Problem Set 9

To be discussed: 13.07.2022

Notation:
For the first two problems, assume V is an n-dimensional vector space equipped with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$, which is used in the definition of the Clifford algebra $\mathrm{Cl}(V)$ and spin group $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)$. We denote by $\mathrm{SO}(V)$ the group of orientation-preserving linear maps $A : V \to V$ that satisfy $\langle Av, Aw\rangle = \langle v, w\rangle$ for all $v, w \in V$.

Problem 1
For any codimension 1 subspace $H \subset V$ on which the restriction of $\langle\ ,\ \rangle$ is nondegenerate, one can define the *reflection about* H as the unique linear map $V \to V$ that fixes every point in H but sends $v \mapsto -v$ for all $v \in H^\perp$. (Note that this definition does not make sense if $\langle\ ,\ \rangle|_H$ is degenerate, because $H^\perp$ is then contained in H; see Lemma 24.7 in the notes from the first semester.)

(a) For $x \in V$ with $\langle x, x\rangle = \pm 1$, show that the reflection $V \to V$ about $x^\perp \subset V$ is given by $v \mapsto -xvx^{-1}$.

(b) Deduce that for each $x \in \mathrm{Spin}(V)$, the transformation $\mathrm{Ad}_x : \mathrm{Cl}(V) \to \mathrm{Cl}(V) : y \mapsto xyx^{-1}$ preserves the subspace $V \subset \mathrm{Cl}(V)$ and acts on it by orientation-preserving orthogonal transformations, defining a group homomorphism $\Phi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$.

Problem 2
Given an orthonormal basis $e_1, \ldots, e_n \in V$, let $\mathfrak{spin}(V) \subset \mathrm{Cl}(V)$ denote the vector space spanned by all products of the form $e_i e_j$ for $i \neq j$. Prove:

(a) $\mathfrak{spin}(V) \subset \mathrm{Cl}(V)$ does not depend on the choice of orthonormal basis $e_1, \ldots, e_n \in V$.

(b) $\mathfrak{spin}(V)$ is a Lie algebra with respect to the commutator bracket $[x, y] := xy - yx$.

(c) For any $v, w \in V$ satisfying $\langle v, v\rangle = \pm 1$, $\langle w, w\rangle = \pm 1$ and $\langle v, w\rangle = 0$, we have $vw \in \mathfrak{spin}(V)$ and $e^{\frac{1}{2}tvw} \in \mathrm{Spin}(V)$ for all $t \in \mathbb{R}$, where for $x \in \mathrm{Cl}(V)$, we define $e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} \in \mathrm{Cl}(V)$.

In the following, $\Phi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is the homomorphism from Problem 1(b).

(d) Under the assumptions of part (c), can you give a geometric interpretation to the family of transformations $\Phi(e^{\frac{1}{2}tvw}) \in \mathrm{SO}(V)$?
Hint: Evaluate $\Phi(e^{\frac{1}{2}vw})$ on v and w and on an arbitrary vector orthogonal to both.

(e) Construct a smooth map $\varphi : \mathfrak{spin}(V) \to \mathrm{Cl}(V)$ whose derivative at $0 \in \mathfrak{spin}(V)$ is the inclusion $\mathfrak{spin}(V) \hookrightarrow \mathrm{Cl}(V)$, such that the image of φ is in $\mathrm{Spin}(V)$ and the derivative of $\Phi \circ \varphi : \mathfrak{spin}(V) \to \mathrm{SO}(V)$ at 0 is a Lie algebra isomorphism $\mathfrak{spin}(V) \to \mathfrak{so}(V)$.
Hint: Using the orthonormal basis $e_1, \ldots, e_n \in V$, first define $\varphi(te_ie_j)$ for each $t \in \mathbb{R}$ and $i \neq j$, then extend it to the rest of $\mathfrak{spin}(V)$ in whatever way is convenient.

Comment: If you find this problem intimidating, try attacking a special case such as $V = \mathbb{R}^2$ or $\mathbb{R}^3$ with the Euclidean inner product. As outlined in the notes, one can combine the result with an algebraic computation of $\ker \Phi$ to prove that $\mathrm{Spin}(V)$ is a Lie group *and* $\Phi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ *is a covering map of degree* 2.

Problem 3
Let $\boldsymbol{\sigma}_i \in \mathbb{C}^{2\times 2}$ for $i = 1, 2, 3$ denote the Pauli matrices defined in §39.2 of the lecture notes, and let $\boldsymbol{\sigma}_0 = \mathbb{1}$. These four matrices form a basis of the real 4-dimensional vector space $H \subset \mathbb{C}^{2\times 2}$ consisting of all Hermitian 2-by-2 matrices. Show that if $\mathbb{R}^4$ is identified with H in this way, then the $\mathrm{SL}(2,\mathbb{C})$-action on $\mathbb{R}^4$ defined by

$$\mathbf{A} \cdot \mathbf{B} := \mathbf{A}\mathbf{B}\mathbf{A}^\dagger \qquad \text{for } \mathbf{A} \in \mathrm{SL}(2,\mathbb{C}) \text{ and } \mathbf{B} \in H$$

defines a Lie group homomorphism $\Phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(1,3) \subset \mathrm{GL}(4,\mathbb{R})$ with $\ker \Phi = \{\pm \mathbb{1}\}$. What does this tell you about the relationship between the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{Spin}(1,3)$? *(Caution:* $\mathrm{SO}(1,3)$ *is not connected!)*
Hint: What is the determinant of a real-linear combination of the $\boldsymbol{\sigma}_\mu$ *for* $\mu = 0, \dots, 3$*?*

Problem 4
Since $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ are naturally isomorphic, the tautological complex line bundle $E \to \mathbb{CP}^n$ with its standard bundle metric can also be viewed as an $\mathrm{SO}(2)$-bundle, meaning an oriented Euclidean vector bundle of rank 2. Show that this bundle does not admit a spin structure. You may use as a black box the following standard fact from covering space theory: if M is simply connected, then every covering map $\widetilde{M} \to M$ is a homeomorphism.

Problem 5
For $n \geqslant 2$, $\mathbb{CP}^n$ is a simply connected $2n$-manifold that is not homeomorphic to S^{2n} or $\mathbb{R}^{2n}$, so by a theorem proved in lecture, it cannot admit any Riemannian metric with constant sectional curvature. Prove however that it does admit a metric that is homogeneous and isotropic.

Problem 6
A *Riemannian symmetric space* is a Riemannian manifold (M, g) such that for every point $p \in M$, there exists an isometry $\psi \in \mathrm{Isom}(M, g)$ with $\psi(p) = p$ and $T_p\psi = -\mathbb{1}$. (Note that unlike the notion of *locally* symmetric Riemannian manifolds we defined in lecture, the isometry ψ is required to be defined globally.) Prove that every Riemannian symmetric space is homogeneous.

Problem 7
Find an explicit example of a closed Riemannian manifold that is homogeneous but not isotropic.

Problem 8
In lecture we proved that every simply connected and complete Riemannian manifold (M, g) with constant positive sectional curvature $K_S = 1/R^2$ is isometric to the sphere S^n_R of radius R in Euclidean space $\mathbb{R}^{n+1}$. Prove that the same conclusion holds if instead of assuming (M, g) is complete, we assume there exists a point $p \in M$ at which the exponential map $\exp_p$ is well defined on a ball $B_r(0) \subset T_pM$ of some radius $r > \pi R$ about the origin.

Problem 9
Suppose (M, g) is a connected Riemannian manifold of dimension $n \geqslant 3$ and $f : M \to \mathbb{R}$ is a smooth function such that the sectional curvature satisfies $K_S(P) = f(p)$ for all $P \subset T_pM$, $p \in M$. Prove that K_S is then constant. (Is this true for $n = 2$?)
Hint: Prove that g *is an Einstein metric.*