PROBLEM SET 8
Due: 22.06.2023

Instructions

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Note: All spaces mentioned on this sheet should be assumed "reasonable", meaning they are path-connected and satisfy the additional hypotheses needed for the lifting theorem and Galois correspondence.

Problems

1. Prove each of the following, assuming $p : Y \to X$ is a covering map with X and Y both path-connected.
   (a) If $\mathcal{U} \subset X$ is evenly covered, then so is every subset of $\mathcal{U}$.
   (b) The map $p : Y \to X$ is open, i.e. it sends open subsets of Y to open subsets of X.
   (c) For every $x \in X$, $f^{-1}(x)$ is a discrete subset of Y.[1]
   (d) If Y is compact, then X is also compact and $\deg(p) < \infty$.
   (e) (∗) The map $p : Y \to X$ is proper[2] if and only if $\deg(p) < \infty$.
   Hint: Showing that properness implies finite degree is easy. For the converse, given a compact set $K \subset X$ and an open cover $f^{-1}(K) \subset \bigcup_\alpha \mathcal{U}_\alpha$, it suffices to find a finite cover of $f^{-1}(K)$ by open sets such that each is contained in some $\mathcal{U}_\alpha$. (Why?) Start by showing that K can be covered by a finite collection of open neighborhoods which are evenly covered and small enough so that their (finitely many!) lifts to Y are each contained in some $\mathcal{U}_\alpha$.
   (f) Deduce from the above that the converse of part (d) also holds: if $\deg(p) < \infty$ and X is compact, then Y is also compact.

2. Assume $p : Y \to X$ is a covering map and X is path-connected.
   (a) (∗) Show that for any two points $x, y \in X$, lifting paths $x \overset{\gamma}{\rightsquigarrow} y$ determines a bijection $\rho_\gamma : p^{-1}(x) \to p^{-1}(y)$ that depends only on the homotopy class of the path γ (with fixed end points).
   (b) Writing $J := p^{-1}(x)$ and applying part (a) in the case $x = y$ gives a map $\rho : \pi_1(X, x) \to S(J)$ sending $[\gamma] \in \pi_1(X, x)$ to ρ_γ, where $S(J)$ is the group of all bijections $J \to J$.[3] Show that this map is a group anti-homomorphism, i.e. it satisfies $\rho_{\alpha \cdot \beta} = \rho_\beta \circ \rho_\alpha$ for all $[\alpha], [\beta] \in \pi_1(X, x)$.[4]
   (c) (∗) Write down the map $\rho : \pi_1(X, x) \to S(J)$ explicitly for the space $(X, x) = (\mathbb{C}^* := \mathbb{C}\backslash\{0\}, 1)$ with covering map $p : \mathbb{C} \to \mathbb{C}^* : z \mapsto e^z$.

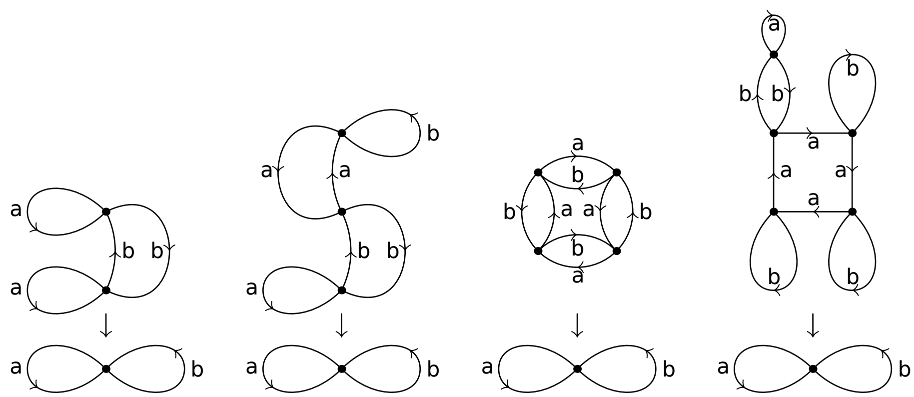
3. (a) Show that every covering map of degree 2 is regular.
   Hint: There is an algebraic way to solve this problem, but a more direct approach is also possible.
   (b) Prove that every covering map of the torus $\mathbb{T}^2 = S^1 \times S^1$ is regular.
   (c) Find all subgroups of $\mathbb{Z}^2$ with index 2.
   Hint: Every subgroup $H \subset \mathbb{Z}^2$ is normal since $\mathbb{Z}^2$ is abelian, and H then has index 2 if and only if the quotient $\mathbb{Z}^2/H$ is isomorphic to $\mathbb{Z}_2$. Consider the images of the two generators $e_1 := (1, 0)$ and

[1] We say that a subset A in a space X is *discrete* if the subspace topology induced by X on A is the same as the discrete topology.
[2] A map $f : X \to Y$ is said to be *proper* (*eigentlich*) if for every compact subset $K \subset Y$, $f^{-1}(K) \subset X$ is also compact.
[3] Notice that if J is a set of n elements, $S(J)$ is isomorphic to the *symmetric group* S_n.
[4] A small correction has been made on this problem sheet, because the original version claimed that ρ is a homomorphism, which is almost but not quite true. It is at least true if $\pi_1(X, x)$ is abelian, and it becomes true if one adopts a slightly unconventional definition of multiplication in $\pi_1(X, x)$, in which paths get concatenated in reverse order.

$e_2 := (0,1)$ *of* $\mathbb{Z}^2$ *under the quotient homomorphism* $\mathbb{Z}^2 \to \mathbb{Z}^2/H$. *Show that there are exactly three possibilities, depending on whether each of* e_1 *or* e_2 *represents the trivial or nontrivial element in the quotient.*

(d) ($*$) Deduce from part (c) that up to isomorphism of covers, $\mathbb{T}^2$ admits exactly three distinct covering maps with degree 2, and write them down explicitly.
Hint: You may have to take an educated guess as to what the covering spaces should be, but notice that part (c) tells you what their fundamental groups are.

4. Convince yourself that the maps depicted in the figure below are covers, and determine their deck transformation groups. Which ones are regular?



5. In this problem, we consider two base-point preserving covering maps

$$\underbrace{(Z, z_0) \xrightarrow{q} (Y, y_0) \xrightarrow{p} (X, x_0)}_{P},$$

whose composition is therefore also a base-point preserving covering map $P : (Z, z_0) \to (X, x_0)$. Let us abbreviate the automorphism groups of P and q by $G := \operatorname{Aut}(P)$ and $H := \operatorname{Aut}(q)$, so for instance if Z is simply connected (though we will not assume this below), then a theorem proved in lecture gives natural isomorphisms $G \cong \pi_1(X, x_0)$ and $H \cong \pi_1(Y, y_0)$. Our goal is to understand what $\operatorname{Aut}(p)$ is.

(a) ($*$) Use the path-lifting property to prove the following lemma: If $F \in G$ and $f \in \operatorname{Aut}(p)$ are deck transformations for which the relation $q \circ F = f \circ q$ holds at the base point $z_0 \in Z$, then it holds everywhere.
Hint: For any $z \in Z$, *choose a path from* z_0 *to* z, *then use* F, f *and the covering projections to cook up other paths in* Z, Y *and* X. *Some of them are lifts of others, and two important ones will turn out to be the same.*

(b) Deduce from part (a) that H is the subgroup of G consisting of all deck transformations $F : Z \to Z$ for P that satisfy $F(z_0) \in q^{-1}(y_0)$.

(c) Show that if $P : Z \to X$ is regular then so is $q : Z \to Y$. Give two proofs: one using the result of part (b), and another using the characterization of regularity in terms of normal subgroups.

(d) The *normalizer* $N(H) \subset G$ of the subgroup H is by definition the largest subgroup of G that contains H as a normal subgroup, i.e.

$$N(H) := \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\}.$$

Show that if the cover $q : Z \to Y$ is regular, then for any $F \in N(H)$, there exists a deck transformation $f : Y \to Y$ of p satisfying the relation $q \circ F = f \circ q$, and it is unique. Moreover, the correspondence $F \mapsto f$ defines a group homomorphism $N(H) \to \operatorname{Aut}(p)$ whose kernel is H.

(e) Show that if the cover $P : Z \to X$ is also regular, then the homomorphism $N(H) \to \operatorname{Aut}(p)$ in part (d) is also surjective, and thus descends to an isomorphism $N(H)/H \xrightarrow{\cong} \operatorname{Aut}(p)$.