frac{1}{p} - \frac{k-a}{n} = \frac{1}{p} - \frac{k}{n} + \frac{a}{n} \le \frac{1}{q} - \frac{m}{n} + \frac{a}{n} \le \frac{1}{q}$$

since $a \le m$, and equality holds only if $a = m$, $b = 0$ and $k - n/p = m - n/q$, which implies $mq > n$. In this case $W^{m-b,q} = W^{m,q} \hookrightarrow L^\infty$, and the pairing (A.1.7) follows because $W^{k-a,p} = W^{k-m,p}$ embeds continuously into L^q: the latter follows from Theorem A.1.6 since $\frac{1}{p} - \frac{k-m}{n} = \frac{1}{q}$.

Finally, when $(k-a)p \le n$ and $(m-b)q \le n$, the hypotheses of the lemma are satisfied since

$$\left(\frac{1}{p} - \frac{k-a}{n}\right) + \left(\frac{1}{q} - \frac{m-b}{n}\right) \le \frac{1}{p} - \frac{k}{n} + \frac{1}{q} - \frac{m}{n} + \frac{m}{n} = \left(\frac{1}{p} - \frac{k}{n}\right) + \frac{1}{q} < \frac{1}{q},$$

where we've used the assumption $kp > n$ and the fact that $a + b \le m$. $\square$

For the next result, we consider functions valued in a Euclidean space $\mathbb{R}^m$, and fix an open subset $\Omega \subset \mathbb{R}^m$. If $kp > n$ so that $W^{k,p}(\mathcal{U}, \mathbb{R}^m) \subset C^0(\overline{\mathcal{U}}, \mathbb{R}^m)$, then

$$W^{k,p}(\mathcal{U}, \Omega) := \left\{ f \in W^{k,p}(\mathcal{U}, \mathbb{R}^m) \mid \overline{f(\mathcal{U})} \subset \Omega \right\}$$

is an open subset of $W^{k,p}(\mathcal{U}, \mathbb{R}^m)$.

THEOREM A.1.18. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $\Omega \subset \mathbb{R}^m$ is an open subset, and $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$. Then for any $N \in \mathbb{N}$, the map*

$$C^k(\Omega, \mathbb{R}^N) \times W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (\Phi, f) \mapsto \Phi \circ f$$

is well defined and continuous, where $C^k(\Omega, \mathbb{R}^N)$ carries the topology of C^k-convergence on compact subsets.

PROOF. We will show first that if $f \in W^{k,p}(\mathcal{U}, \Omega)$ is smooth, then $\Phi \circ f$ belongs to $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ for every $\Phi \in C^k(\Omega, \mathbb{R}^N)$. Note that since $\overline{f(\mathcal{U})} \subset \Omega$ is necessarily compact, we lose no generality by shrinking Ω if necessary to a subset that still contains $\overline{f(\mathcal{U})}$ but has compact closure so that Φ belongs to the Banach space $C^k(\overline{\Omega}, \mathbb{R}^N)$. Then

$$|\Phi \circ f(x) - \Phi(0)| \le \|\Phi\|_{C^1(\Omega)} \cdot |f(x)|$$

for all $x \in \mathcal{U}$, implying

$$\|\Phi \circ f - \Phi(0)\|_{L^p} \le \|\Phi\|_{C^1(\Omega)} \cdot \|f\|_{L^p},$$

hence $\Phi \circ f \in L^p(\mathcal{U}, \mathbb{R}^N)$.

For $\ell = 1, \dots, k$, we can regard the ℓth derivative of Φ as a bounded and uniformly continuous map from Ω into the vector space of symmetric k-multilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^N$, denoting this by

$$D^\ell \Phi : \Omega \to \operatorname{Hom}((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^N).$$

Now for any multiindex α with $|\alpha| \le k$, the derivative $\partial^\alpha(\Phi \circ f)$ is a linear combination of terms of the form

$$(D^\ell \Phi \circ f)(\partial^{\beta_1} f, \dots, \partial^{\beta_\ell} f), \tag{A.1.8}$$

where $\ell \in \{1, \dots, |\alpha|\}$ and $|\beta_1| + \ldots + |\beta_\ell| = |\alpha|$. This expression satisfies

$$\left\|(D^\ell \Phi \circ f)(\partial^{\beta_1} f, \dots, \partial^{\beta_\ell} f)\right\|_{L^p(\mathcal{U})} \le \|D^\ell \Phi\|_{C^0(\Omega)} \cdot \left\| \prod_{j=1}^{\ell} |\partial^{\beta_j} f| \right\|_{L^p(\mathcal{U})}$$

if the product on the right hand side has finite L^p-norm. The latter is trivially true if $\ell = 1$. To deal with the $\ell \ge 2$ case, note that $\partial^{\beta_j} f \in W^{k-|\beta_j|,p}(\mathcal{U})$ for each $j = 1, \dots, \ell$, so the necessary bound will follow from the existence of a continuous product map

$$W^{k-m_1,p}(\mathcal{U}) \times \ldots \times W^{k-m_\ell,p}(\mathcal{U}) \to L^p(\mathcal{U})$$

for $m_j := |\beta_j|$, and we claim that such a product map does exist whenever $kp > n$ and $m_1, \dots, m_\ell \ge 0$ are integers satisfying $m_1 + \ldots + m_\ell \le k$. To see this, note first that since $W^{k-m_j,p} \hookrightarrow L^\infty$ whenever $(k - m_j)p > n$, it suffices to prove the claim under the assumption that $(k - m_j)p \le n$ for every $j = 1, \dots, \ell$. In this case, Lemma A.1.17 provides the desired product map if the condition

$$\sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) < \frac{1}{p} \le \sum_{j=1}^{\ell} \frac{1}{p}$$

is satisfied. And it is: using $kp > n$, $\ell \ge 2$ and $m_1 + \ldots + m_\ell \le k$, we find

$$\begin{aligned} \sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) &= \ell \left(\frac{1}{p} - \frac{k}{n} \right) + \frac{m_1 + \ldots + m_\ell}{n} \\ &\le \frac{1}{p} + (\ell - 1) \left(\frac{1}{p} - \frac{k}{n} \right) < \frac{1}{p}. \end{aligned}$$

We've shown that $\Phi \circ f \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$, and moreover,

$$\|\Phi \circ f - \Phi(0)\|_{W^{k,p}} \le \|\Phi\|_{C^k(\Omega)} \cdot C(\|f\|_{W^{k,p}})$$

for some continuous function $C : [0, \infty) \to [0, \infty)$ with $C(0) = 0$.

Next, suppose $f \in W^{k,p}(\mathcal{U}, \Omega)$ is not necessarily smooth but $f_i \in W^{k,p}(\mathcal{U}, \Omega)$ is a sequence of smooth functions converging to f in $W^{k,p}$, while $\Phi_i \in C^k(\overline{\Omega}, \mathbb{R}^N)$ converges to $\Phi \in C^k(\overline{\Omega}, \mathbb{R}^N)$ in C^k. Then f_i is also C^0-convergent, thus $D^\ell \Phi_i \circ f_i$ converges to $D^\ell \Phi \circ f$ in $C^0(\overline{\mathcal{U}}, \mathbb{R}^N)$, and each of the derivatives $\partial^{\beta_j} f_i$ appearing in (A.1.8) also converges in $L^p(\mathcal{U})$. In light of the continuous product maps discussed above, it follows that each derivative $\partial^\alpha(\Phi_i \circ f_i)$ for $|\alpha| \le k$ is L^p-convergent, and its limit is necessarily the corresponding weak derivative $\partial^\alpha(\Phi \circ f)$, hence (see Exercise A.1.19 below) $\Phi \circ f \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ and $\Phi_i \circ f_i \xrightarrow{W^{k,p}} \Phi \circ f$. □

EXERCISE A.1.19. Show that if f_i is a sequence of smooth functions on an open set $\mathcal{U} \subset \mathbb{R}^n$ with $f_i \xrightarrow{L^p} f$ and $\partial^\alpha f_i \xrightarrow{L^p} g$ for some multiindex α and functions $f, g \in L^p(\mathcal{U})$, then $\partial^\alpha f = g$ in the sense of distributions.

The following result on coordinate transformations of the domain can be proved in an analogous way to Theorem A.1.18, though it is considerably easier since there is no need to worry about Sobolev product maps (and thus no need to assume $kp > n$ or impose regularity conditions on the domain).

THEOREM A.1.20 ([**AF03**, §3.41]). *Assume $k \in \mathbb{N}$, $1 \le p \le \infty$, and $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open subsets with a C^k-smooth diffeomorphism $\varphi : \mathcal{U} \to \mathcal{U}'$ such that all derivatives of φ and φ^{-1} up to order k are bounded and uniformly continuous. Then there is a well-defined Banach space isomorphism*

$$W^{k,p}(\mathcal{U}') \to W^{k,p}(\mathcal{U}) : f \mapsto f \circ \varphi.$$

□

REMARK A.1.21. One should be careful about applying too much topological intuition to Theorem A.1.20. For example, there exists a diffeomorphism φ from the open unit ball B^n to $\mathbb{R}^n$, but $f \mapsto f \circ \varphi$ cannot define a surjective map $W^{k,p}(\mathbb{R}^n) \to W^{k,p}(B^n)$; in particular the image of this map cannot include functions in $C^k(\overline{B^n})$ that fail to decay to zero at the boundary. The problem here is that while φ may be a diffeomorphism, it is not bounded and neither are its derivatives, i.e. $\varphi \notin C^k(\overline{B^n}, \mathbb{R}^n)$, although $\varphi^{-1} \in C^k(\overline{\mathbb{R}^n}, B^n)$.

We conclude this section with a rescaling lemma that is useful for the proof of local nonlinear regularity in §2.11. We denote by

$$B^n, B^n_\epsilon \subset \mathbb{R}^n$$

the open balls of radius 1 and ϵ respectively about the origin.

LEMMA A.1.22. *Assume $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, and for each $f \in W^{k,p}(B^n)$ and $\epsilon \in (0, 1]$, define $f_\epsilon \in W^{k,p}(B^n)$ by*

$$f_\epsilon(x) := f(\epsilon x).$$

Then there exist constants $C > 0$ and $r > 0$ such that for every $f \in W^{k,p}(B^n)$,

$$\|f_\epsilon - f(0)\|_{W^{k,p}(B^n)} \le C\epsilon^r \|f - f(0)\|_{W^{k,p}(B^n)} \quad \text{for all } \epsilon \in (0, 1].$$

PROOF. Let β denote a multiindex of order $|\beta| = k$. Then using a change of variables, we have

$$\begin{aligned}\|\partial^\beta(f_\epsilon - f(0))\|^p_{L^p(B^n)} &= \epsilon^{kp} \int_{B^n} |\partial^\beta f(\epsilon x)|^p = \epsilon^{kp-n} \int_{B^n_\epsilon} |\partial^\beta f(x)|^p \\ &\le \epsilon^{kp-n} \|\partial^\beta f\|_{L^p(B^n)} \le \epsilon^{kp-n} \|f - f(0)\|_{W^{k,p}(B^n)},\end{aligned}$$

and $\epsilon^{kp-n} \to 0$ as $\epsilon \to 0$ since $kp - n > 0$.

Next, suppose $|\beta| = m \in \{1, \dots, k-1\}$. Then $\partial^\beta f$ and $\partial^\beta f_\epsilon$ are in $W^{k-m,p}(B^n)$, and if $(k-m)p < n$, Theorem A.1.6 gives a continuous inclusion

$$W^{k-m,p}(B^n) \hookrightarrow L^q(B^n) \tag{A.1.9}$$

with $q > p$ satisfying $1/q + (k-m)/n = 1/p$. Likewise, if $(k-m)p \ge n$, then (A.1.9) is a continuous inclusion for arbitrarily large choices of $q \ge p$. We will therefore assume in general that (A.1.9) holds with $q \in (p, \infty)$ satisfying

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p},$$

where $r = \frac{n}{k-m}$ if $(k-m)p < n$ and otherwise $r = p + \delta$ for some $\delta > 0$ which may be chosen arbitrarily small. Given this, we use Hölder's inequality and find

$$\begin{aligned}
\|\partial^\beta(f_\epsilon - f(0))\|_{L^p(B^n)} &= \epsilon^{mp} \int_{B^n} |\partial^\beta f(\epsilon x)|^p = \epsilon^{mp-n} \int_{B^n_\epsilon} |\partial^\beta f(x)|^p \\
&\le \epsilon^{mp-n} \|\partial^\beta f\|^p_{L^q(B^n_\epsilon)} \|1\|^p_{L^r(B^n_\epsilon)} \\
&\le \epsilon^{mp-n} \left[\mathrm{Vol}(B^n_\epsilon)\right]^{p/r} \|\partial^\beta f\|^p_{L^q(B^n)} \\
&\le c\epsilon^{mp-n} \left[\mathrm{Vol}(B^n_\epsilon)\right]^{p/r} \|\partial^\beta f\|^p_{W^{k-m,p}(B^n)} \\
&\le c\epsilon^{mp-n} \left[\mathrm{Vol}(B^n_\epsilon)\right]^{p/r} \|f - f(0)\|^p_{W^{k,p}(B^n)}
\end{aligned}$$

for some constant $c > 0$. Writing $\mathrm{Vol}(B^n_\epsilon) = C\epsilon^n$ for a suitable constant $C > 0$, the exponent on ϵ in this expression becomes

$$mp - n + \frac{np}{r},$$

which is positive whenever $r = p + \delta$ with $\delta > 0$ sufficiently small since $m \ge 1$, and in the case $r = n/(k-m)$, it becomes simply $kp - n > 0$.

Finally, to bound the L^p-norm of $f_\epsilon - f(0)$ itself, we can use the fact that $f \in W^{k,p}$ is Hölder continuous, i.e. it satisfies

$$|f(x) - f(0)| \le c\|f - f(0)\|_{W^{k,p}(B^n)} |x|^\alpha \quad \text{for all } x \in B^n$$

for suitable constants $c > 0$ and $\alpha \in (0,1)$. Thus

$$\begin{aligned}
\|f_\epsilon - f(0)\|^p_{L^p(B^n)} &= \int_{B^n} |f(\epsilon x) - f(0)|^p \le c^p \|f - f(0)\|^p_{W^{k,p}} \int_{B^n} |\epsilon x|^{\alpha p} \\
&= c^p \|f - f(0)\|^p_{W^{k,p}} \epsilon^{\alpha p} \int_{B^n} |x|^{\alpha p} \\
&= \epsilon^{\alpha p} \frac{c^p \,\mathrm{Vol}(S^{n-1})}{\alpha p + n} \|f - f(0)\|^p_{W^{k,p}}.
\end{aligned}$$

□

A.1.4. Difference quotients. If f is a function on $\mathbb{R}^n$, then for every $i = 1, \ldots, n$ and $h \in \mathbb{R} \setminus \{0\}$, the **difference quotient**

$$D_i^h f(x_1, \ldots, x_n) := \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

defines a function $D_i^h f$ on $\mathbb{R}^n$. The **total difference quotient** of f is then the n-tuple of functions

$$D^h f := (D_1^h f, \ldots, D_n^h f),$$

so for example if $f : \mathbb{R}^n \to \mathbb{R}^m$, then $D^h f : \mathbb{R}^n \to \mathbb{R}^{mn}$. The transformation $f \mapsto D_i^h f$ is obviously linear for any fixed number h, and it satisfies a Leibniz rule

$$D_i^h(fg) = D_i^h f \cdot g + f \cdot D_i^h g$$

whenever pointwise products of f and g can be defined (e.g. if both are real or complex valued). It also commutes with differentiation

$$D_i^h(\partial_j f) = \partial_j(D_i^h f)$$

on any function f for which $\partial_j f$ can be defined (weakly or strongly). Clearly if $f \in W^{k,p}(\mathbb{R}^n)$, then $D^h f \in W^{k,p}(\mathbb{R}^n)$ for every $h \in \mathbb{R} \setminus 0$, and if f is supported in an open subset $\mathcal{U} \subset \mathbb{R}^n$, then $D^h f$ is supported in an arbitrarily small neighborhood of $\overline{\mathcal{U}}$ for sufficiently small $|h|$. Moreover, if f is a function defined only on $\mathcal{U} \subset \mathbb{R}^n$, then on any open subset $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$, $D^h f$ can be defined on $\mathcal{V}$ for $|h|$ sufficiently small.

The following result about difference quotients is useful for proving local regularity of solutions to PDEs, as in Chapter 2.

THEOREM A.1.23. *Assume $\mathcal{V} \subset \mathcal{U} \subset \mathbb{R}^n$ are open subsets with $\mathcal{V}$ having compact closure contained in $\mathcal{U}$, $1 \le p < \infty$, and $k \in \mathbb{N}$.*

1. *If $f \in W^{k,p}(\mathcal{U})$, then $D^h f$ converges in $W^{k-1,p}(\mathcal{V})$ to ∇f as $h \to 0$.*
2. *Suppose $p > 1$, $f \in W^{k-1,p}(\mathcal{U})$ and the difference quotients $D^h f$ satisfy a uniform bound in $W^{k-1,p}(\mathcal{V})$ as $h \to 0$. Then $f|_{\mathcal{V}} \in W^{k,p}(\mathcal{V})$.*

The next few results are intended as preparation for the proof of Theorem A.1.23.

LEMMA A.1.24. *For any open subset $\mathcal{U} \subset \mathbb{R}^n$ and continuously differentiable function f on $\mathcal{U}$, the difference quotients $D_i^h f$ converge to $\partial_i f$ uniformly on compact subsets as $h \to 0$.*

PROOF. Fix a compact subset $\mathcal{K} \subset \mathcal{U}$. Then for every $x \in \mathcal{K}$ and $h \in \mathbb{R} \setminus \{0\}$ sufficiently small, the mean value theorem gives

$$D_i^h f(x) = \partial_i f(x')$$

where

$$x' := (x_1, \ldots, x_{i-1}, x_i + th, x_{i+1}, \ldots, x_n) \in \mathcal{U}$$

for some $t \in [0, 1]$, so in particular, $|x' - x| \le |h|$. We then have $|\partial_i f(x) - D_i^h f(x)| = |\partial_i f(x) - \partial_i f(x')|$, and the result follows since both x and x' may be assumed to lie in a compact subset of $\mathcal{U}$, on which $\partial_i f$ is uniformly continuous. □

PROPOSITION A.1.25. *Suppose $1 \le p < \infty$, $\mathcal{U} \subset \mathbb{R}^n$ is an open subset and $f \in W^{1,p}(\mathcal{U})$. Then on any open subset $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$, $D^h f \to \nabla f$ in $L^p(\mathcal{V})$ as $h \to 0$.*

PROOF. We show first that for any $f \in W^{1,p}(\mathcal{U})$,

$$\|D_i^h f\|_{L^p(\mathcal{V})} \le \|\partial_i f\|_{L^p(\mathcal{U})}, \qquad i = 1, \dots, n \tag{A.1.10}$$

for every $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$ and every $h \ne 0$ sufficiently small such that $D_i^h f$ is defined on $\mathcal{V}$. Indeed, if $f \in W^{1,p}(\mathcal{U}) \cap C^\infty(\mathcal{U})$, then denoting the standard basis of $\mathbb{R}^n$ by $(e_1, \dots, e_n)$, we have

$$\begin{aligned}
\left|D_i^h f(x)\right| = \left|\frac{f(x+he_i) - f(x)}{h}\right| &= \left|\frac{1}{h}\int_0^1 \frac{d}{dt} f(x + the_i)\, dt\right| \\
&= \left|\int_0^1 \partial_i f(x + the_i)\, dt\right| \le \int_0^1 |\partial_i f(x + the_i)|\, dt.
\end{aligned}$$

Then since any measurable function $g : [0,1] \to \mathbb{R}$ satisfies

$$\left(\int_0^1 |g(t)|\, dt\right)^p \le \int_0^1 |g(t)|^p\, dt$$

by Jensen's inequality, this gives

$$\begin{aligned}
\|D_i^h f\|_{L^p(\mathcal{V})}^p &= \int_{\mathcal{V}} \left|D_i^h f(x)\right|^p d\mu(x) \le \int_{\mathcal{V}} \left(\int_0^1 |\partial_i f(x + the_i)|\, dt\right)^p d\mu(x) \\
&\le \int_{\mathcal{V}} \int_0^1 |\partial_i f(x + the_i)|^p\, dt\, d\mu(x) = \int_0^1 \int_{\mathcal{V}} |\partial_i f(x + the_i)|^p\, d\mu(x)\, dt \\
&\le \int_0^1 \|\partial_i f\|_{L^p(\mathcal{U})}^p\, dt = \|\partial_i f\|_{L^p(\mathcal{U})}^p.
\end{aligned}$$

This estimate extends to every $f \in W^{1,p}(\mathcal{U})$ by density of smooth functions.

Next, suppose $f \in W^{1,p}(\mathcal{U})$ and $\epsilon > 0$ is given. Choose a smooth approximation $f_\epsilon \in W^{1,p}(\mathcal{U}) \cap C^\infty(\mathcal{U})$ with $\|f - f_\epsilon\|_{W^{1,p}(\mathcal{U})} < \epsilon/3$. By Proposition A.1.25, $D_i^h f_\epsilon \to \partial_i f_\epsilon$ in C^0_{loc} on $\mathcal{U}$ as $h \to 0$, and since $\mathcal{V}$ has finite measure, this implies we can find $\delta > 0$ such that $|h| < \delta$ implies $\|D_i^h f_\epsilon - \partial_i f_\epsilon\|_{L^p(\mathcal{V})} < \epsilon/3$. Now by (A.1.10),

$$\|D_i^h f_\epsilon - D_i^h f\|_{L^p(\mathcal{V})} \le \|\partial_i f_\epsilon - \partial_i f\|_{L^p(\mathcal{U})} \le \|f_\epsilon - f\|_{W^{1,p}(\mathcal{U})} < \epsilon/3,$$

so combining these estimates gives $\|D_i^h f - \partial_i f\|_{L^p(\mathcal{V})} < \epsilon$ whenever $|h| < \delta$. □

The proof of the next proposition will require the following standard result from real analysis, known as the **Banach-Alaoglu theorem**; see for instance [**LL01**, §2.18].

THEOREM A.1.26 (Banach-Alaoglu). *For any measurable subset $\mathcal{U} \subset \mathbb{R}^n$, if $1 < p < \infty$, then every bounded sequence $f_j \in L^p(\mathcal{U})$ has a weakly convergent subsequence, i.e. after passing to a subsequence, one can find a function $f_\infty \in L^p(\mathcal{U})$ such that for every $\varphi \in L^q(\mathcal{U})$ with $1/p + 1/q = 1$, $\int_{\mathcal{U}} f_j \varphi \to \int_{\mathcal{U}} f_\infty \varphi$.* □

PROPOSITION A.1.27. *Suppose $\mathcal{V} \subset \mathcal{U} \subset \mathbb{R}^n$ are open subsets such that $\mathcal{V}$ has compact closure contained in $\mathcal{U}$, $1 < p < \infty$, f is a measurable function on $\mathcal{U}$ with $\|f\|_{L^p(\mathcal{V})} < \infty$, and there exist constants $C > 0$ and $\delta > 0$ such that*

$$\|D_i^h f\|_{L^p(\mathcal{V})} \le C \qquad \text{whenever } 0 < |h| < \delta.$$

Then $f|_{\mathcal{V}}$ has a weak partial derivative $\partial_i f$ in $L^p(\mathcal{V})$.

PROOF. For any sequence $h_j \to 0$ of sufficiently small nonzero real numbers, the sequence $D_i^{h_j} f$ is bounded in $L^p(\mathcal{V})$, thus the Banach-Alaoglu theorem implies that after passing to a subsequence, one finds a function $g \in L^p(\mathcal{V})$ such that

$$\int_{\mathcal{V}} (D_i^{h_j} f)\varphi \to \int_{\mathcal{V}} g\varphi$$

for all $\varphi \in L^q(\mathcal{V})$, where $1/p + 1/q = 1$. In particular, this is true for all test functions $\varphi \in C_0^\infty(\mathcal{V})$, and in this case there is an "integration by parts" relation

$$\begin{aligned} \int_{\mathcal{V}} (D_i^{h_j} f)\varphi &= \int_{\mathcal{V}} \frac{f(x + h_j e_i) - f(x)}{h_j} \varphi(x)\, d\mu(x) \\ &= -\int_{\mathcal{V}} f(x) \frac{\varphi(x - h_j e_i) - \varphi(x)}{-h_j}\, d\mu(x) = -\int_{\mathcal{V}} f D_i^{-h_j} \varphi. \end{aligned}$$

By Lemma A.1.24, $D_i^{-h_j}\varphi \to \partial_i \varphi$ uniformly on $\mathcal{V}$ and thus also in $L^q(\mathcal{V})$, so taking the limit of the integrals, we've shown

$$\int_{\mathcal{V}} g\varphi = -\int_{\mathcal{V}} f \partial_i \varphi \qquad \text{for all } \varphi \in C_0^\infty(\mathcal{V}),$$

in other words, $\partial_i f = g \in L^p(\mathcal{V})$. ☐

PROOF OF THEOREM A.1.23. The two statements in the theorem follow by applying Propositions A.1.25 and A.1.27 respectively to $\partial^\alpha f$ for every multiindex α with $|\alpha| \le k - 1$, using the fact that $D^h(\partial^\alpha f) = \partial^\alpha(D^h f)$. ☐

A.2. Compact manifolds

In this and the next section, suppose M is a smooth n-dimensional manifold, possibly with boundary, and $\pi : E \to M$ is a smooth complex vector bundle of rank m. This comes with a "bundle atlas" $\mathcal{A}(\pi)$, a set whose elements $\alpha \in \mathcal{A}(\pi)$ consist of the following data:

(1) An open subset $\mathcal{U}_\alpha \subset M$;
(2) A smooth local coordinate chart $\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha$, where Ω_α is an open subset of $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge 0\}$;
(3) A smooth local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{C}^m$.

Smoothness of φ_α and Φ_α means as usual that for every pair $\alpha, \beta \in \mathcal{A}(\pi)$, the coordinate transformations

$$\psi_{\beta\alpha} := \varphi_\beta^{-1} \circ \varphi_\alpha : \Omega_{\alpha\beta} \xrightarrow{\cong} \Omega_{\beta\alpha}, \qquad \Omega_{\alpha\beta} := \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

and transition maps

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{C}) \quad \text{such that} \quad \begin{array}{l} \Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_{\beta\alpha}(x)v) \\ \text{for } x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta,\ v \in \mathbb{C}^m \end{array}$$

are smooth, and we shall assume the bundle atlas is maximal in the sense that any triple $(\mathcal{U}, \varphi, \Phi)$ that is smoothly compatible with every $\alpha \in \mathcal{A}(\pi)$ also belongs to $\mathcal{A}(\pi)$.

Any $\alpha \in \mathcal{A}(\pi)$ now associates to sections $\eta : M \to E$ their local coordinate representatives

$$\eta^\alpha := \mathrm{pr}_2 \circ \Phi_\alpha \circ \eta \circ \varphi_\alpha^{-1} : \Omega_\alpha \to \mathbb{C}^m,$$

where $\mathrm{pr}_2 : \mathcal{U}_\alpha \times \mathbb{C}^m \to \mathbb{C}^m$, and the representatives with respect to two distinct $\alpha, \beta \in \mathcal{A}(\pi)$ are related by

$$\eta^\beta = (g_{\beta\alpha} \circ \varphi_\beta^{-1})(\eta^\alpha \circ \psi_{\alpha\beta}) \quad \text{on } \Omega_{\beta\alpha}.$$

For $p \in [1, \infty]$ and each integer $k \geq 0$, we then define the topological vector space of sections of class $W^{k,p}_{\mathrm{loc}}$ by

$$W^{k,p}_{\mathrm{loc}}(E) := \big\{ \eta : M \to E \;\big|\; \text{sections such that } \eta^\alpha \in W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{C}^m) \text{ for all } \alpha \in \mathcal{A}(\pi) \big\},$$

where convergence $\eta_j \to \eta$ in $W^{k,p}_{\mathrm{loc}}(E)$ means that $\eta_j^\alpha \to \eta^\alpha$ in $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{C}^m)$ for all $\alpha \in \mathcal{A}(\pi)$. Note that Ω_α is not necessarily an open subset of $\mathbb{R}^n$ since it may contain points in $\partial \mathbb{R}^n_+ = \{0\} \times \mathbb{R}^{n-1}$, but its interior $\mathring{\Omega}_\alpha$ is open in $\mathbb{R}^n$, and $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha)$ is thus defined as in §A.1.1.

It turns out that $W^{k,p}_{\mathrm{loc}}(E)$ can be given the structure of a Banach space if M is compact. This results from the fact that M can then be covered by a finite subset of the atlas $\mathcal{A}(\pi)$, but we must be a little bit careful: not all charts in $\mathcal{A}(\pi)$ are equally suitable for defining $W^{k,p}$-norms on sections, because e.g. even a nice smooth section $\eta \in \Gamma(E)$ may have $\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)} = \infty$ if $\Omega_\alpha \subset \mathbb{R}^n_+$ is unbounded. One way to deal with this is as follows: we will say that $\alpha \in \mathcal{A}(\pi)$ is a **precompact chart** if there exists $\alpha' \in \mathcal{A}(\pi)$ and a compact subset $\mathcal{K} \subset M$ such that

$$\mathcal{U}_\alpha \subset \mathcal{K} \subset \mathcal{U}_{\alpha'}.$$

When this is the case, $\Omega_\alpha \subset \mathbb{R}^n_+$ is necessarily bounded, and the transition maps between two precompact charts necessarily have bounded derivatives of all orders, as they are restrictions to precompact subsets of maps that are smooth on larger domains. If M is compact, then one can always find a finite subset $I \subset \mathcal{A}(\pi)$ such that $M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$.

DEFINITION A.2.1. Suppose $E \to M$ is a smooth vector bundle over a compact manifold M, and $I \subset \mathcal{A}(\pi)$ is a finite set of precompact charts such that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of M. We then define $W^{k,p}(E)$ as the vector space of all sections $\eta : M \to E$ for which the norm

$$\|\eta\|_{W^{k,p}} := \|\eta\|_{W^{k,p}(E)} := \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}$$

is finite.

The norm in the above definition depends on auxiliary choices, but it is easy to see that the resulting definition of the space $W^{k,p}(E)$ and its topology do not. In fact:

LEMMA A.2.2. *If M is compact, then $W^{k,p}(E) = W^{k,p}_{\mathrm{loc}}(E)$, and a sequence η_j converges to η in $W^{k,p}_{\mathrm{loc}}(E)$ if and only if the norm given in Definition A.2.1 satisfies $\|\eta_j - \eta\|_{W^{k,p}(E)} \to 0$.*

PROOF. It suffices to show that if $\|\cdot\|_{W^{k,p}(E)}$ is given by Definition A.2.1, then for every precompact chart $\beta \in \mathcal{A}(\pi)$, there is a constant $c > 0$ such that

$$\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} \le c\|\eta\|_{W^{k,p}(E)}.$$

To see this, choose a partition of unity $\{\rho_\alpha : M \to [0,1]\}_{\alpha\in I}$ subordinate to the finite open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$. Now $\eta = \sum_{\alpha\in I}\rho_\alpha\eta$, and each $\rho_\alpha\eta$ is supported in $\mathcal{U}_\alpha$, so $(\rho_\alpha\eta)^\beta$ has support in $\Omega_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus using Theorem A.1.20 with the fact that $g_{\beta\alpha}$, φ_β^{-1}, $\psi_{\alpha\beta}$ and $\psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1}$ are all smooth functions with bounded derivatives of all orders on the domains in question, we find

$$\begin{aligned}
\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} &= \left\|\sum_{\alpha\in I}(\rho_\alpha\eta)^\beta\right\|_{W^{k,p}(\mathring{\Omega}_\beta)} \le \sum_{\alpha\in I}\|(\rho_\alpha\eta)^\beta\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \\
&= \sum_{\alpha\in I}\|(\rho_\alpha\circ\varphi_\beta^{-1})(g_{\beta\alpha}\circ\varphi_\beta^{-1})(\eta^\alpha\circ\psi_{\alpha\beta})\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \\
&\le c\sum_{\alpha\in I}\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_{\alpha\beta})} \le c\sum_{\alpha\in I}\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}.
\end{aligned}$$

□

COROLLARY A.2.3. *If M is compact, the norm on $W^{k,p}(E)$ given by Definition A.2.1 is independent of all auxiliary choices up to equivalence of norms.*

With these fundamentals in place, it is an easy exercise to generalize most of the results of §A.1 to the setting of a bundle over a compact manifold.

continue from here

A.3. Domains with cylindrical ends

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