PROBLEM SET 3
Due: 11.05.2023

Instructions

Problems marked with (∗) will be graded.[1] Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Problems

1. In lecture we defined S^1 as the unit circle in $\mathbb{R}^2$ with the subspace topology (induced by the Euclidean metric on $\mathbb{R}^2$). Show that the following spaces with their natural quotient topologies are both homeomorphic to S^1:

   (a) $\mathbb{R}/\mathbb{Z}$, meaning the set of equivalence classes of real numbers where $x \sim y$ means $x - y \in \mathbb{Z}$.

   (b) (∗) $[0,1]/\sim$, where $0 \sim 1$.[2]

   For the next example, we introduce a convenient piece of standard notation. The quotient of a space X by a subset $A \subset X$ is defined as $X/A := X/\sim$ with the quotient topology, where the equivalence relation is defined such that $x \sim y$ for every $x, y \in A$ and otherwise $x \sim x$ for all $x \in X$. In other words, X/A is the result of modifying X by "collapsing A to a point".

   (c) Convince yourself that for every $n \in \mathbb{N}$, S^n is homeomorphic to $\mathbb{D}^n/S^{n-1}$, where $\mathbb{D}^n$ denotes the closed unit disk $\{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \le 1\}$.
   Remark: Part (b) becomes a special case of part (c) if we replace $[0,1]$ *by* $\mathbb{D}^1 = [-1,1]$.

2. Suppose X and Y are topological spaces, $x \in X$, and $K \subset Y$ is a compact subset.

   (a) (∗) Prove that every neighborhood of $\{x\} \times K$ in $X \times Y$ contains $\mathcal{V} \times K$ for some neighborhood $\mathcal{V} \subset X$ of x.

   (b) Find an example showing that the statement in part (a) is not always true if K is not compact.

3. Recall that $[0,1]^{\mathbb{R}}$ denotes the set of all functions $f : \mathbb{R} \to [0,1]$, with the topology of pointwise convergence. Tychonoff's theorem implies that $[0,1]^{\mathbb{R}}$ is compact, but one can show that it is not first countable, so it need not be sequentially compact.

   (a) For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x_{(n)} \in \{0, \dots, 9\}$ denote the nth digit to the right of the decimal point in the decimal expansion of x. Now define a sequence $f_n \in [0,1]^{\mathbb{R}}$ by setting $f_n(x) = \frac{x_{(n)}}{10}$. Show that for any subsequence f_{k_n} of f_n, there exists $x \in \mathbb{R}$ such that $f_{k_n}(x)$ does not converge, hence f_n has no pointwise convergent subsequence.
   Food for thought: Could you do this if you also had to assume that x is rational? Presumably not, because $[0,1]^{\mathbb{Q}}$ *is a product of countably many second countable spaces, and we proved in lecture that such products are second countable (unlike* $[0,1]^{\mathbb{R}}$*). This implies that since* $[0,1]^{\mathbb{Q}}$ *is compact, it must also be sequentially compact.*

   (b) (∗) The compactness of $[0,1]^{\mathbb{R}}$ does imply that every sequence has a convergent *subnet*, or equivalently, a cluster point. Use this to deduce that for any given sequence $f_n \in [0,1]^{\mathbb{R}}$, there exists a function $f \in [0,1]^{\mathbb{R}}$ such that for every finite subset $X \subset \mathbb{R}$, some subsequence of f_n converges to f at all points in X.

[1] For the first few problem sets in this semester we do not yet have a grader, so for each starred problem you will be given a pass/fail mark based on whether an obvious effort has been made.

[2] To clarify: in situations like this we always mean the *smallest* equivalence relation for which the stated equivalence holds, e.g. in this case, the unstated equivalences $0 \sim 0$, $1 \sim 1$ and $1 \sim 0$ are also assumed to hold since $\sim$ is required to be reflexive, symmetric and transitive.

Achtung: Pay careful attention to the order of quantifiers here. We're claiming that the element f exists independently of the finite set $X \subset \mathbb{R}$ on which we want some subsequence to converge to f. (If you could let f depend on the choice of subset X, this would be easy—but that is not allowed.) On the other hand, the actual choice of subsequence is allowed to depend on the subset X.

Challenge: Find a direct proof of the statement in part (b), without passing through Tychonoff's theorem. I do not know of any way to do this that isn't approximately as difficult as actually proving Tychonoff's theorem—I conjecture that it cannot be done without the axiom of choice, but I would be interested to know if I am wrong!

4. Consider the space $X = \left\{ f \in [0,1]^{\mathbb{R}} \;\middle|\; f(x) \neq 0 \text{ for at most countably many points } x \in \mathbb{R} \right\}$, with the subspace topology that it inherits from $[0,1]^{\mathbb{R}}$.

   (a) Show that X is sequentially compact.
   Hint: For any sequence $f_n \in X$, the set $\bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} \mid f_n(x) \neq 0\}$ is also countable.

   (b) For each $x \in \mathbb{R}$, define $\mathcal{U}_x = \{f \in X \mid -1 < f(x) < 1\}$. Show that the collection $\{U_x \subset X \mid x \in \mathbb{R}\}$ forms an open cover of X that has no finite subcover, hence X is not compact.

5. (a) Show that a finite topological space satisfies the axiom T_1 if and only if it carries the discrete topology.

   (b) Show that X is a T_2 space (i.e. Hausdorff) if and only if the *diagonal* $\Delta := \{(x,x) \in X \times X\}$ is a closed subset of $X \times X$.

   (c) Show that every compact Hausdorff space is regular, i.e. compact + $T_2 \Rightarrow T_3$.
   Hint: The argument needed for this was already used in lecture to prove something else.

   (d) $(*)$ Show that every metrizable space satisfies the axiom T_4 (i.e. it is *normal*).
   Hint: Given disjoint closed sets $A, A' \subset X$, each $x \in A$ admits a radius $\epsilon_x > 0$ such that the ball $B_{\epsilon_x}(x)$ is disjoint from A', and similarly for points in A' (why?). The unions of all these balls won't quite produce the disjoint neighborhoods you want, but try cutting their radii in half.

6. Each of the following should be relatively easy, but they are worth thinking about at least once.

   (a) Show that if X is a Hausdorff space, then every subset $A \subset X$ becomes a Hausdorff space when assigned the subspace topology.

   (b) Find an example of a non-Hausdorff space X with a subset $A \subset X$ that is Hausdorff with the subspace topology.

   (c) In Problem Set 1 #1(e), we considered the space $X = (-1,1)^2 \setminus \{(0,0)\} \subset \mathbb{R}^2$ with an equivalence relation $\sim$ and endowed the set $X/\!\sim$ of equivalence classes with the quotient topology (though we could not call it that at the time). Was that quotient space Hausdorff? Why or why not?

   Now consider an arbitrary collection $\{X_\alpha\}_{\alpha \in I}$ of topological spaces. Prove:

   (d) The disjoint union $\coprod_{\alpha \in I} X_\alpha$ is Hausdorff if and only if X_α is Hausdorff for every $\alpha \in I$.

   (e) The product $\prod_{\alpha \in I} X_\alpha$ is Hausdorff if and only if X_α is Hausdorff for every $\alpha \in I$.

7. Suppose X is a Hausdorff space and $\sim$ is an equivalence relation on X. Let $X/\!\sim$ denote the quotient space equipped with the quotient topology and denote by $\pi : X \to X/\!\sim$ the canonical projection. Given a subset $A \subset X$, we will sometimes also use the notation X/A explained in Problem 1.

   (a) A map $s : X/\!\sim \to X$ is called a *section* of π if $\pi \circ s$ is the identity map on $X/\!\sim$. Show that if a continuous section exists, then $X/\!\sim$ is Hausdorff.

   (b) Show that if X is also regular and $A \subset X$ is a closed subset, then X/A is Hausdorff.

   (c) $(*)$ Consider $X = \mathbb{R}$ with the non-closed subset $A = (0,1]$. Which of the separation axioms $T_0, \ldots, T_4$ does X/A satisfy?

   Just for fun: think about some other examples of Hausdorff spaces X with non-Hausdorff quotients $X/\!\sim$. What stops you from constructing continuous sections $X/\!\sim \to X$?