PROBLEM SET 10
Due: 20.07.2023

Instructions

Problems marked with $(*)$ will be graded—and since it is the end of the semester, **there are only three** of those this week. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Notation: In cases where the choice of coefficient group G does not matter, we will abbreviate $H_k(X) := H_k(X;G)$ and $H_k(X,A) := H_k(X,A;G)$.

Problems

1. In lecture, we used an algebraic result about short exact sequences of chain complexes with chain maps in order to derive the long exact sequence of a pair (X,A) in relative singular homology, and in particular the existence of the connecting homomorphisms $\partial_* : H_n(X,A) \to H_{n-1}(A)$ that fit into that exact sequence.

   (a) Show that for any class $[c] \in H_n(X,A)$ represented by a relative n-cycle $c \in C_n(X)$,[1] the map ∂_* is given explicitly by the formula
   $$\partial_*[c] = [\partial c],$$
   where on the right hand side, ∂c is regarded as an $(n-1)$-cycle in $C_{n-1}(A)$ and thus represents a homology class in $H_{n-1}(A)$. Achtung: the n-chain $c \in C_n(X)$ need not be contained in A, so its boundary $\partial c \in C_{n-1}(A)$ could represent a nontrivial homology class in $H_{n-1}(A)$, even though the class it represents in $H_{n-1}(X)$ is definitely trivial.
   Advice: You will need to work through a portion of the diagram-chasing argument that we used for defining the connecting homomorphism—we did this in lecture for an arbitrary collection of chain complexes and chain maps forming a short exact sequence, but the situation here is more specific.

   (b) Deduce the following "naturality" property: for any map of pairs $f : (X,A) \to (Y,B)$, the long exact sequences of (X,A) and (Y,B) form the rows of a commutative diagram
   $$\begin{tikzcd} \ldots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \ldots \\ \ldots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \ldots \end{tikzcd}$$

2. Let $i : A \hookrightarrow X$ denote the inclusion map for a pair (X,A). Show that the induced homomorphism $i_* : H_n(A) \to H_n(X)$ is an isomorphism for all n if and only if the relative homology groups $H_n(X,A)$ vanish for all n.

3. Suppose the following diagram of abelian groups and homomorphisms commutes, and that both of its rows are exact sequences, meaning $\operatorname{im} f = \ker g$, $\operatorname{im} g' = \ker h'$ and so forth:[2]
   $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "i"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E' \end{tikzcd}$$

[1] Recall that being a "relative n-cycle" in (X,A) means that the boundary of c is contained in A; in brief, $\partial c \in C_{n-1}(A)$.

[2] The lack of any additional arrows to the left of A and A' or to the right of E and E' means that there is no assumption about the kernels of f and f' or the images of i and i'.

(a) ($*$) Prove that if α, β, δ and ε are all isomorphisms, then so is γ.
*This result is known as the **five-lemma**.*

(b) Here is an application: given a map of pairs $f : (X, A) \to (Y, B)$, show that if any two of the induced homomorphisms $H_k(X) \to H_k(Y)$, $H_k(A) \to H_k(B)$ and $H_k(X, A) \to H_k(Y, B)$ are isomorphisms for every k, then so is the third.
Hint: You need the result of Problem 1(b) for this.

4. For any space X and integer $k \geqslant 1$, there is an isomorphism

$$S_* : H_k(X) \to H_{k+1}(SX),$$

where $SX := C_+X \cup_X C_-X$ denotes the suspension of X, defined by gluing together two homeomorphic copies of its cone CX. Letting $p_- \in C_-X \subset SX$ denote the tip of the bottom cone, one can construct S_* out of the following diagram:

$$\begin{tikzcd}
H_k(X) \arrow[rr, "S_*"] & & H_{k+1}(SX) \arrow[d, "k_*"] \\
H_{k+1}(C_+X, X) \arrow[u, "\partial_*"] \arrow[r, "i_*"] & H_{k+1}(SX \backslash \{p_-\}, C_-X \backslash \{p_-\}) \arrow[r, "j_*"] & H_{k+1}(SX, C_-X)
\end{tikzcd}$$

Here ∂_* is the connecting homomorphism from the long exact sequence of the pair (C_+X, X) and is an isomorphism due to the fact that the terms $H_k(C_+X)$ and $H_{k+1}(C_+X)$ in that sequence vanish, since C_+X is contractible. The other maps are all induced by the obvious inclusions of pairs and they are all also isomorphisms: i_* because i is a homotopy equivalence of pairs (see Example 25.1 in the lecture notes), j_* by the excision theorem, and k_* due to the fact that $H_k(C_-X) = H_{k+1}(C_-X) = 0$ in the long exact sequence of (SX, C_-X). One can use the diagram to write down a formula for S_*, but it isn't an immediately useful formula since it involves k_*^{-1} and ∂_*^{-1}, which we cannot so easily write in terms of cycles. But we can still characterize S_* in terms of cycles as follows:

(a) ($*$) Show that for any k-cycle $b \in C_k(X) \subset C_k(SX)$, there exist two $(k+1)$-chains $c_\pm \in C_{k+1}(C_\pm X) \subset C_{k+1}(SX)$ such that

$$\partial c_+ = -\partial c_- = b \tag{1}$$

and S_* satisfies

$$S_*[b] = [c_+ + c_-]. \tag{2}$$

Moreover, show that the formula (2) holds for any pair of $(k+1)$-chains $c_\pm \in C_{k+1}(C_\pm X)$ satisfying (1).
Hint: Start with a relative cycle representing $[c_+] \in H_{k+1}(C_+X, X)$, use the formula in Problem 1(a) for ∂_*, then follow the arrows wherever they lead you. When you get to the map k_*, deduce whatever you can from the fact that it is an isomorphism.

(b) ($*$) In the special case $X = S^k$, we have $SS^k \cong S^{k+1}$ and $H_1(S^1; \mathbb{Z}) \cong \pi_1(S^1) \cong \mathbb{Z}$, so induction on k gives $H_k(S^k; \mathbb{Z}) \cong \mathbb{Z}$ for all $k \in \mathbb{N}$. In this case we can define a distinguished generator $[S^k] \in H_k(S^k; \mathbb{Z})$, the *fundamental class* of S^k, inductively as follows:

- $[S^1] \in H_1(S^1; \mathbb{Z})$ is the image under the isomorphism $\pi_1(S^1) \to H_1(S^1; \mathbb{Z})$ of the canonical generator of $\pi_1(S^1)$ defined by the loop $\mathrm{Id} : S^1 \to S^1$.
- $[S^{k+1}] := S_*[S^k] \in H_{k+1}(S^{k+1}; \mathbb{Z})$ for $k \geqslant 1$.

Prove by induction that for each $k \in \mathbb{N}$, the class $[S^k] \in H_k(S^k; \mathbb{Z})$ can be represented by a cycle of the form $\sum_i \epsilon_i \sigma_i$, where $\epsilon_i = \pm 1$ and $\sigma_i : \Delta^k \to S^k$ are parametrizations of the k-simplices in an oriented triangulation of S^k.
Hint: There may be multiple valid ways to do this, but in my solution, I end up with 2^k simplices of dimension k in the triangulation of S^k.

5. Work through the rest of the proof of Theorem 23.5 in the lecture notes, the existence of the long exact sequence resulting from any short exact sequence of chain complexes. Stop when you either finish, get tired, or simply decide that you believe the theorem.